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$A \geq B \geq 0$ ensures $(B^rA^pB^r)^{1/q} \geq (B^rB^pB^r)^{1/q}$ for $r \geq 0$, $p \geq 0$, $q \geq 1$ with $(1+2r)q \geq p+2r$ and its applications

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In what follows, capital letter means a bounded linear operator on a Hilbert space.

An operator T is said to be positive (in symbol : $T \geq 0$) if $(Tx, x) \geq 0$ for all $x \in H$. Also an operator T is strictly positive (in symbol : $T > 0$) if T is positive and invertible.

As an extension of the Löwner-Heinz theorem [17][20], we established the Furuta inequality [6] which reads as follows. If $A \geq B \geq 0$, then for each $r \geq 0$ (i) $(B^rA^pB^r)^{1/q} \geq (B^rB^pB^r)^{1/q}$ and (ii) $(A^rA^pA^r)^{1/q} \geq (A^rB^pA^r)^{1/q}$ hold for p and q such that $p \geq 0$ and $q \geq 1$ with $(1+2r)q \geq p+2r$. We remark that the Furuta inequality yields the Löwner-Heinz theorem when we put $r = 0$ in (i) or (ii) stated above : if $A \geq B \geq 0$ ensures $A^\alpha \geq B^\alpha$ for any $\alpha \in [0,1]$. Alternative proofs of the Furuta inequality are given in [3][8][18] and an elementary proof is shown in [9].

Theorem A (Löwner-Heinz 1934). *If $A \geq B \geq 0$ ensures $A^\alpha \geq B^\alpha$ for any $\alpha \in [0,1]$.*

Related to Theorem A, the following result is well known.

Proposition. *If $A \geq B \geq 0$ does not always ensure $A^p \geq B^p$ for any $p > 1$.*

As a generalization of Theorem A and related to Proposition, we established the following result.

Theorem B (Furuta 1987). *If $A \geq B \geq 0$, then for each $r \geq 0$*

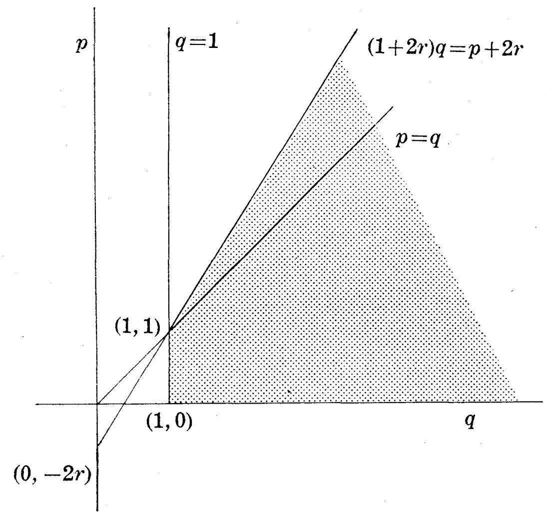
(i)
$$(B^rA^pB^r)^{1/q} \geq (B^rB^pB^r)^{1/q}$$

and

(ii)
$$(A^rA^pA^r)^{1/q} \geq (A^rB^pA^r)^{1/q}$$

hold for each p and q such that $p \geq 0$, $q \geq 1$ and $(1+2r)q \geq p+2r$.

Inequalities (i) and (ii) in Theorem B hold for the points on p, q and r belong to the oblique lines in the following figure.



Figure

In this paper, we cite several applications of Theorem B as follows.

Applications of Theorem B

(A) **Operator inequalities**

(1) Characterizations of operators satisfying $logA \geq logB$

(2) Generalizations of Ando's theorem

(3) Applications to the relative operator entropy

(4) Applications to other operator inequalities

(5) Applications to the Log-Majorization by Ando and Hiai

(6) Application to p-hyponormal operators for $0 < p < 1$

....

