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A Note on Woodford's Conjecture: Constructing Stationary Sunspot Equilibria in a Continuous Time Model*

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Abstract

We show how to construct stationary sunspot equilibria in a continuous time model, where equilibrium is indeterminate near either a steady state or a closed orbit. Woodford's conjecture that the indeterminacy of equilibrium implies the existence of stationary sunspot equilibria remains valid in a continuous time model.

Introduction

If for given equilibrium dynamics there exist a continuum of non-stationary perfect foresight equilibria all converging asymptotically to a steady state (a deterministic cycle resp.), we say the equilibrium dynamics is indeterminate near the steady state (the deterministic cycle resp.). Suppose that the fundamental characteristics of an economy are deterministic, but that economic agents believe nevertheless that equilibrium dynamics is affected by random factors apparently irrelevant to the fundamental characteristics (sunspots). This prophecy could be self-fulfilling, and one will get a sunspot equilibrium, if the resulting equilibrium dynamics is subject to a nontrivial stochastic process and confirms the agents' belief. See Shell [19], and Cass-Shell [3].

Woodford [23] suggested that there exists a close relation between the indeterminacy of equilibrium near a deterministic steady state and the existence of stationary sunspot equilibria in the immediate vicinity of it. See also Azariadis [1]. We might summarize Woodford's conjecture as what follows: "Let $\bar{x}$ be a steady state of a deterministic model which has a continuum of non stationary perfect foresight equilibria all converging asymptotically to the steady state. Then given any neighborhood $U(\bar{x})$ of it, there exist stationary sunspot equilibria with a support in $U(\bar{x})$."

Azariadis [1], Farmer-Woodford [9], Grandmont [10], Guesnerie [11], Woodford [24], and Peck [18] have shown that the conjecture holds good in various kinds of models. Woodford [25], Spear-Srivastatva-Woodford [22], and Chiappori-Geoffard-Guesnerie [4] investigate the connection between the local indeterminacy of equilibria and the existence of local stationary sunspot equilibria thoroughly and show the conjecture holds good in extremely general situations. However the existing results supporting Woodford's conjecture are all derived from discrete time models. See Chiappori-Guesnerie [5] and Guesnerie-Woodford [12] for thorough surveys on the existing sunspot literature. The purpose of this note is to show that Woodford's conjecture extends to a continuous time model. We present the method of constructing stationary sunspot equilibria near a steady state (a closed orbit resp.) in a continuous time model, where equilibrium is indeterminate near the steady state (the closed orbit resp.). One can use our method to show there exist stationary sunspot equilibria in such models as treated by Howitt-McAfee [15], Hammour [13], Diamond-Fudenberg [6], Benhabib-Farmer [2], and Drazen [8]. The models treated by [13, 6, 8] include a stable limit cycle, where equilibrium is indeterminate around the stable limit cycle. One can use our method to construct stationary sunspot equilibria around the stable limit cycle in these models.

Earlier results on the existence of sunspot equilibria are based on the overlapping generations model, where fluctuations exhibited all occur on time scale too long compared to the life times of agents. However, as shown by Woodford [24], Spear [21], and Kehoe-Levine-Romer [17], frictions like cash-in-advance constraints, externalities, and proportional taxation can generate market dynamics amenable to the construction of sunspot equilibria in otherwise well-behaved models having finitely-many infinite-lived agents. See Kehoe-Levine-Romer [16] for the well-behaved case. The models treated by [15, 13, 6, 2, 8] also include infinite-lived agents, and, in spite of this, generate indeterminate equilibria through various kinds of market imperfections.

The note is composed of four sections. Section 1 presents our model. Section 2 describes deterministic equilibrium dynamics. Section 3 specifies a Markov process which generates sunspot variables. Section 4 proves the existence of stationary sunspot equilibria.

1 The Model

Let $\begin{bmatrix} \dot{K}_t \\ (E_t dq_t)/dt \end{bmatrix} = F(K_t, q_t) \in R^2 \qquad (1)$

be a first order condition of some intertemporal optimization problem with market equilibrium conditions incorporated. F is assumed to be a continuously differentiable function (i.e. a C^1 function). K_t is a predetermined variable. q_t is a forwardlooking variable. E_t is a conditional expectation operator. We assume fundamental characteristics of an economy such as preferences, technologies, and endowments are deterministic. In other words, there is no intrinsic uncertainty. Any random factor is irrelevant to the fundamentals (sunspot). That is, the only uncertainty is extrinsic.

