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Existence of Feasible Potentials on Networks

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1. Introduction and problem setting

Let $G := \{X, Y, K\}$ be a directed, locally finite graph without self-loops. Here X is the countable set of nodes, Y is the countable set of arcs, and K is the node-arc incidence function of G. For every arc $y \in Y$ the initial node $x^-(y)$ and the terminal node $x^+(y)$ are uniquely defined by the relations $K(x^-(y), y) = -1$, $K(x^+(y), y) = +1$. Local finiteness of G means that, for every $x \in X$, $K(x, \cdot)$ has finite support in Y.

Denote by $L(X)$ the space of all real-valued functions defined on X, by $L_0(X)$ the set of all $f \in L(X)$ having finite support. Let $L^+(X)$(resp. $L_0^+(X)$) denote the subset of $L(X)$(resp. $L_0(X)$) which consists of the non-negative functions. For $f \in L(X)$ and $g \in L_0(X)$, let

$$< f, g > := \sum_{x\in X} f(x)g(x).$$

Replacing X by Y, we define $L(Y)$, $L_0(Y)$, $L^+(Y)$, $L_0^+(Y)$ and the bilinear functional $< \cdot, \cdot >$ on $L(Y) \times L_0(Y)$ similarly.

To the function $K(x, y)$, we associate two mappings K from $L(X)$ to $L(Y)$ and K^* from $L_0(Y)$ to $L_0(X)$ through

$$(Kf)(y) := \sum_{x\in X} K(x, y)f(x) = f(x^+(y)) - f(x^-(y)) \text{ (tension of } f),$$

$$(K^*w)(x) := \sum_{y\in Y} K(x, y)w(y) \text{ (divergence of } w).$$

Note that $< Kf, w > = < f, K^*w >$ for all $f \in L(X)$ and $w \in L_0(Y)$.

Let $A, B \in L(Y)$ be given such that $A(y) \leq B(y)$ on Y. Let $\beta \in L(X)$ be given such that $0 \leq \beta(x)$ on X. We are concerned with the following feasible potential problem:

(FPP) Find a function $u \in L(X)$ (potential function) satisfying the constraints:

(1) $0 \leq u(x) \leq \beta(x)$ on X,

(2) $A(y) \leq Ku(y) \leq B(y)$ on Y.

We want to find a necessary and sufficient condition for the existence of the solution of (FPP).

2. Existence result

For $w \in L(Y)$, let $w^+(y) := \max\{0, w(y)\}$, $w^-(y) := \max\{0, -w(y)\}$. Similarly we define f^+, f^- for $f \in L(X)$. Let us define functionals ℓ and ρ on $L_0(Y)$ by

$$\ell(w) := < A, w^+ > - < B, w^- >$$

$$\rho(w) := < \beta, (K^*w)^+ >.$$

Notice that $-\ell$ and ρ are sublinear. In particular, we have

$$\rho(w_1 + w_2) \leq \rho(w_1) + \rho(w_2),\ \ell(w_1 + w_2) \geq \ell(w_1) + \ell(w_2).$$

Now we have the following result.

Theorem 1. (FPP) *has a solution if, and only if, the following condition holds*:

(N1) $\ell(w) \leq \rho(w)$ *for all* $w \in L_0(Y)$.

Proof. To prove the necessity, let $u \in L(X)$ be a solution of (FPP). Then for every $w \in L_0(Y)$ we have

$$\ell(w) \leq < Ku, w^+ > - < Ku, w^- > = < Ku, w >$$

$$= < u, K^*w > \leq < u, (K^*w)^+ > \leq < \beta, (K^*w)^+ > = \rho(w).$$

Thus (N1) holds. We turn to the sufficiency.