(B) **Norm inequalities**

(1) Several type generalizations of Heinz-Kato theorem

(2) Generalizations of some folk theorem on norm

(C) **Operator equations**

(1) Generalizations of Pedersen-Takesaki theorem and related results

Among applications of Theorem B states above, we cite [2][4][5][10] and [11] for (A) operator inequalities and also we cite [12][13][14] and [16] for (B) norm inequalities and finally we cite [7] for (C) operator equations.

Ando-Hiai [1] have established a lot of useful and beautiful results on log-majorization and we are really impressed with these beautiful and useful results. The purpose of this paper is to announce new application [15] of Theorem B to the log-majorization by Ando-Hiai [1]. Precisely speaking, we can interpolate Theorem B and this log-majorization.

§1. AN EXTENSION OF THE FURUTA INEQUALITY

First of all, we state the following extension of the Furuta inequality.

Theorem 1.1. *If $A \geq B \geq 0$ with $A > 0$, then for each $t \in [0,1]$ and $p \geq 1$,*

$$F_{p,t}(A,B,r,s) = A^{-r/2}\{A^{r/2}(A^{-t/2}B^pA^{-t/2})^sA^{r/2}\}^{\frac{1-t+r}{(p-t)s+r}}A^{-r/2}$$

is a decreasing function of both r and s for any $s \geq 1$ and $r \geq t$ and the following inequality holds

$$\begin{aligned} A^{1-t} &= F_{p,t}(A,A,r,s) \\ &\geq F_{p,t}(A,B,r,s) \end{aligned} \tag{1.10}$$

for any $s \geq 1, p \geq 1$ and r such that $r \geq t \geq 0$.

Corollary 1.2. *If $A \geq B \geq 0$ with $A > 0$, then for each $t \in [0,1]$,*

$$\{A^{r/2}(A^{-t/2}A^pA^{-t/2})^sA^{r/2}\}^{\alpha} \geq \{A^{r/2}(A^{-t/2}B^pA^{-t/2})^sA^{r/2}\}^{\alpha}$$

holds for any $s \geq 0$, $p \geq 0$, $0 \leq \alpha \leq 1$ and $r \geq t$ with $(s-1)(p-1) \geq 0$ and $1-t+r \geq ((p-t)s+r)\alpha$.

Remark 1.1. In the case $t = 0$ in Corollary 1.2 ,we may not assume $A > 0$. Putting $t = 0$ and $s = 1$ in Corollary 1.2, we have (ii) of Theorem B. Hence Corollary 1.2 can be considered as an extension of Theorem B since (i) is equivalent to (ii) in Theorem B.

Corollary 1.2 easily implies the following result when we put $t = 1$.

Corollary 1.3. *If $A \geq B \geq 0$ with $A > 0$, then*

$$A^r \geq \{A^{r/2}(A^{-1/2}B^pA^{-1/2})^sA^{r/2}\}^{\frac{r}{(p-1)s+r}}$$

holds for any $s \geq 1, p \geq 1$ and $r \geq 1$.

When we put $s = r$ in Corollary 1.3, we have the following Theorem C obtained by Ando and Hiai [1,Theorem 3.5].

Theorem C [1]. *If* $A \geq B \geq 0$ *with* $A > 0$, *then*

$$A^r \geq \{A^{r/2}(A^{-1/2}B^pA^{-1/2})^rA^{r/2}\}^{1/p}$$

holds for any $p \geq 1$ *and* $r \geq 1$.

Corollary 1.4. *If* $A \geq B \geq 0$ *with* $A > 0$, *then for each* $t \in [0,1]$

(i) $$A^{1+t} \geq (A^{t/2}B^{2p-t}A^{t/2})^{\frac{1+t}{2p}} \geq |A^{-t/2}B^pA^{t/2}|^{\frac{1+t}{p}}$$

and

(ii) $$A^2 \geq (A^{1/2}B^{2p-t}A^{1/2})^{\frac{2}{2p+1-t}} \geq |A^{-t/2}B^pA^{1/2}|^{\frac{4}{2p+1-t}}$$

hold for any $2p \geq 1+t$.