Our equilibrium stochastic process is described by

$$\begin{bmatrix} \dot{K}_t dt \\ dq_t \end{bmatrix} = F(K_t, q_t)dt + s\begin{bmatrix} 0 \\ d\varepsilon_t \end{bmatrix}, \qquad (2)$$

where we assume $\lim_{h \to +0} E(\varepsilon_{t+h} - \varepsilon_t | \varepsilon_s, s \leq t)/h$ is well defined and equal to 0 so that $(E_t d\varepsilon_t)/dt = 0$. $s \in (-\underline{\eta}, \overline{\eta})$, $\underline{\eta}$ and $\overline{\eta}$ are sufficiently small positive numbers. dt is a Lebesgue measure. dq_t and $d\varepsilon_t$ are Lebesgue-Stieltjes signed measures with respect to t. We assume $d\varepsilon_t$ is a "singular" signed measure of t relative to the Lebesgue measure dt. $\int_t^{t+h} d\varepsilon_s = \varepsilon_{t+h} - \varepsilon_t$ and ε_t is a random variable irrelevant to fundamentals (i.e. a sunspot variable). For $s = 0$, the system is deterministic, whereas it is stochastic for $s \neq 0$.

We define a sunspot equilibrium as follows. A sunspot equilibrium is a stochastic process $\{(K_t, q_t, \varepsilon_t)\}_{t \geq 0}$ with a compact support such that $\{(K_t, q_t)\}_{t \geq 0}$ is a solution of the stochastic differential equation (2) with $s \neq 0$. If the sunspot equilibrium is a stationary stochastic process, we call it a stationary sunspot equilibrium.

2. Deterministic Dynamics

We assume the deterministic equilibrium dynamics, where sunspots do not matter, satisfies the following condition.

Assumption 1. $(\dot{K}, \dot{q}) = F(K, q)$ is a C^1 vector field defined on an open subset W on R^2. W includes a compact convex subset D with nonempty interior points such that the vector field $(\dot{K}, \dot{q})$ points inward on the boundary, ∂D, of D. (See Figure 1.)

Under Assumption 1 the differential equation has a unique forward solution for any initial condition located on D. Let $x_t = \phi(t, x)$ be a solution of $\dot{x} = F(x)$ with an initial condition $x_0 = x \in D$. $\phi : [0, +\infty) \times D \rightarrow D$ is a well defined continuously differentiable function.

If there exist $x \in D$ and a monotonically increasing sequence $t_n \rightarrow \infty$, $n = 1, 2, \ldots$ such that $\lim_{n \rightarrow \infty} \phi(t_n, x) = y \in D$, y is called an ω-limit point of x. For $t > 0$, $x \in D$, define $\phi(-t, x) \in D$ as an inverse image z of $x = \phi(t, z)$, if the latter is well defined. Suppose $\phi(-t, x) \in D$ is well defined for $\forall t > 0$ for some $x \in D$. If there exists a monotonically increasing sequence $t_n \rightarrow \infty$, $n = 1, 2, \ldots$ such that $\lim_{n \rightarrow \infty} \phi(-t_n, x) = y \in D$, we call y an α-limit point of x. A limit set of D is defined as a set of all points in D such that each of them is either an ω-limit or an α-limit point of some x in D respectively. The structure of a limit set of a planar dynamical system is very simple. The limit set is composed of steady states (Figure 2), closed orbits (Figure 3), and trajectories joining steady states (Figure4). If a steady state (a closed orbit resp.) is stable, the equilibrium is indeterminate near the steady state (the closed orbit resp.). (See Figures 2 and 3.)

As shown below, the stochastic differential equation (2) generates a family of perturbations of the deterministic equilibrium dynamics $(\dot{K}, \dot{q}) = F(K, q)$. To talk about "perturbation" precisely, we introduce the following functional space endowed with the C^1-topology. $\mathrm{C}(W) = \{g : g\colon W \rightarrow R^2.\ g \text{ is a } C^1 \text{ function.}\}$ Note that $F \in \mathrm{C}(W)$. A perturbation of F is an element of some neighborhood of F in $\mathrm{C}(W)$ with respect to the C^1-topology. The following proposition is an obvious consequence of the structural stability (Hirsch-Smale [14, Theorem 16.3.2]), where int X and ∂X denote a set of all interior points and the boundary of some closed set X, respectively.

