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ON THE ASYMPTOTIC BEHAVIOR OF SOLUTIONS FOR THE DISCRETE BOLTZMANN EQUATION WITH LINEAR AND QUADRATIC TERMS

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In this paper, we study the discrete Boltzmann equation in one-dimensional space with linear and quadratic terms. This system, which is different from the usual one by the intervention of linear terms, describes the gas motion of molecules which take only a finite number of velocities under the interactions between particles represented by the quadratic terms and also under the reflection of molecules at the inner wall of an infinite thin tube, represented by the linear terms which we treated in the papers [13], [14], [15], [16], [17], [18].

$$\text{(B)} \qquad \begin{cases} \dfrac{\partial u_i}{\partial t} + c_i \dfrac{\partial u_i}{\partial x} = Q_i(u) + L_i(u) \ , \\ u_i|_{t=0} = u_i^0(x) \ . \end{cases}$$

The physical theory imposes to this system the natural conditions :

Conditions .—

$$(*) \qquad \begin{cases} A_{ij}^{k\ell} \geqq 0, \quad A_{ij}^{k\ell} = A_{ji}^{k\ell} = A_{ij}^{\ell k} \ , \\ A_{k\ell}^{ij} \neq 0 \quad \Rightarrow \quad i \neq j \ , \\ \alpha_i^k \geqq 0 \quad \text{and} \quad \alpha_i^i = 0 \qquad \text{for all} \quad i, k \ . \end{cases}$$

This linear terms are more general than the ones which are obtained by considering solutions around constant stationary solutions, called constant Maxwellian. We suppose furthermore the conservation of momentum in the course of interactions and also reflections

Condition mvQ.—

$$A_{ij}^{k\ell} \neq 0 \quad \Rightarrow \quad c_i + c_j = c_k + c_\ell \ .$$

and

Condition mvL.—

$$\forall i \in I\ ,\ \sum_{k\in I} \alpha_k^i (c_k - c_i) = 0\ .$$

Remarque : We omit here all detailed explanations of these conditions which are described in [18].

Remarque : We know already that for bounded, summable and positive Cauchy data, the solutions exist globally in time, they are positive and bounded on $[0,\infty)$ [13], [16], [18].

Under these assumptions, we show 'asymptotic' behaviors of solutions which means that, for bounded and summable Cauchy data, the solutions $\sum_{c_a=c_i} u_i(x+c_it,t)$ tend almost everywhere and in $L^q(q \in [1,\infty))$ to a function $\varphi_a(x)$. Furthermore, supposing that the velocities are mutually different, we prove that, for i belonging to a subset I_0, the $u_i(x+c_it,t)$ converge in L^∞ to $\varphi_i(x) \equiv 0$. Finally we treat the small data case, supposing that "sufficiently" reflection coefficients α_i^k are non zero, which is incompatible with the momentum conservation for the linear terms (mvL). Then we show, according to the argument due to Shizuta and Kawashima [9], [11], the decay in $(1+t)^{-\frac{1}{4}}$ of solutions. At the end, we prove that this assumption is also necessary for the decay of solutions.

Definition.— *We define a subset I_0 as follows :*

$$I_0 = \{i\ :\exists j \text{ such that } \alpha_j^i > 0\}. \tag{1}$$

We denote $\gamma_0 = \max_{i\in I} c_i$ and $E_0 = \{i\ : c_i = \gamma_0\}$, then $\gamma_1 = \max_{i\in I\setminus E_0} c_i$ and $E_1 = \{i\ : c_i = \gamma_1\}, \cdots$. In this way, we have the decreasing sequence of velocities $\gamma_0 > \gamma_1 > \cdots$ and $E_a = \{i\ : c_i = \gamma_a\}$. We put then

$$U_a(x,t) = \sum_{c_i=\gamma_a} u_i(x,t) \tag{2}$$

and

$$U_a^0(x,t) = \sum_{c_i=\gamma_a} u_i^0(x,t) \quad \text{and } \mu = \sum_{i\in I} \int_{\mathbf{R}} u_i^0(x)dx. \tag{3}$$