Corollary 1.5. *If* $A \geq B \geq 0$ *with* $A > 0$, *then*

$$A^2 \geq (A^{1/2}B^{2p-1}A^{1/2})^{1/p} \geq |A^{-1/2}B^pA^{1/2}|^{2/p} \text{ for any } p \geq 1.$$

Corollary 1.6 [4][10][11]. *If* $A \geq B \geq 0$, *then*

$$G(p,r) = A^{-r/2}(A^{r/2}B^pA^{r/2})^{(1+r)/(p+r)}A^{-r/2}$$

is a decreasing function of both p *and* r *for* $p \geq 1$ *and* $r \geq 0$.

§2. THE LOG-MAJORIZATION EQUIVALENT TO AN EXTENSION OF THE FURUTA INEQUALITY

Throughout this section , a capital letter means $n \times n$ matrix.

Following after Ando and Hiai [1], let us write $A \underset{(\log)}{\prec} B$ for positive semidefinite matrices $A, B \geq 0$ and call the *log-majorization* if

$$\prod_{i=1}^{k} \lambda_i(A) \leq \prod_{i=1}^{k} \lambda_i(B), \qquad k = 1, 2, ..., n-1,$$

and

$$\prod_{i=1}^{n} \lambda_i(A) = \prod_{i=1}^{n} \lambda_i(B), \quad \text{i.e. } \det A = \det B,$$

where $\lambda_1(A) \geq \lambda_2(A) \geq ... \geq \lambda_n(A)$ and $\lambda_1(B) \geq \lambda_2(B) \geq ... \geq \lambda_n(B)$ are the eigenvalues of A and B respectively arranged in decreasing order. Note that when $A, B > 0$ (strictly positive) the log-majorization $A \underset{(\log)}{\prec} B$ is equivalent to $logA \prec logB$. Also $A \underset{(\log)}{\prec} B$ ensures $||A|| \leq ||B||$ holds for any unitarily invariant norm.

Definition 1. When $0 \leq \alpha \leq 1$, the α-power mean of $A, B > 0$ is defined by

$$A \ \#_\alpha B = A^{1/2}(A^{-1/2}BA^{-1/2})^\alpha A^{1/2}.$$

Further $A\#_\alpha B$ for $A, B \geq 0$ is defined by

$$A\#_\alpha B = \lim_{\epsilon \downarrow 0}(A + \epsilon I)\#_\alpha(B + \epsilon I).$$

This α-power mean is the operator mean corresponding to the operator monotone function t^α. We can see [19] for general theory of operator means.

For the sake of convenience for symbolic expression , we define $A \natural_s B$ for any $s \geq 0$ and for $A > 0$ and $B \geq 0$ by the following

$$A \natural_s B = A^{1/2}(A^{-1/2}BA^{-1/2})^s A^{1/2}.$$

$A \natural_\alpha B$ in the case $0 \leq \alpha \leq 1$ just coincides with the usual α-power mean denoted by $A\#_\alpha B$.

We can transform (1.10) of Theorem 1.1 into the following log-majorization inequality by using the method by Ando and Hiai [1].

Theorem 2.1. *For every $A > 0$, $B \geq 0$, $0 \leq \alpha \leq 1$ and each $t \in [0, 1]$*

(2.1)
$$(A\#_\alpha B)^h \underset{(\log)}{\succ} A^{1-t+r}\#_\beta\,(A^{1-t} \natural_s B)$$

holds for $s \geq 1$, and $r \geq t \geq 0$, where $\beta = \dfrac{\alpha(1-t+r)}{(1-\alpha t)s + \alpha r}$ *and* $h = \dfrac{(1-t+r)s}{(1-\alpha t)s + \alpha r}$.