Proposition 1. (1) (Figure 1) There is a neighborhood $\mathrm{V}(W) \subset \mathrm{C}(W)$ of F such that for $\forall g \in \mathrm{V}(W)$, $\dot{x} = g(x)$ points inward on ∂D.

(2) (Figure 2) Suppose the limit set of D is composed of a unique stable steady state $\bar{x}$, and fix some open neighborhood $U(\bar{x}) \subset D$ of $\bar{x}$. Then there exists a neighborhood

$M(W) \subset C(W)$ of F with the following property. For $\forall g \in M(W)$, the limit set of the dynamical system $\dot{x} = g(x)$ on D is composed of a unique stable steady state $\bar{x}(g)$ such that $\bar{x}(g) \in U(\bar{x})$, and there exists a compact subset $X(\bar{x})$ of $U(\bar{x})$ such that for $\forall g \in M(W)$, $\bar{x}(g) \in \mathrm{Int}\ X(\bar{x})$ and $\dot{x} = g(x)$ points inward on $\partial X(\bar{x})$.

(3) (Figure 3) Suppose the limit set of D is composed of a unique unstable steady state $\bar{x}$ and a unique stable limit cycle γ, and fix some open neighborhood $U(\gamma) \subset D$ of γ. Then there exists a neighborhood $N(W) \subset C(W)$ of F with the following property. For $\forall g \in N(W)$, the limit set of the dynamical system $\dot{x} = g(x)$ on D is composed of a unique unstable steady state $\bar{x}(g)$ and a unique stable limit cycle $\gamma(g)$ such that $\gamma(g) \subset U(\gamma)$, and there exists a compact subset $X(\gamma)$ of $U(\gamma)$ such that for $\forall g \in N(W)$, $\gamma(g) \subset \mathrm{Int}\ X(\gamma)$ and $\dot{x} = g(x)$ points inward on $\partial X(\gamma)$.

3. Stochastic Process

We specify a stochastic process $\{\varepsilon_t\}_{t \geq 0}$ generating sunspot variables in a way consistent with the formulation in the equations (1) and (2).

We assume the sunspot process takes a finite number of values and is subject to a continuous time Markov process with a stationary transition matrix. Let Z be defined as $Z = \{z_1, z_2, ..., z_N\}$, where $-\underline{\varepsilon} \leq z_1 < z_2 < ... z_N \leq \bar{\varepsilon}$ with sufficiently small positive constants $\underline{\varepsilon}$ and $\bar{\varepsilon}$, and with a positive but finite integer N. Let $[\{\varepsilon_t(\omega)\}_{t \geq 0}, (\Omega, B_\Omega, P)]$ be a continuous time stochastic process, where $\omega \in \Omega$, B_Ω is a σ-field in Ω, P is a probability measure, and $\varepsilon_t(\ ) : \Omega \to Z$ is a random variable for $\forall t \geq 0$. Let $P(h) = [p_{ij}(h)]_{1 \leq i,j \leq N}$, $h \geq 0$, denote an $N \times N$ stationary transition probability matrix, where $p_{ij}(h)$ is the conditional probability that $\varepsilon_t(\omega)$ moves from $\varepsilon_t(\omega) = z_i$ to $\varepsilon_{t+h}(\omega) = z_j$ through the length of time h under the condition $\varepsilon_t(\omega) = z_i$. $\sum_{j=1}^{N} p_{ij}(h) = 1$ for $\forall i$, $\forall h \geq 0$. We assume:

Assumption 2. (1) $\{\varepsilon_t(\omega)\}_{t \geq 0}$ is a continuous time Markov process with a stationary transition probability matrix $P(h) = [p_{ij}(h)]_{h \geq 0\ 1 \leq i,j \leq N}$.

(2) The transition matrix satisfies the following continuity condition.

