数理解析研究所講究録
第 860 巻 1994 年 7-11

DILATN13.TEX; 1994/01/31

COMMUTING CONTRACTIONS の SIMULTANEOUS UNITARY DILATION

山形大学理学部　岡安隆照　(Takateru OKAYASU)

The following matter is realy fundamental:

Sz.-Nagy's Unitary Dilation Theorem. Let T be a contraction on a Hilbert space $\mathcal{H}$. Then, there exist an enlarged Hilbert space $\mathcal{K} \supseteq \mathcal{H}$ and a unitary U, called a unitary dilation of T, on $\mathcal{K}$, such that

$$T^m = PU^m|\mathcal{H} \quad \text{for} \quad m = 0, 1, 2, \cdots,$$

where P is the projection on $\mathcal{K}$ onto $\mathcal{H}$.

This yields, and, is yielded by, the so-called

von Neumann Inquality. Let T be a contraction on a Hilbert space. Then,

$$||p(T)|| \leq ||p|| = \sup_{z \in \mathbf{T}} |p(z)|$$

holds for any polynomial p with complex coefficients.

The "logical equivalence" is accompanied by the the following

Theorem [6]. If a set of commuting contractions on a Hilbert space $\mathcal{H}$, $T_1, T_2, \cdots, T_n$, admits a simultaneous unitary dilation, namely, there exist a Hilbert space $\mathcal{K} \supseteq \mathcal{H}$ and commuting unitaries $U_1, U_2, \cdots, U_n$ on $\mathcal{K}$, such that

$$T_1^{m_1} T_2^{m_2} \cdots T_n^{m_n} = P U_1^{m_1} U_2^{m_2} \cdots U_n^{m_n} | \mathcal{H}$$

for $m_1, m_2, \cdots, m_n = 0, 1, 2, \cdots$, where P is the projection on $\mathcal{K}$ onto $\mathcal{H}$, then $T_1, T_2, \cdots, T_n$ enjoys the von Neumann ineqality, namely,

$$||(p_{ij}(T_1, T_2, \cdots, T_n))|| \leq ||(p_{ij})|| = \sup_{z_1, z_2, \cdots, z_n \in \mathrm{T}} ||(p_{ij}(z_1, z_2, \cdots, z_n))||$$

holds for any $m \times m$ matrix (p_{ij}) whose entries are polynomials with complex coeffieients; and *vice versa.*

On the other hand, the following theorems are known:

Andô's Theorem [1]. Any pair of commuting contractions on a Hilbert space admits a simultaneous unitary dilation.

Andô's Theorem [2]. Any triple of commuting contractions on a Hilbert space, one of which duoble commutes with others, admits a simultaneous unitary dilation.

We, aside, have examples of triples of commuting contractions which do not admit a simultaneous unitary dilation, [4], [8] and [9].

In [6] we gave the following theorem and corollary:

Theorem. Suppose each of sets of commuting contractions, S_1, S_2, $\cdots$, S_m and $T_1, T_2, \cdots, T_n$, on a Hilbert space, admits a simultaneous unitary dilation, and every S_j double commutes with all T_k. If the set $S_1, S_2, \cdots, S_m$ generates a nuclear C^* algebra, then the set $S_1, S_2, \cdots$, S_m, T_1, T_2, $\cdots$, T_n admits a simultaneous unitary dilation.

Collorary. Suppose S is a GCR contraction, i.e., a contraction which generates a GCR (postliminal) algebra, $T_1, T_2, \cdots, T_n$ commuting contractions, on a Hilbert space, the set $T_1, T_2, \cdots, T_n$ admits a simultaneous unitary dilation. and S double commutes with all T_k. Then the set $S, T_1, T_2, \cdots, T_n$ admits a simultaneous unitary dilation.

The following, furtheremore, turned out to be true [7]:

Theorem. Suppose each of sets of commuting contractions, S_1, S_2, $\cdots$, S_m and T_1, T_2, $\cdots$, T_n, on a Hilbert space, admits a simultaneous unitary dilation, and every S_j double commutes with all T_k. If the set $S_1, S_2, \cdots, S_m$ generates an injective von Neumann algebra, then the set $S_1, S_2, \cdots, S_m, T_1, T_2, \cdots, T_n$ admits a simultaneous unitary dilation.