Corollary 2.2. *For every $A, B \geq 0$ and $0 \leq \alpha \leq 1$,*

$$(2.3) \qquad (A\#_\alpha B)^h \underset{(\log)}{\succ} A^r \#_{\frac{h\alpha}{s}} B^s \qquad \text{for } r \geq 1 \text{ and } s \geq 1$$

where $h = [\alpha s^{-1} + (1-\alpha)r^{-1}]^{-1}$.

The above log-majorization is equivalent to any one of the following (2.4) ,(2.5) and (2.6) :

$$(2.4) \qquad (A^r \#_\alpha B^r)^{1/r} \underset{(\log)}{\succ} (A^q \#_{\frac{k\alpha}{p}} B^p)^{1/k} \qquad \text{for } 0 < r \leq q \text{ and } 0 < r \leq p,$$

where $k = [\alpha p^{-1} + (1-\alpha)q^{-1}]^{-1}$.

$$(2.5) \qquad (A^r \#_\alpha B^q)^{1/s} \underset{(\log)}{\succ} (A^p \#_{\frac{l\alpha}{r}} B^p)^{1/p} \qquad \text{for } 0 < r \leq p \text{ and } 0 < q \leq p\ ,$$

where $s = \alpha q + (1-\alpha)r$ and $l = [\alpha r^{-1} + (1-\alpha)q^{-1}]^{-1}$.

$$(2.6) \qquad (A^r \#_\alpha B^q)^{1/u} \underset{(\log)}{\succ} (A^q \#_\beta B^p)^{1/p} \qquad \text{for } 0 < r \leq q \leq p,$$

where $u = \dfrac{\alpha q^2 + (1-\alpha)pr}{q}$ and $\beta = \dfrac{\alpha q^2}{\alpha q^2 + (1-\alpha)pr}$.

Remark 2.1. We remark that $h = [\alpha s^{-1} + (1-\alpha)r^{-1}]^{-1}$ in Corollary 2.2 is a generalized harmonic mean of r and s and when $\alpha = 1/2$, h is the usual harmonic mean of r and s. Also l in (2.5) is a generalized harmonic one of r and q, while s in (2.5) is a generalized arithmetic mean of q and r.

Corollary 2.2 yields the following result [1, Theorem 2.1].

Theorem D [1]. *For every $A, B \geq 0$ and $0 \leq \alpha \leq 1$,*

$$(A\#_\alpha B)^r \underset{(\log)}{\succ} A^r \#_\alpha B^r \qquad \text{for } r \geq 1$$

or equivalently

$$(A^q \#_\alpha B^q)^{1/q} \underset{(\log)}{\succ} (A^p \#_\alpha B^p)^{1/p} \qquad \text{for } 0 < q \leq p.$$

Also we can transform Corollary 1.2 into the following log-majorization.

Theorem 2.3. *If $A > 0$ and $B \geq 0$, then for each $t \in [0,1]$ and $0 \leq \alpha \leq 1$*

$$(A^{1/2}BA^{1/2})^{\alpha ps} \underset{(\log)}{\succ} A^{\frac{1}{2}\alpha((p-t)s+r)}(A^{\frac{-(r-t)}{2}}(A^{t} \natural_{s} B^{p})A^{\frac{-(r-t)}{2}})^{\alpha} A^{\frac{1}{2}\alpha((p-t)s+r)}$$

$$= A^{\frac{q}{2}}[A^{r-t} \#_{\alpha} (A^{-t} \natural_{s} B^{p})]A^{\frac{q}{2}}$$

holds for any nonnegative numbers s, p and r such that $r \geq t$ and $(s-1)(p-1) \geq 0$ with $1 - t + r \geq ((p-t)s + r)\alpha$ where $q = \alpha(p-t)s + \alpha r - r + t$.