$\lim_{h\to+0} p_{ij}(h) = 1$, for $i = j$, and $= 0$, for $i \neq j$.

(3) The stochastic process $\{\varepsilon_t(\omega)\}_{t\geq 0}$ is "separable".

See Doob [7] for the concept of separability. Under Assumption 2, we have the following two observations about the sunspot process, where $\varepsilon_t(\omega) = \varepsilon(t, \omega)$.

Observation 1. (Doob [7, Theorem 6.1.2])

(1) The limit $\lim\limits_{t\to+0} \dfrac{1-p_{ii}(t)}{t} = q_i < +\infty$ exists for all i.

(2) $P\{\varepsilon(t, \omega) \equiv z_i$, for all $t_0 \leq t \leq t_0+\alpha \mid \varepsilon(t_0, \omega) = z_i\} = e^{-\alpha q_i}$, and if $\varepsilon(t_0, \omega_0) = z_i$, $\varepsilon(t, \omega_0) = z_i$ in some neighborhood of t_0 (whose size depends on ω_0.) with probability one.

A function $g(\ )$ will be called a step function, if it has only finitely many points of discontinuity in every finite closed interval, if it is identically constant in every open interval of continuity points and if, when t_0 is a point of discontinuity,

$g(t_0-) \leq g(t_0) \leq g(t_0+)$, or $g(t_0-) \geq g(t_0) \geq g(t_0+)$.

A function $g(\ )$ will be said to have a jump at a point t_0, if it is discontinuous there, and if the onesided limits $g(t_0-)$ and $g(t_0+)$ exist and satisfy one of the two preceding inequalities.

Observation 2. (Doob [7, Theorems 6.1.3, and 6.1.4])

(1) The limit $\lim\limits_{t\to+0} \dfrac{p_{ij}(t)}{t} = q_{ij} \quad i \neq j$ exists, and $\sum\limits_{j\neq i} q_{ij} = q_i$.

(2) If $q_i > 0$ and if $\varepsilon(t, \omega) = z_i$, there is with probability 1 a sample function discontinuity, which is a jump; if $0 < \alpha \leq \infty$, the probability that if there is a discontinuity in the interval $[t_0, t_0+\alpha)$ the first jump is a jump to z_j is q_{ij}/q_i.

(3) Under Assumption 2, the sample functions of $\{\varepsilon_t(\omega)\}_{t\geq 0}$ are almost all step functions.

Observation 2-3 implies for arbitrarily large but finite $T > 0$, the sample paths $\varepsilon_t = \varepsilon_t(\omega)$ are step functions and include only finitely many discontinuous jumps over $[0, T)$ with probability one. We assume:

Assumption 3. (1) For $\forall i, j = 1,2,\ldots,N$, $q_i > 0$, and $q_{ij} > 0$, where q_i and q_{ij} are specified as in Observations 1, and 2.
(2) The sample paths are continuous on the right at each jump discontinuity with probability one.

A typical sample path of the sunspot process $\{\varepsilon_t(\omega)\}_{t\geq 0}$ is depicted in Figure 5.

4 Stationary Sunspot Equilibria

We have sufficient preparation to prove that the model specified in section 1 has stationary sunspot equilibria under Assumptions 1, 2, and 3.

4-1. On the Solution of (2)

We use Assumptions 2 and 3 in the present subsection explicitly, and show how to construct a solution of the stochastic differential equation (2).

We can rewrite (2) as

$$\begin{bmatrix} \dot{K}dt \\ dq - sd\varepsilon \end{bmatrix} = F(K, q)dt, \quad s \neq 0.$$

In what follows, $s \in (-\underline{\eta}, \ \overline{\eta})$ is a fixed parameter and $s \neq 0$, unless stated otherwise. Let y be defined as $y \equiv q - s\varepsilon$. Then we have

$$\begin{bmatrix} \dot{K}dt \\ dy \end{bmatrix} = F(K, y + s\varepsilon)dt.$$

Let G be defined as $G(K, y, \varepsilon : s) \equiv F(K, y+s\varepsilon)$. Then we have

$$\begin{bmatrix} \dot{K}dt \\ dy \end{bmatrix} = G(K, y, \varepsilon : s)dt. \qquad (3)$$

We interpret the stochastic differential equation (3) as an ordinary differential equation for each fixed value ε.