Collorary. Suppose S is a type I contraction, i.e., a contraction which generates a type I von Neumann algebra, $T_1, T_2, \cdots, T_n$ commuting contractions, on a Hilbert space, the set $T_1, T_2, \cdots, T_n$ admits a simultaneous unitary dilation and S double commutes with all T_k. Then, the set $S, T_1, T_2, \cdots, T_n$ admits a simultaneous unitary dilation.

We here will improve the theorem, by making the assumption thin as the following

Theorem. Suppose each of sets of commuting contractions, S_1, S_2, $\cdots$, S_m and $T_1, T_2, \cdots, T_n$, on a Hilbert space, admits a simultaneous unitary dilation, and every S_j double commutes with all T_k. Then, the set $S_1, S_2, \cdots, S_m, T_1, T_2, \cdots, T_n$ admits a simultaneous unitary dilation.

This is the aimed theorem of ours. A proof of this is given, on acount of the Steinspring representation of completely positive maps, by the preceding theorem and the

Arveson Theorem [3, Theorem 1.3.1]. Let $\mathcal{H}$, $\mathcal{K}$ be Hilbert spaces, V a bounded operator from $\mathcal{H}$ into $\mathcal{K}$, and $\mathcal{B}$ a *subalgebra of $\mathcal{B}(\mathcal{K})$, the full operator algebra, which satisfies that $[\mathcal{B}V\mathcal{H}] = \mathcal{K}$. Then, for every $T \in (V^*\mathcal{B}V)'$ there exists a unique $\tilde{T} \in \mathcal{B}'$ such that $\tilde{T}V = VT$, and the mapping $(\ )^{\sim} : (V^*\mathcal{B}V)' \longrightarrow \mathcal{B}'$ is a σ weakly continuous *homomorphism.

We have as well

Collorary. Suppose each of pairs of commuting contractions, S_1, S_2, and T_1, T_2, on a Hilbert space, admits a simultaneous unitary dilation, and each of S_1, S_2 double commutes with T_1, T_2. Then, the set S_1, S_2, T_1, T_2 admits a simultaneous unitary dilation.

Our theorem, of course, gives a good understanding to Andô's "triple" assertion; on the Andô's "pair" assertion, the next matter sheds light:

Theorem [5, Theorem 6]. Let T be a contraction on a Hilbert space $\mathcal{H}$, U the minimal unitary dilation of T. Then for every $S \in \{T\}'$ there exists $\tilde{S} \in \{U\}'$ such that $S = P\tilde{S}|\mathcal{H}$ and $||\tilde{S}|| = ||S||$.

References

1. T. Andô, On a pair of commuting contractions, Acta Sci. Math. 24(1963), 88-90.

2. T. Andô, Unitary dilation for a triple of commuting contractions, Bull. Acad. Polonaise Math. 24(1976), 851-853.

3. W. B. Arveson, Subalgebras of C^* algebras, Acta Math. 123(1969), 141-224.

4. M. J. Crabb, and A. M. Davie, von Neumann's inequality for Hilbert space operators, Bull. London Math. Soc. 7(1975), 49-50.

5. R. G. Douglas, P. S. Muhley, and C. Pearcy, Lifting commuting operators, Michgan Math. Journ. 15(1968), 385-395.

6. T. Okayasu, The von Neumann inequality and dilation theorems for contractions, Operator Theory: Adv. Appl. 59(1992), 285-291.

7. 岡安隆照, The von Neumann inequality, 日本数学会秋期総合分科会, 1992/10/6.

8. I. Parrott, Unitary dilations for commuting contractions, Pacif. Jorn. Math. 34(1970), 481-490.

9. N. Th. Varopoulos, On an inequality of von Neumann and an application of metric theory of tensor products to operator theory, Journ. Funct, Analy. 16(1974), 83-100.