Theorem 2.4. *If $A > 0$ and $B \geq 0$, then for each $t \in [0,1]$ and $0 \leq \alpha \leq 1$*

$$A^{1/2}(A^{p} \#_{\alpha} B^{p})^{q/p} A^{1/2}$$

$$\underset{(\log)}{\succ} A^{\frac{1}{2}(1-\frac{qt}{p}+\frac{rq}{ps})}\{A^{-r/2}[A^{\frac{t}{2}}(A^{p} \#_{\alpha} B^{p})A^{\frac{t}{2}}]^{s} A^{-r/2}\}^{\frac{q}{sp}} A^{\frac{1}{2}(1-\frac{qt}{p}+\frac{rq}{ps})}$$

holds for every $p \geq q > 0$, $r \geq t$ and $s \geq 1$.

When $t = 0$ Theorem 2.4 becomes the following result.

Corollary 2.5 . *If $A > 0$ and $B \geq 0$, then for every $0 \leq \alpha \leq 1$*

$$A^{1/2}(A^{p} \#_{\alpha} B^{p})^{q/p} A^{1/2}$$

$$\underset{(\log)}{\succ} A^{\frac{1}{2}(1+\frac{rq}{ps})}\{A^{-r/2}(A^{p} \#_{\alpha} B^{p})^{s} A^{-r/2}\}^{\frac{q}{sp}} A^{\frac{1}{2}(1+\frac{rq}{ps})}$$

holds for every $p \geq q > 0$, $r \geq 0$ and $s \geq 1$.

When $s = 1$ and $r = p$ Corollary 2.5 yields the following Theorem E [1,Theorem **3.3**].

Theorem E [1] . *If $A > 0$ and $B \geq 0$, then*

$$A^{1/2}(A^{p} \#_{\alpha} B^{p})^{q/p} A^{1/2}$$

$$\underset{(\log)}{\succ} A^{\frac{1+q}{2}}(A^{-p/2}B^{p}A^{-p/2})^{\frac{\alpha q}{p}} A^{\frac{1+q}{2}}$$

for every $0 \leq \alpha \leq 1$ and $0 < q \leq p$.

Taking $s = 2$ and $r = p$ in Corollary 2.5 we have

Corollary 2.6 . *If* $A > 0$ *and* $B \geq 0$, *then for every* $0 \leq \alpha \leq 1$

$$A^{1/2}(A^p \#_\alpha B^p)^{q/p} A^{1/2}$$

$$\underset{(\log)}{\succ} A^{\frac{1}{2}(1+\frac{q}{2})}\{(A^{-p/2}B^pA^{-p/2})^\alpha A^p (A^{-p/2}B^pA^{-p/2})^\alpha\}^{\frac{q}{2p}} A^{\frac{1}{2}(1+\frac{q}{2})}$$

holds for any $0 < q \leq p$.

Corollary 2.7. *If* $A > 0$ *and* $B \geq 0$, *then for every* $0 \leq r \leq 1$

$$A^{r/2}B^rA^{r/2} \underset{(\log)}{\succ} A^{\frac{r(1+\alpha)}{2}}(A^{-1/2}B^{1/\alpha}A^{-1/2})^{\alpha r}A^{\frac{r(1+\alpha)}{2}}$$

holds for every $0 < \alpha \leq 1$.

Corollary 2.8. *If* $A > 0$ *and* $B \geq 0$, *then for every* $0 \leq r \leq 1$

$$(A^{1/2}BA^{1/2})^r \underset{(\log)}{\succ} A^{\frac{\alpha u + r}{2}}(A^{-u/2}B^{r/\alpha}A^{-u/2})^\alpha A^{\frac{\alpha u + r}{2}}$$

holds for every $0 < \alpha \leq 1$ *and* $u \geq 0$.

Corollary 2.7 and Corollary 2.8 imply the following known result [1, Corollary 3.4].

Corollary F [1]. *If* $A > 0$ *and* $B \geq 0$, *then for every* $0 \leq r \leq 1$

$$(A^{1/2}BA^{1/2})^r \underset{(\log)}{\succ} A^{r/2}B^rA^{r/2} \underset{(\log)}{\succ} A^r(A^{-1/2}BA^{-1/2})^rA^r.$$

§3. LOGARITHMIC TRACE INEQUALITIES AS AN APPLICATION OF LOG-MAJORIZATION IN §2

Throughout this section, a capital letter means $n \times n$ matrix.