$$\begin{bmatrix} \dot{K} \\ \dot{y} \end{bmatrix} = G(K, y, \varepsilon : s). \qquad (4)$$

Since $\varepsilon_t(\omega)$ takes N values, $z_1, z_2,...z_N$, this generates a family of N ordinary differential equations,

$$\begin{bmatrix} \dot{K} \\ \dot{y} \end{bmatrix} = G(K, y, z_i : s),\ s \neq 0,\ i = 1,2,\ldots,N. \quad (5)$$

The family constitutes a set of C^1-perturbations of $(\dot{K}, \dot{q}) = F(K, q)$, $s = 0$.

We construct the solution of the stochastic differential equation (5) as follows. Under Assumptions 2 and 3, for almost every ω, the sample path of $\varepsilon_t(\omega) = \varepsilon(t, \omega)$ is a right hand continuous step function, for $t \in [0, +\infty)$, assumes only N different values, $z_1, z_2,...z_N$, and for arbitrarily large but finite T, includes at most finitely many discontinuous jumps over $[0, T)$ with probability one. Fix $\omega = \omega_0$, and suppose the sample path of $\varepsilon_t(\omega_0) = \varepsilon(t, \omega_0)$, for $t \in [t_1, t_2)$, includes one and only one discontinuous jump at t_1+h from z_i to z_j, where $z_i \neq z_j$. See Figure 6. Then the system is subject to the differential equation, $(\dot{K}, \dot{y}) = G(K, y, z_i : s)$ during $t \in [t_1, t_1+h)$, and then subject to $(\dot{K}, \dot{y}) = G(K, y, z_j : s)$ during $t \in [t_1+h, t_2)$. Let $x_{t+u} = \phi(u, x_t, z_i : s)$ be a solution of the ordinary differential equation, $\dot{x} = G(x, z_i\colon s)$, with an initial condition $x = x_t$, where u is the length of time during which x moves from x_t to x_{t+u}, and z_i is fixed. Let $x_{t1} = (K(t_1), y(t_1))$. If $\dot{x} = G(x, z_i\colon s)$ has a solution for an initial value $x = (K(t_1), y(t_1))$, which implies $x = \phi(h, K(t_1), y(t_1), z_i\colon s)$ exists, and if $\dot{x} = G(x, z_j\colon s)$ has a solution for the initial value, $x = \phi(h, K(t_1), y(t_1), z_i\colon s)$, then for the ω_0 fixed above, we have $(K_t, y_t) = \phi(t-t_1, K(t_1), y(t_1), z_i\colon s)$, during $t \in [t_1, t_1+h)$, and $(K_t, y_t) = \phi(t-(t_1+h), \phi(h, K(t_1), y(t_1), z_i\colon s), z_j\colon s)$, during $t \in [t_1+h, t_2)$.

Suppose that the N differential equations (5) have a common compact support X, and that the N vector fields all point inward on the boundary ∂X of X. Then $\phi(t, x, z_i\colon s)$ is well defined, and belongs to X, for any $x \in X$, for any $i=1,2,\ldots,N$, and for any $t\geq 0$. Hence, it is the case that for almost every ω, and for any fixed initial condition $x \in X$, we can construct a solution of the stochastic differential equation (4) with $s \neq 0$ by means of successive applications of the above method, since for almost every ω the sample path of $\varepsilon_t(\omega) = \varepsilon(t, \omega)$ is a step function of $t \in [0, +\infty)$. If $\{\varepsilon_t\}_{t\in[0,T)}$ includes m (> 0) discontinuities at $t = t_1, t_2, \ldots t_m$ ($0< t_1< t_2\ldots<t_m<T$), then we have

$$x_T = \phi(T-t_m, \phi(t_m-t_{m-1}, \phi(\ldots \phi(t_1, x_0, \varepsilon_0\colon s),\ldots\ldots),\varepsilon_{tm-1}\colon s),\varepsilon_{tm}\colon s),$$

where $x_0 = (K_0, y_0)$, and $x_T = (K_T, y_T)$.

