数理解析研究所講究録
第 862 巻 1994 年 105-110

NAVIER–STOKES EQUATIONS WITH DISTRIBUTIONS AS INITIAL DATA

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§1 Introduction.

Let Ω be an exterior domain in $\mathbf{R}^n(n \geq 3)$,i.e., a domain having a compact complement $\mathbf{R}^n \backslash \Omega$, and assume that the boundary $\partial\Omega$ is of class $C^{2+\mu}(0 < \mu < 1)$. The motion of the incompressible fluid occupying Ω is governed by the Navier-Stokes equations:

$$
(S) \qquad \begin{cases} -\Delta w + w \cdot \nabla w + \nabla \pi = \operatorname{div} F & \text{in} \quad \Omega, \\ \operatorname{div} w = 0 \quad \text{in} \quad \Omega, \\ w = 0 \quad \text{on} \quad \partial\Omega, \quad w(x) \to 0 \quad \text{as} \quad |x| \to \infty, \end{cases}
$$

where $w = w(x) = (w^1(x), \cdots, w^n(x))$ and $\pi = \pi(x)$ denote the velocity vector and the pressure of the fluid at point $x \in \Omega$, respectively, while $F = F(x) = \{F_{ij}(x)\}_{i,j=1,\cdots,n}$ is the given $n \times n$ matrics with div F the external force. In the previous paper [14], the first author and Ogawa showed the stability in L^n of solutions w in the class

$$
(CL) \qquad w \in L^n(\Omega) \quad \text{and} \quad \nabla w \in L^{n/2}(\Omega).
$$

In case $n \geq 4$ we can show the existence and uinqueness for solutions w of (S) with (CL). In the three dimensional case, however, the solution in the class (CL) yields that the net force exerted to the body is equal to zero:

$$
\int_{\partial\Omega} (T(w,\pi) + F) \cdot \nu dS = 0,
$$

where $T(w,\pi) = \{\partial w^i/\partial x^j + \partial w^j/\partial x^i - \delta_{ij}\pi\}_{i,j=1,\cdots,n}$ and ν denote the stress strain and the unit outer normal to $\partial\Omega$, respectively(see Kozono-Sohr [16]). Introducing another class

$$
(CL') \qquad \sup_{x\in\Omega} |x||w(x)| + \sup_{x\in\Omega} |x|^2 |\nabla w(x)| \equiv C_w < \infty
$$

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Borchers-Miyakawa [3] constructed the solution with (CL') and showed that if C_w is small, then w is stable under the initial disturbance in weak- L^n space $L^{n,\infty}(\Omega)$.

The purpose of this note is to find a larger class of stable flows than (CL'). Indeed, we shall show that stationary flows in the class

$$(CL") \qquad w \in L^{n,\infty}(\Omega)$$

are stable under such perturbation as Borchers-Miyakawa's [3]. As a result, we shall obtain the *same class* of stable solutions and initial disturbances. More precisely, if w is perturbed by a, then the perturbed flow $v(x,t)$ is governed by the following *non-stationary* Navier-Stokes equations:

$$(N-S) \qquad \begin{cases} \dfrac{\partial v}{\partial t} - \Delta v + v\cdot\nabla v + \nabla q = f \quad \text{in } \Omega, t>0, \\ \text{div } v = 0 \quad \text{in } \Omega, t>0, \\ v = 0 \quad \text{on } \partial\Omega, t>0, \quad v(x,t)\to 0 \quad \text{as} \quad |x|\to\infty, \\ v(x,0) = w(x) + a(x) \quad \text{for} \quad x\in\Omega. \end{cases}$$