Lemma 1.— *Suppose the conditions (mvQ) and (mvL). Let u_i^0 be positive, summable and bounded Cauchy data. Then we have*

$$U_a(x+\gamma_a t,t) = U_a^0(x) + \int_0^t p_a(x+\gamma_a s,s)ds - \int_0^t n_a(x+\gamma_a s,s)ds \tag{4}$$

with $p_a(x,t) \geqq 0$, $n_a(x,t) \geqq 0$ and $\int_0^\infty \int_{\mathbf{R}} p_a(x,t)dxdt \leqq C(\mu^2+\mu)$, where C is a constant depending only on the equations.

Preuve. Integrating the sum of equations for $c_i = \gamma_a$, we have

$$U_a(x+\gamma_a t, t) = U_a^0(x) + \int_0^t p_a(x+\gamma_a s, s)ds - \int_0^t n_a(x+\gamma_a s, s)ds \tag{5}$$

where

$$p_a(x,t) = \sum{}' A_{ij}^{k\ell} u_k u_\ell + \sum{}'' \alpha_i^k u_k, \tag{6}$$

$$n_a(x,t) = \sum_{c_i=\gamma_a} A_{k\ell}^{ij} u_i u_j + \sum_{c_i=\gamma_a} \sum_k \alpha_k^i u_i, \tag{7}$$

where $\sum''$[resp. $\sum'$] is the summation which operates only for $c_k \neq c_i = \gamma_a$ with $\alpha_i^k \neq 0$ [resp. for $c_k \neq c_\ell$ or $c_k = c_\ell$ such that there exist i and j with $A_{ij}^{k\ell} \neq 0$ and $c_k \neq c_i = \gamma_a$]. To prove the global existence of solutions, we knew [18] that $\int_0^\infty \int_{\mathbf{R}} p_a(x,t)dxdt \leqq C(\mu^2 + \mu)$.

Theorem 2.— *Suppose the conditions (mvQ) and (mvL). Let u_i^0 be positive, summable and bounded Cauchy data. Then the $U_a(x+\gamma_a t, t)$ converge almost everywhere and in L^q with $q \in [1,\infty)$ to a function $\varphi_a(x)$ positive, summable and bounded.*

Furthermore, if we suppose the

Condition vd.—

$$i \neq j \Rightarrow c_i \neq c_j \ ,$$

then the $u_i(x+c_i t, t)$ converge almost everywhere and in L^q with $q \in [1,\infty)$ to a function $\varphi_i(x)$.

Preuve. By virtue of Lemma 1, we have

$$U_a(x+\gamma_a t, t) \leqq G_a(x) \text{ almost everywhere} \tag{8}$$

where $G_a(x) = U_a^0(x) + \int_0^\infty p_a(x+\gamma_a t, t)dt \geqq 0$. Then we have $G_a \in L^1$. We see easily that

$$\begin{aligned} 0 &\leqq \int_0^t n_a(x+\gamma_a s, s)ds \\ &\leqq U_a^0(x) + \int_0^t p_a(x+\gamma_a s, s)ds \\ &\leqq G_a(x). \end{aligned} \tag{9}$$

Since G_a is summable, we have, for almost all $x \in \mathbf{R}$,

$$\int_0^\infty p_a(x+\gamma_a t, t)dt < \infty \tag{10}$$

and

$$\int_0^\infty n_a(x+\gamma_a t,t)dt < \infty. \tag{11}$$

Furthermore we have

$$\int_0^\infty \int_{\mathbf{R}} n_a(x+\gamma_a t,t)dtdx < \infty. \tag{12}$$

We see that

$$\lim_{t\to+\infty} U_a^0(x) + \int_0^t p_a(x+\gamma_a s,s)ds - \int_0^t n_a(x+\gamma_a s,s)ds, \tag{13}$$

denoted by $\varphi_a(x)$, exists for almost all $x \in \mathbf{R}$. We obtain

$$U_a(\cdot+\gamma_a t,t) \overset{t\to+\infty}{\longrightarrow} \varphi_a(\cdot x) \text{ almost everywhere.} \tag{14}$$