Let $\{(x_t, \varepsilon_t)\}_{t\geq 0} = \{(K_t, y_t, \varepsilon_t)\}_{t\geq 0}$ be a constructed solution of (4) together with $\{\varepsilon_t\}_{t\geq 0}$, for some fixed $x = x_0 = (K_0, y_0) \in X$, and for some fixed ω. For the fixed ω and the fixed $(K_0, q_0) = (K_0, y_0+s\varepsilon_0)$, we have $\{(K_t, q_t)\}_{t\geq 0} = \{(K_t, y_t+s\varepsilon_t)\}_{t\geq 0}$ as a solution of the stochastic differential equation (2). Note that if $\{(x_t(\omega), \varepsilon_t(\omega))\}_{t\geq 0} = \{(K_t(\omega), y_t(\omega), \varepsilon_t(\omega))\}_{t\geq 0}$ is stationary, then $\{(K_t(\omega), q_t(\omega), \varepsilon_t(\omega))\}_{t\geq 0} = \{(K_t(\omega), y_t(\omega)+s\varepsilon_t(\omega), \varepsilon_t(\omega))\}_{t\geq 0}$ is also stationary.

We can show that together with sunspot variables, the constructed solution $\{(x_t, \varepsilon_t)\}_{t\geq 0}$ is subject to a Markov process with a stationary transition probability and a compact support $X \times Z$ (Shigoka [20, Proposition 4]). We can also show the Markov operator associated with this transition probability maps an arbitrary continuous function defined on $X \times Z$ into some continuous function on it ([20, Proposition 6]). Therefore by Yoshida [26, Theorem 13.4.1], there exists an invariant probability measure on $X \times Z$ such that if we assign this measure to $X \times Z$ as an initial probability measure, then the resulting Markov process $\{(x_t(\omega), \varepsilon_t(\omega))\}_{t\geq 0}$ is stationary. Indeed it could be ergodic ([26, Theorem 13.4.3]).

4-2. Existence of Stationary Sunspot Equilibria.

We use Assumptions 1, 2, and 3 in the present subsection explicitly. We can establish a main result.

Choose $\underline{\eta}$, $\overline{\eta}$, $\underline{\varepsilon}$, $\overline{\varepsilon}$, $s \in (-\underline{\eta},\ \overline{\eta})$, $s \neq 0$, and $-\underline{\varepsilon} \leq z_1 < z_2 < \ldots z_N \leq \overline{\varepsilon}$ in such a way that $G(\ \ , z_i{:}\ s)$ i=1,2,...,N all belong to V(W) in Proposition 1-1. Then we can take D in Assumption 1 as X in Subsection 4-1. Hence we can construct stationary sunspot equilibria by means of the method described there. By construction, $(x_t, \varepsilon_t) = (K_t, y_t, \varepsilon_t) \in D \times Z$. So if we choose V($W$) sufficiently small, then $(K_t, q_t) = (K_t, y_t+s\varepsilon_t) \in W$, where W is as in Assumption 1.

Next, suppose that the assumption in Proposition 1-2 is satisfied. Choose $\underline{\eta}$, $\overline{\eta}$, $\underline{\varepsilon}$, $\overline{\varepsilon}$, $s \in (-\underline{\eta},\ \overline{\eta})$, $s \neq 0$, and $-\underline{\varepsilon} \leq z_1 < z_2 < \ldots z_N \leq \overline{\varepsilon}$ in such a way that $G(\ \ , z_i{:}\ s)$ i=1,2,...,N all belong to M(W) in Proposition 1-2. Then we can take $X(\overline{x})$ in Proposition 1-2 as X in Subsection 4-1. Hence we can construct stationary sunspot equilibria by means of the method described there. By construction, $(x_t, \varepsilon_t) = (K_t, y_t, \varepsilon_t) \in X(\overline{x}) \times Z$. So, if we choose $X(\overline{x})$ and M(W) sufficiently small, then $(K_t, q_t) = (K_t, y_t+s\varepsilon_t) \in U(\overline{x})$, where $U(\overline{x})$ is as in Proposition 1-2.