Theorem 3.1 . *If* $A > 0$ *and* $B \geq 0$, *then for every* $0 \leq \alpha \leq 1$ *and* $t \in [0,1]$

$$s\mathrm{Tr}A\log(A^p\#_\alpha B^p) - \mathrm{Tr}A\log\{A^{-r/2}[A^{t/2}(A^p\#_\alpha B^p)A^{t/2}]^sA^{-r/2}\}$$

$$\geq (r - st)\mathrm{Tr}A\log A$$

holds for any $s \geq 1$, $r \geq t$ and $p \geq 0$.

When $t = 0$ Theorem 3.1 yields the following result.

Corollary 3.2 . *If* $A > 0$ *and* $B \geq 0$, *then for every* $0 \leq \alpha \leq 1$

$$s\mathrm{Tr}Alog(A^p\#_\alpha B^p) - \mathrm{Tr}Alog\{A^{-r/2}[A^p\#_\alpha B^p]^s A^{-r/2}\}$$

$$\geq r\mathrm{Tr}AlogA$$

holds for any $s \geq 1$, $r \geq 0$ and $p \geq 0$.

Taking $s = 1$ and $r = p > 0$ in Corollary 3.2 we have the following result [1,Theorem 5.3].

Theorem G [1] . *If* $A \geq 0$ *and* $B > 0$, *then for every* $0 \leq \alpha \leq 1$ *and* $p > 0$

$$\frac{1}{p}\ \mathrm{Tr}Alog(A^p\#_\alpha B^p) + \frac{\alpha}{p}\mathrm{Tr}Alog(A^{p/2}B^{-p}A^{p/2})$$

$$\geq \mathrm{Tr}AlogA.$$

Corollary 3.3 . *If* $A > 0$ *and* $B > 0$, *then for every* $0 \leq \alpha \leq 1$

$$\mathrm{Tr}Alog(A^p\#_\alpha B^p) + \mathrm{Tr}Alog\{A^{q/2}[A^{-p}\#_\alpha B^{-p}]A^{q/2}\}$$

$$\geq q\mathrm{Tr}AlogA$$

holds for any $p \geq 0$ and $q \geq 0$.

We remark that Corollary 3.3 yields Theorem G stated above taking $q = p$.

Also taking $s = 2$, $t = 0$ and $r = p \geq 0$ in Theroem 3.1 we have :

Corollary 3.4. *If* $A > 0$ *and* $B > 0$, *then for every* $0 \leq \alpha \leq 1$

$$\mathrm{Tr}Alog(A^p\#_\alpha B^p)^2 + \mathrm{Tr}Alog\{(A^{p/2}B^{-p}A^{p/2})^\alpha A^{-p}(A^{p/2}B^{-p}A^{p/2})^\alpha\}$$

$$\geq p\mathrm{Tr}AlogA$$

holds for any $p \geq 0$.

At the end of this early announcement, we summarize the following implication relations among results in this paper.

$$\begin{tikzcd}
 & \text{(1.10) in } \textbf{Theorem 1.1} \arrow[dl, "t=0"'] \arrow[d, "t=1"] \arrow[r, Leftrightarrow] & \textbf{Theorem 2.1} \arrow[d, "t=1"] \\
\textbf{Corollary 1.6} \arrow[dd] & \textbf{Corollary 1.3} \arrow[dd, "r=s"] \arrow[r, Leftrightarrow] & \textbf{Corollary 2.2} \arrow[dd, "r=s"] \\
 & & \\
\textbf{Theorem B } [6] & \textbf{Theorem C } [1] \arrow[r, Leftrightarrow] & \textbf{Theorem D } [1] \\
\text{(Furuta inequality)} & & \text{(log-majorization)}
\end{tikzcd}$$

The details , proofs and related results in this paper will appear in [15].

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