We see easily that φ_a is positive, summable and bounded. We have

$$\begin{aligned} 0 &\leqq U_a^0(x) + \int_0^t p_a(x+\gamma_a s,s)ds - \int_0^t n_a(x+\gamma_a s,s)ds \\ &\leqq G_a(x) \in L^1. \end{aligned} \tag{15}$$

By virtue of Lebesgue's theorem, we deduce that

$$U_a^0(\cdot) + \int_0^t p_a(\cdot+\gamma_a s,s)ds - \int_0^t n_a(\cdot+\gamma_a s,s)ds \tag{16}$$

converges to φ_a in $L^1(\mathbf{R})$.

To prove the convergence in L^q with $q \in [1,\infty)$, we have only to use the interpolation between L^1 and L^∞. Indeed we have

$$\left\| \sum_{i\in E_a} u_i(\cdot+c_i t,t) - \varphi_a(\cdot) \right\|_{L^\infty} \leqq C \sup_{\mathbf{R}\times[0,\infty)} \sum_i u_i < \infty. \tag{17}$$

In the above proof, we showed the

Corollary 3.— *Suppose the conditions (mvQ) and (mvL). Let u_i^0 be positive, summable and bounded Cauchy data. Then there exists a function G_a positive and summable such that we have*

$$U_a(x,t) \leqq G_a(x-\gamma_a t) \tag{18}$$

for all $(x,t) \in \mathbf{R}\times[0,\infty)$.

To study better the asymptotic behavior of solutions, we exclude henceforth the case of multiple velocities. The final aim is to show that, for $i \in I_0$, the $u_i(x+c_i t,t)$ tend to $\varphi_i(x)$ uniformly in $x \in \mathbf{R}$. First we have

Proposition 4.— *Suppose the conditions (vd), (mvQ) and (mvL). Let u_i^0 be positive, summable and bounded Cauchy data. Then, for all $\varepsilon > 0$, there exists a big T such that we have*

$$\int_T^\infty \int_{\mathbf{R}} u_i(x,t)u_j(x,t)dxdt \leqq \varepsilon \tag{19}$$

if $i \neq j$,

$$\int_T^\infty \int_{\mathbf{R}} u_i(x,t)dxdt \leqq \varepsilon \tag{20}$$

if $i \in I_0$, and

$$\int_{\mathbf{R}} u_i(x,T)dx \leqq \varepsilon \tag{21}$$

if $i \in I_0$.

Preuve. By virtue of the remarks concerning the global existence of solutions, we deduce that, for all $\varepsilon > 0$, there exists a big T^0 such that we have

$$\int_{T^0}^\infty \int_{\mathbf{R}} u_i(x,t)u_j(x,t)dxdt \leqq \varepsilon \tag{22}$$

if $i \neq j$,

$$\int_{T^0}^\infty \int_{\mathbf{R}} u_i(x,t)dxdt \leqq \varepsilon \tag{23}$$

if $i \in I_0$. Since we have

$$\int_{T^0}^{T^0+1} \int_{\mathbf{R}} u_i(x,t)dxdt \leqq \varepsilon, \tag{24}$$

there exists a $T \in [T^0, T^0+1]$ such that we have

$$\int_{\mathbf{R}} u_i(x,T)dx \leqq \varepsilon. \tag{25}$$

We have then the third inequality.