Finally, suppose that the assumption in Proposition 1-3 is satisfied. Choose $\underline{\eta}$, $\overline{\eta}$, $\underline{\varepsilon}$, $\overline{\varepsilon}$, $s \in (-\underline{\eta},\ \overline{\eta})$, $s \neq 0$, and $-\underline{\varepsilon} \leq z_1 < z_2 < \ldots z_N \leq \overline{\varepsilon}$ in such a way that $G(\ \ , z_i{:}\ s)$ i=1,2,...,N all belong to N(W) in Proposition 1-3. Then we can take $X(\gamma)$ in Proposition 1-3 as X in Subsection 4-1. We can proceed in exactly the same way as above. Then we have shown the following theorem. See Shigoka [20] for the application of the theorem to models due to Benhabib-Farmer [2] and Diamond-Fudenberg [6].

Theorem 1. Let $\begin{bmatrix} \dot{K} \\ (E_t dq)/dt \end{bmatrix} = F(K,q)$ be a first order condition of some intertemporal optimization problem with market equilibrium conditions incorporated. There is no intrinsic uncertainty, so fundamental characteristics of an economy are deterministic. Extrinsic uncertainty (sunspot), if any, alone exists. Suppose the deterministic equilibrium dynamics $(\dot{K}, \dot{q}) = F(K, q)$ satisfies Assumption 1.

(1) Global Sunspot Equilibria. We can construct stationary sunspot equilibria with a support of the endogenous variable (K_t, q_t) in W.

(2) Local Sunspot Equilibria near a Steady State. Suppose the assumption of Proposition 1-2 is satisfied, so equilibrium is indeterminate near the steady state $\bar{x}$. For any neighborhood $U(\bar{x})$ of it, we can construct stationary sunspot equilibria with a support of the endogenous variable (K_t, q_t) in $U(\bar{x})$.

(3) Local Sunspot Equilibria around a Limit Cycle. Suppose that the assumption of Proposition 1-3 is satisfied, so equilibrium is indeterminate around the limit cycle γ. For any neighborhood $U(\gamma)$ of it, we can construct stationary sunspot equilibria with a support of the endogenous variable (K_t, q_t) in $U(\gamma)$.

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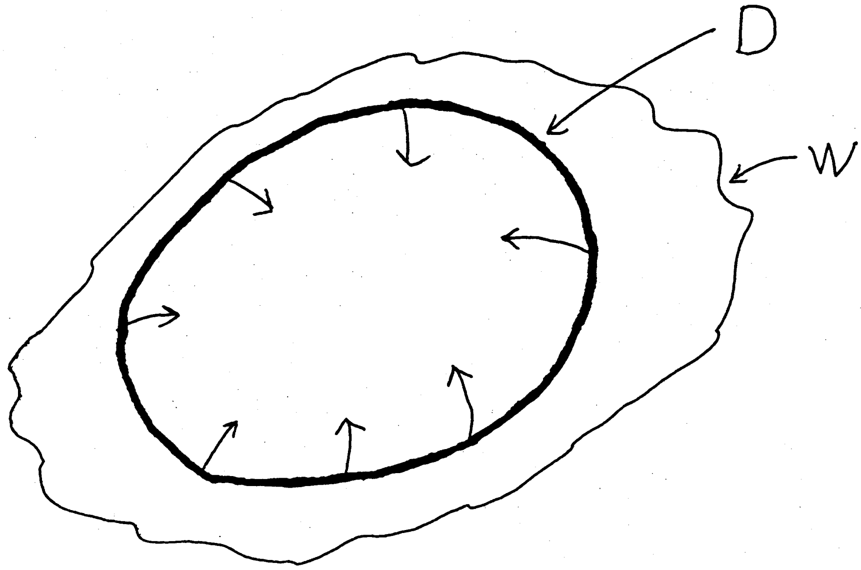