In this note we shall show: if the stationary flow w and the initial disturbance a are both small enough in $L^{n,\infty}(\Omega)$, then there is a unique *global strong solution* v of (N-S) such that the integrals

$$\int_\Omega |v(x,t) - w(x)|^r dx \quad \text{for} \quad n<r<\infty$$

converges to zero with *definite decay rates* as $t\to\infty$. Let w and v be solutions of (S) and (N-S), respectively. Then the pair of functions $u \equiv v - w, p \equiv q - \pi$ satisfies

$$(N-S') \qquad \begin{cases} \dfrac{\partial u}{\partial t} - \Delta u + w\cdot\nabla u + u\cdot\nabla w + u\cdot\nabla u + \nabla p = 0 \quad \text{in } \Omega, t>0, \\ \text{div } u = 0 \quad \text{in } \Omega, t>0, \\ u = 0 \quad \text{on } \partial\Omega, t>0, \quad u(x,t)\to 0 \quad \text{as} \quad |x|\to\infty, \\ u|_{t=0} = a. \end{cases}$$

Hence our problem on the stability for (S) can now be reduced to investigation into asymptotic behaviour of the solution u of (N-S'). In a three-dimensional exterior domain, Heywood [10,11] and Masuda [18] considered inhomogeneous boundary condition at infinity like $w(x)\to w^\infty$ as $|x|\to\infty$, where w^∞ is a prescribed non-zero constant vector in $\mathbf{R}^3$. They showed the stability for such solutions in L^2-spaces. On account of the parabolically wake region behind obstacles, their decay rates are slower than that of our solutions. To obtain sharper decay rates in L^r-spaces of the solutions of (N-S') with the initial data in weak- L^n space, we need to establish $L^{p,\infty} - L^r$-estimates for the semigroup e^{-tL_r}, where L_r is the operator defined by

$$L_r u \equiv A_r u + P_r(w\cdot\nabla u + u\cdot\nabla w).$$

Here P_r is the projection operator from $L^r(\Omega)$ onto $L^r_\sigma(\Omega)$ and $A_r \equiv -P_r\Delta$ denotes the Stokes operator in $L^r_\sigma(\Omega)$.

In case $w \equiv 0$, we have $L_r = A_r$ and hence our problem coincides with obtaining a global strong solution and its decay properties of the Navier-Stokes equations in exterior domains. Since the pioneer work of Kato [13] and Ukai [23], many efforts have been made to get $L^p - L^r$-estimates for the Stokes semigroup e^{-tA_r} in *exterior* domains and there are mainly two methods. One is to characterize the domain $D(A^\alpha_r)$ of fractional powers $A^\alpha_r (0 < \alpha < 1)$ due to Giga [7],Giga-Sohr [9] and Borchers-Miyakawa [2] and another is to obtain asymptotic expansion of the resolvent $(A_r+\lambda)^{-1}$ near $\lambda = 0$ due to Iwashita [12]. In our case, since L_r is the operator with *variable* coffecients, both of these methods seem to be difficult to get the same asymptotic behavior of e^{-tL_r} as that of e^{-tA_r} as $t \to \infty$. If we restrict ourselves to the case $n/(n-1) < r < \infty$, however, then L_r can be treated as a perturbation of A_r, and for such r, we can get satisfactory $L^{p,\infty} - L^r$-estimates of e^{-tL_r}, which is enough to construct the global strong solution of (N-S'). Our proof needs neither estimates of the purely imaginary powers $L^{is}_r (s \in \mathbf{R})$ of L_r nor asymptotic expansion of $(L_r+\lambda)^{-1}$ near $\lambda = 0$; we need only such a standard resolvent estimate of elliptic differential operators as Agmon's [1].

On account of the restriction $n/(n-1) < r < \infty$, we cannot construct the strong solution directly in the same way as Giga-Miyakawa [8] and Kato [13]. Therefore, we need to first introduce a *mild solution* which is an intermediate between weak and strong solutions (see Definition below). This procedure is due to Kozono-Ogawa [14]. Then we shall show the existence and uniqueness of the *global* mild solution u of (N-S') in the class $C((0,\infty); L^{n,\infty}(\Omega))$ with decay property $\|u(t)\|_r = O(t^{-1/2+n/2r})$ as $t \to \infty$ for $n < r < \infty$. Using a similar uniqueness criterion to Serrin [21] and Masuda [19], we may identify the mild solution with the strong solution. As a result, it will be clarified that the restriction on r causes no obstruction for our purpose.