Proposition 5.— *Suppose the conditions (vd), (mvQ) and (mvL). Let u_i^0 be positive, summable and bounded Cauchy data. Then, for all $\varepsilon > 0$, there exists a big T such that we have, for c_i, c_k, c_ℓ mutually different,*

$$\operatorname*{ess\,sup}_x \int_T^\infty u_k u_\ell(x + c_i t, t)dt \leqq \varepsilon \tag{26}$$

and, for $k \in I_0$ such that $i \neq k$,

$$\operatorname*{ess\,sup}_x \int_T^\infty u_k(x + c_i t, t)dt \leqq \varepsilon. \tag{27}$$

Preuve. We knew that the $G_i(x) = u_i^0(x,t) + \int_0^\infty p_i(x + c_i t, t)dt$ are positive and summable and that $u_i(x,t) \leqq G_i(x - c_i t, t)$. For all $\varepsilon > 0$, there exists a closed interval $K \subseteq \mathbf{R}$ such that $\int_{\mathbf{R}\setminus K} G_i(x)dx < \varepsilon$ for all $i \in I$. We put

$$H_i(x) = \begin{cases} 0, & \text{on } K, \\ G_i(x), & \text{otherwise.} \end{cases} \tag{28}$$

Then we have $\|H_k\|_{L^1} < \varepsilon$. In the outside of the compact set $L = \{(x,t) : x - c_k t, x - c_\ell t \in K\}$, we see that

$$u_k u_\ell(x,t) \leqq M\left(H_k(x - c_k t) + H_\ell(x - c_\ell t)\right), \tag{29}$$

where we put

$$M = \sup_{(x,t)\in\mathbf{R}\times[0,\infty)} \sum_{i\in I} u_i(x,t) < \infty. \tag{30}$$

There exists a big T such that

$$\mathbf{R} \times [T,\infty) \cap L \neq \emptyset. \tag{31}$$

Then we obtain

$$\begin{aligned} &\int_T^\infty u_k u_\ell(x + c_i t, t)dt \\ &\leqq M \int_T^\infty \left(H_k(x - c_k t) + H_\ell(x - c_\ell t)\right) dt \\ &\leqq 2M\varepsilon. \end{aligned} \tag{32}$$

Therefore we showed the first inequality.

Concerning to the second inequality, we have first, for $t > T$,

$$\begin{aligned} u_k(x + c_i t, t) \leqq\; & u_k(x + c_i t - c_k(t - T), T) \\ & + C \sum_{p\neq q} \int_t^T u_p u_q(x + c_i t - c_k(t-\tau), \tau)d\tau \\ & + \sum_{p\neq k} \alpha_k^p \int_T^t u_p(x + c_i t - c_k(t-\tau), \tau)d\tau. \end{aligned} \tag{33}$$

Then we have

$$\begin{aligned} \int_T^\infty u_k(x + c_i t, t)dt \leqq\; & C \int_{\mathbf{R}} u_k(x,T)dx \\ & + C \sum_{p\neq q} \int_T^\infty \int_{\mathbf{R}} u_p u_q dxdt \\ & + \sum_{p\neq k} \alpha_k^p \int_T^\infty \int_{\mathbf{R}} u_p dxdt. \end{aligned} \tag{34}$$

The two first terms in the right-hand side are less that ε by virtue of the Proposition 4. Concerning to the third term, the summation operates only for $p \neq k$ such that $\alpha_k^p \neq 0$, i.e. for $p \in I_0$. This term is then less also than ε by virtue of the Proposition 4.

Now we state the theorem which describes more precisely the asymptotic behavior of solutions.

Theorem 6.— *Suppose the conditions (vd), (mvQ) and (mvL). Let u_i^0 be positive, summable and bounded Cauchy data. Then, for $i \in I_0$, the $u_i(x + c_i t, t)$ tend to 0 uniformly as $t \to +\infty$. In particular, we have $\varphi_i(x) \equiv 0$.*

Preuve. We have

$$\begin{aligned}\frac{d}{dt}u_i(x+c_it,t) \leqq -au_i(x+c_it,t) + C\sum_{p\neq q} u_pu_q(x+c_it,t)\\ + C\sum_{p\in I_0} u_p(x+c_it,t)\end{aligned} \tag{35}$$

with $a > 0$. By virtue of the Proposition 5, for all $\varepsilon > 0$, there exists a big T such that

$$\operatorname*{ess\,sup}_x \int_T^t \sum_{p\neq q} u_pu_q(x+c_i\tau,\tau)d\tau < \varepsilon, \tag{36}$$

$$\operatorname*{ess\,sup}_x \int_T^t \sum_{p\in I_0} u_p(x+c_i\tau,\tau)d\tau < \varepsilon. \tag{37}$$