Figure 1

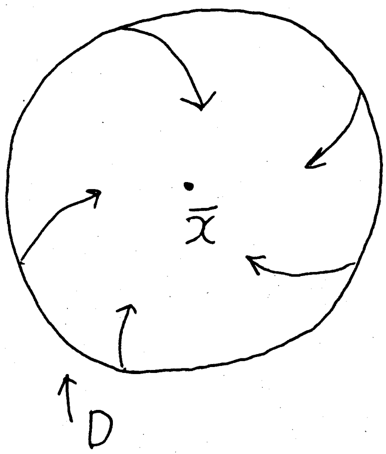
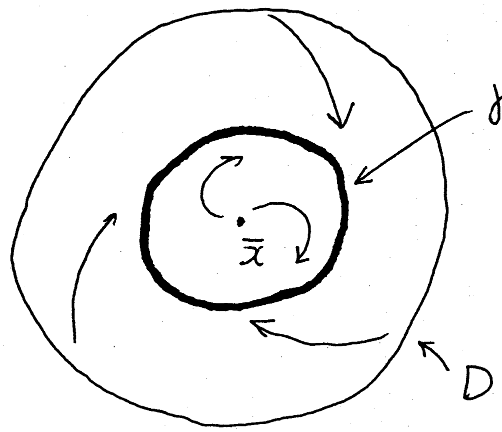


Figure 2



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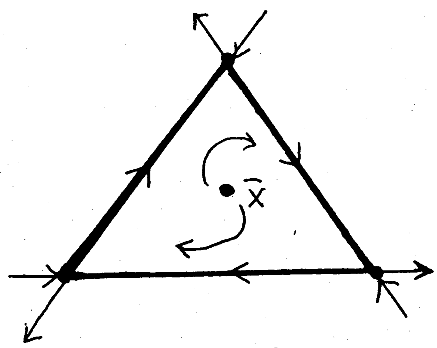

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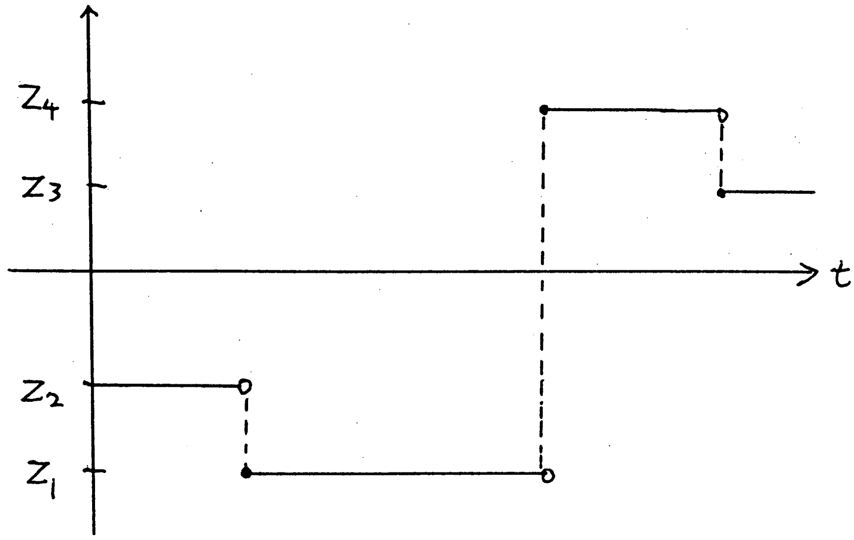

Figure 5

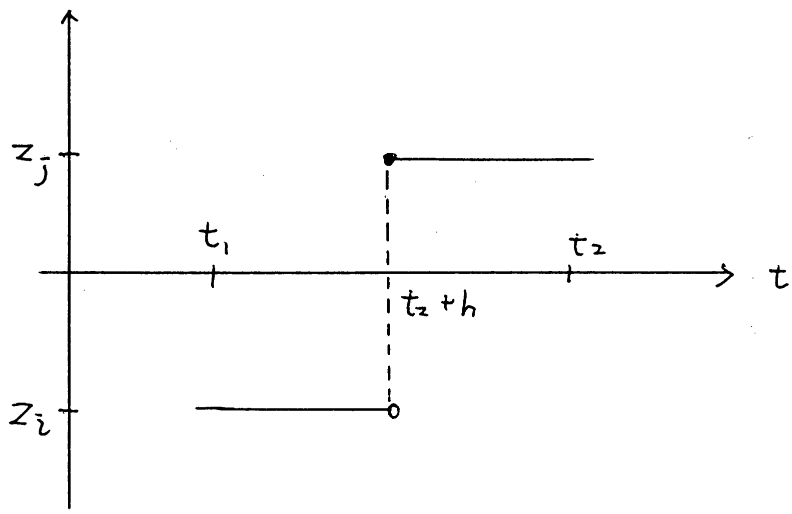

Figure 6