§2 Results.

Before stating our results, we introduce some notations and function spaces and then give our definition of mild solutions of (N-S'). Let $C^\infty_{0,\sigma}$ denote the set of all C^∞-real vector functions $\phi = (\phi^1, \cdots, \phi^n)$ with compact support in Ω, such that div $\phi = 0$. L^r_σ is the closure of $C^\infty_{0,\sigma}$, with respect to the L^r-norm $\|\ \ \|_r$; $(\cdot,\cdot)$ denotes the L^2-inner product and the duality pairing between L^r and $L^{r'}$, where $1/r + 1/r' = 1$. L^r stands for the usual(vector-valued)L^r-space over Ω, $1 < r < \infty$. $H^{1,r}_{0,\sigma}$ denotes the closure of $C^\infty_{0,\sigma}$ with respect to the norm

$$\|\phi\|_{H^{1,r}} = \|\phi\|_r + \|\nabla\phi\|_r,$$

where $\nabla\phi = (\partial\phi^i/\partial x_j; i,j = 1,\cdots,n)$. When X is a Banach space, its norm is denoted by $\|\cdot\|_X$. Then $C^m((t_1,t_2);X)$ is a usual Banach space, where $m = 0,1,2,\cdots$ and t_1 and t_2 are real numbers such that $t_1 < t_2$. $BC^m((t_1,t_2);X)$ is the set of all functions $u \in C^m((t_1,t_2);X)$ such that $\sup_{t_1<t<t_2} \|\frac{d^m u(t)}{dt^m}\|_X < \infty$.

Let us recall the Helmholtz decomposition:

$$L^r = L^r_\sigma \oplus G^r \quad (\text{direct sum}), \quad 1 < r < \infty,$$

where $G^r = \{\nabla p \in L^r; p \in L^r_{loc}(\overline{\Omega})\}$. For the proof, see Fujiwara-Morimoto[6], Miyakawa[20] and Simader-Sohr[22]. P_r denotes the projection operator from L^r onto L^r_σ along G^r. The Stokes operator A_r on L^r_σ is then defined by $A_r = -P_r\Delta$ with domain $D(A_r) = \{u \in H^{2,r}(\Omega); u|_{\partial\Omega} = 0\} \cap L^r_\sigma$. It is known that

$$(L^r_\sigma)^*(\text{the dual space of } \ L^r_\sigma) = L^{r'}_\sigma, \quad A^*_r(\text{the adjoint operator of } \ A_r) = A_{r'},$$

where $1/r + 1/r' = 1$.

Furthermore, for $1 < r < \infty$ and $1 \le q \le \infty$, $L^{r,q}$ denotes the Lozentz space over Ω with norm $\|\cdot\|_{r,q}$. Then we define $L^{r,q}_\sigma$ as

$$L^{r,q}_\sigma \equiv \{u \in L^{r,q}; \text{div } u = 0 \quad \text{in} \quad \Omega, u \cdot \nu = 0 \quad \text{on} \quad \partial\Omega\}.$$

Let us introduce the operator L_r in L^r_σ. To this end, we make the following assumption on w.

Assumption. *w is a smooth solenoidal vector function on $\overline{\Omega}$ with $w|_{\partial\Omega} = 0$ in the class $w \in L^{n,\infty}_\sigma$*

For the existence of such solutions w of (S), see Finn [4] and Fujita [5]. Under this assumption, we define the operator B_r on L^r_σ by

$$B_r u \equiv P_r(w \cdot \nabla u + u \cdot \nabla w) \quad \text{with domain} \quad D(B_r) = H^{1,r}_{0,\sigma}.$$

L_r is now defined by

$$D(L_r) = D(A_r) \quad \text{and} \quad L_r \equiv A_r + B_r.$$

Since div $w = 0$ in Ω, we can easily verify that the operator L' defined by

$$L'_r u = A_r u - P_r(w \cdot \nabla u + \sum_{j=1}^{n} w^j \nabla u^j), \quad D(L'_r) = D(A_r)$$

is the adjoint operator of $L_{r'}$ on $L^{r'}_\sigma$. It should be noted that the operator L' contains no derivative $\partial w/\partial x^j (j = 1, \cdots, n)$ of w in its coefficients.