Integrating the inequality (35), we have, for $t > T$,

$$u_i(x+c_it,t) \leqq Me^{-a(t-T)} + 2\varepsilon \tag{38}$$

with

$$M = \sup_{\mathbf{R}\times[0,\infty)} \sum_{i\in I} u_i(x,t) < \infty. \tag{39}$$

There exists a T^0 such that $Me^{-a(t-T)} < \varepsilon$ for all $t > T^0$. We have therefore, for $t > T^0$, $u_i(x+c_it,t) < 3\varepsilon$. The function $u_i(x+c_it,t)$ tends, in L^∞−norm, to 0 i.e. $\varphi_i(x) \equiv 0$.

Finally we apply the argument due to Shizuta and Kawashima [9], [11] to show the decay in $(1+t)^{-\frac{1}{4}}$ of solutions in case "sufficiently" reflection coefficients α_i^k are non zero, which is incompatible with (mvL) and also necessary for the decay of solutions. First of all, we state some conditions.

Condition $\perp$.— If $\mu_i \in \mathbf{R}$ verify $\sum_i \mu_i L_i(u) = 0$ for all u, then we have $\sum_i \mu_i Q_i(u) = 0$ for all u.

Condition dsp.— There is no eigenvector λ of $\mathcal{C} = \operatorname{diag}(c_1, \cdots, c_N)$ such that λ is in the kernel of $\mathcal{L}^t$, where $\mathcal{L} = (\alpha_i^j - \delta_{ij}\sum_k \alpha_k^i)_{ij}$ and $N = \sharp I$.

Remarque : The condition (mvL) is incompatible with the condition (dsp). Indeed, the vector (μ_i) such that $\mu_i = 1$ if $i \in E_0$ and $\mu_i = 0$ otherwise, is an eigenvector of $\mathcal{C}$ and is in the kernel of $\mathcal{L}^t$.

Corollary 7.— *The conditions* ($\perp$) *and (dsp) are verified if*
1)there is at least two distinct velocities and 0 *is a simple eigenvalue of* $\mathcal{L}$.
or
2)(very particular case)there is at least two distinct velocities and all $\alpha_i^k (i \neq k)$ *are non zero.*

Condition μrL'.—

$$\alpha_i^k = \alpha_k^i \quad \text{for all } i \text{ and } k$$

(microreversibility of reflection).

Shizuta and Kawashima [9], [11] proved the

Proposition 8.— *Suppose the conditions (dsp) and (μrL'). We put* $\mathcal{S} = \mathcal{L} - \mathcal{C}\dfrac{\partial}{\partial x}$. *Let be* $u^0 \in H^s \cap L^1(\mathbf{R})$ $(s > \frac{1}{2})$. *Then we have*

$$\|e^{t\mathcal{S}}u^0\|_{H^s} \leqq C(1+t)^{-\frac{1}{4}}\|u^0\|_{H^s\cap L^1}\,. \tag{40}$$

Furthermore, if u^0 *is orthogonal to the kernel of* $\mathcal{L}$, *then we have*

$$\|e^{t\mathcal{S}}u^0\|_{H^s} \leqq C(1+t)^{-\frac{3}{4}}\|u^0\|_{H^s\cap L^1}\,. \tag{41}$$

We show now the decay of solutions of our nonlinear system.