Our definition of mild solutions of (N-S') is as follows:

Definition. *Let $a \in L^{n,\infty}_\sigma$ and let w satisfy the Assumption. Suppose that $n < r < \infty$. A measurable function u defined on $\Omega \times (0,\infty)$ is called a mild solution of (N-S') in L^r_σ if*

(1) *$u \in BC((0,\infty); L^{n,\infty}_\sigma)$ and $t^{(1-n/r)/2}u(\cdot) \in BC((0,\infty); L^r_\sigma)$;*

(2)

$$(u(t), \phi) = (e^{-tL}a, \phi) + \int_0^t (u(s) \cdot \nabla e^{-(t-s)L'}\phi, u(s))ds$$

for all $\phi \in C^\infty_{0,\sigma}$ and all $0 < t < \infty$.

Our results now read:

Theorem 1. *(1)(existence) Let $a \in L^{n,\infty}_\sigma$ and let w satisfy the Assumption. Then for every $n < r < \infty$, there is a positive number $\lambda = \lambda(n,r)$ such that if*

$$\|a\|_{n,\infty} \leq \lambda, \quad \|w\|_{n,\infty} \leq \lambda,$$

there exists a mild solution u of (N-S') in L^r_σ such that

$$u(t) \to a \quad \text{weakly} * \text{in} \quad L^{n,\infty}_\sigma \quad \text{as} \quad t \downarrow +0.$$

(2) (uniqueness) There is a constant $k = k(n,r)$ such that any mild solution u of (N-S') in L^r_σ with

$$\limsup_{t\to+0} t^{\frac{n}{2}(\frac{1}{n}-\frac{1}{r})}\|u(t)\|_r \leq k$$

is unique.

Concerning the regularity of the solution, we have

Theorem 2. *The mild solution u given in Theorem 1 is actually a stronng solution in the following sense:*

(1) $u \in C^1((0,\infty); L^r_\sigma)$;
(2) $u(t) \in D(L_r)$ *for* $t \in (0,\infty)$ *and* $L_r u \in C((0,\infty); L^r_\sigma)$;
(3) u *satisfies*

$$\frac{du}{dt} + L_r u + P_r(u\cdot\nabla u) = 0, \quad t > 0 \quad \text{in} \quad L^r_\sigma$$

Remarks. *(1) The above theorems show that the space $L^{n,\infty}_\sigma$ is the class of stable stationary flows and that it is the same class as that of initial disturbances. Borchers-Miyakawa [3] obtained, among others, similar results to ours including the uniform L^∞ estimate in time. They make, however, such a stronger assumption as $\sup_{x\in\Omega}|x||w(x)| + \sup_{x\in\Omega}|x|^2|\nabla w(x)|$ is small enough. On the other hand, our theorems assert that the assumption on the spacial decay of $\nabla w(x)$ as $|x| \to \infty$ is not necessary. Moreover, the class of the space $L^{n,\infty}$ is larger than that of functions w such that $\sup_{x\in\Omega}|x||w(x)| < \infty$.*

(2) Since the semigroup $\{e^{-tL}\}_{t\geq 0}$ is not strongly continuous in $L^{n,\infty}_\sigma$, we cannot *assure whether our solution u satisfies*

$$\lim_{t\to+0} t^{\frac{n}{2}(\frac{1}{n}-\frac{1}{r})}\|u(t)\|_r = 0.$$

(3) When $\Omega = \mathbf{R}^n (n \geq 3)$, without assuming any regularity on the stationary flow w, Kozono-Yamazaki [17] obtained a similar strong solution with a uniform decay estimate.

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