Theorem 9.— *Suppose the conditions* ($\perp$), *(dsp) and (μrL'). Let* u^0 *be a positive Cauchy data and in* $H^s \cap L^1(\mathbf{R})(s > \frac{1}{2})$. *Let* $u(t)$ *be a solution of*

$$\left(\frac{\partial}{\partial t} + \mathcal{C}\frac{\partial}{\partial x}\right)u = Q(u) + L(u), \tag{42}$$

with the Cauchy data u^0, *where* $Q(u) = (Q_i(u))_i$, $L(u) = (L_i(u))_i$. *If* u^0 *is sufficiently small in* $H^s \cap L^1(\mathbf{R})$, *then we have the global existence of solution and the decay of solution in* $H^s(\mathbf{R})$:

$$\|u(t)\|_{L^\infty} \leqq \|u(t)\|_{H^s} \leqq C(1+t)^{-\frac{1}{4}}\|u(t)\|_{H^s\cap L^1} \tag{43}$$

where the constant C *depends only on the equations.*

Preuve. Owing to the usual argument, it is sufficient to show the estimate (43) up to the time-existence of solution T^*. By virtue of the condition ($\perp$), $Q(u)$ is orthogonal to the kernel of $\mathcal{L}^t$. We have

$$u(t) = e^{t\mathcal{S}}u^0 + \int_0^t e^{(t-\tau)\mathcal{S}}Q(u)(\tau)d\tau. \tag{44}$$

Remarking that $H^s(\mathbf{R})$ forms an algebra for $s > \frac{1}{2}$, we obtain, by the Proposition 8,

$$\begin{aligned}
&\|u(t)\|_{H^s}\\
&\leqq C(1+t)^{-\frac{1}{4}}(\|u^0\|_{H^s} + \|u^0\|_{L^1})\\
&+ C\int_0^t (1+t-\tau)^{-\frac{3}{4}}(\|Q(u)(\tau)\|_{H^s} + \|Q(u)(\tau)\|_{L^1})d\tau\\
&\leqq CU_0(1+t)^{-\frac{1}{4}} + C\int_0^t (1+t-\tau)^{-\frac{3}{4}}\|u(\tau)\|_{H^s}{}^2 d\tau,
\end{aligned} \tag{45}$$

where we put $U_0 = \left\|u^0\right\|_{H^s} + \left\|u^0\right\|_{L^1}$. Denoting $U(t) = \sup_{\tau\in[0,t]} (1+\tau)^{\frac{1}{4}} \|u(\tau)\|_{H^s}$, we have

$$
\begin{aligned}
U(t) &\leqq CU_0 + C(1+t)^{\frac{1}{4}} U(t)^2 \int_0^t (1+t-\tau)^{-\frac{3}{4}} (1+\tau)^{-\frac{1}{2}} d\tau \\
&\leqq CU_0 + CU(t)^2.
\end{aligned} \tag{46}
$$

We remark here that the equation $X = CX_0 + CX^2$ admits two real roots α and β ($\alpha \leqq \beta$) for sufficiently small X_0 with $\alpha = O({X_0}^2)$. By the continuity of $U(t)$, the value $U(t)$ is included in the interval $[0, \alpha]$, which completes the proof.

At the end, we show that the condition (dsp) is also necessary for the decay of solutions under the condition ($\perp$).

Theorem 10.— *Suppose the condition ($\perp$). Furthermore we suppose that the condition (dsp) is not verified. Let u_i^0 be positive, summable and bounded Cauchy data. Then the solutions do not tend to* 0 *except when the data verify a linear relation well precise.*

Preuve. We know already [13] that the solutions $u_i(x,t)$ are positive, summable in x and locally bounded in $\mathbf{R} \times \mathbf{R}_+$. By hypothesis, there exists $\mu = (\mu_i) \in \ker \mathcal{L}^t$ and $\gamma \in \mathbf{R}$ such that $\mu_i \neq 0 \Rightarrow c_i = \gamma$. Then we have $\sum_i \mu_i L_i(u) \equiv 0$ and, by the condition ($\perp$), $\sum_i \mu_i Q_i(u) \equiv 0$. We have then

$$
\left(\frac{\partial}{\partial t} + \gamma \frac{\partial}{\partial x}\right) \left(\sum_i \mu_i u_i\right) = 0. \tag{47}
$$

Therefore $\sum_i \mu_i u_i$ is a conservative quantity which moves at the velocity γ. Except when $\sum_i \mu_i u_i^0 \equiv 0$, the solutions u_i can not tend to 0.

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