If $w_1 \in L_0^+(Y)$, $w_2 \in L_0^+(Y)$ and $w = w_1 - w_2$, then
$< A, w^+ > - < B, w^- > \geq < A, w_1 > - < B, w_2 >$.
Moreover, if $f \in L_0^+(X)$, $w \in L_0(Y)$ and $f \geq K^*w$, then $\rho(w) \leq < \beta, f >$.
Therefore from (N1) we obtain:

(N1#) $0 \leq < \beta, f > - < A, w_1 > + < B, w_2 >$ for all $f \in L_0^+(X)$, $w_1 \in L_0^+(Y)$, $w_2 \in L_0^+(Y)$ satisfying $f - K^*w_1 + K^*w_2 \in L_0^+(X)$.

To proceed with the proof, we need a certain generalization of Farkas' Lemma [2; p. 134]. For a real topological vector space E, let E^* be its continuous dual. For a convex cone $P \subset E$, let P^O be the polar cone defined by

$$P^O := \{\xi \in E^*;\ < e, \xi > \geq 0 \text{ for all } e \in P\}.$$

Lemma 1. *Let the following assumptions hold:*

- E *is a real topological vector space,* F *is a locally convex topological vector space;*
- $P \subset E$ *and* $Q \subset F$ *are convex cones,* $P^O \subset E^*$ *and* $Q^O \subset F^*$ *are their polar cones;*
- T *is a linear mapping from* E *to* F, T^* *is the linear mapping from* F^* *to* E^* *such that*
  $< Te, \varphi > = < e, T^*\varphi >$ *for all* $e \in E$, $\varphi \in F^*$;
- $f_0 \in F$ *is a fixed element;*
- $T(P) + Q$ *is closed in* F.

Then $f_0 \in T(P) + Q$ *if the following condition holds:*

(F) $< f_0, \varphi > \geq 0$ *for all* $\varphi \in Q^O$ *with* $T^*\varphi \in P^O$.

Proof. Assume, for contradiction, that f_0 is not an element of the closed convex cone $T(P) + Q$. Then from the strong separation theorem in a locally convex space [3; p. 65] there exists $\varphi \in F^*$ such that

$$< f_0, \varphi > < 0 \leq < Tp + q, \varphi > = < p, T^*\varphi > + < q, \varphi >$$

for all $p \in P$, $q \in Q$. It follows that $T^*\varphi \in P^o$ and $\varphi \in Q^o$. Thus (F) does not hold.

We continue with the proof of Theorem 1. In order to apply Lemma 1 we provide $L(Y) = \mathbb{R}^Y$ with the product topology. Then $L_0(Y)$ can be identified with the topological dual of $L(Y)$ and $(L^+(Y))^o = L_0^+(Y)$. Likewise $L(X)$ is provided with the product topology. We substitute in Lemma 1:

$E := L(X)$, $F := L(X) \times L(Y) \times L(Y)$,

$P := L^+(X)$, $Q := L^+(X) \times L^+(Y) \times L^+(Y)$,

$T := (I, - K, K)$ (I denotes the identity mapping of $L(X)$),

$f_0 := (\beta, - A, B)$,

$P^o = L_0^+(X)$, $Q^o = L_0^+(X) \times L_0^+(Y) \times L_0^+(Y)$,

$T^*(f, w_1, w_2) = f - K^*w_1 + K^*w_2$.

Notice that (N1#) implies (F). The conclusion of Lemma 1 gives

$$(\beta, - A, B) \in (I, - K, K)(L^+(X)) + L^+(X) \times L^+(Y) \times L^+(Y).$$

So there exists $u \in L^+(X)$ such that

$$\beta \in u + L^+(X),\ - A \in - Ku + L^+(Y),\ B \in Ku + L^+(Y).$$

Then u is a solution of (FPP).

It remains to verify the closedness of the set $T(P) + Q$ in Lemma 1.

Lemma 2. *The set* $C := (I, - K, K)(L^+(X)) + L^+(X) \times L^+(Y) \times L^+(Y)$ *is closed in* $L(X) \times L(Y) \times L(Y)$.

Proof. Recall that $L(X)$ and $L(Y)$ are endowed with the product topology. So the convergence in $L(X)$ and $L(Y)$ means the pointwise convergence. Let (a, b, c) be an element of the closure of C. Since $L(X)$ and $L(Y)$ have countable neighborhood bases of zero, there exist countable sequences $\{f_n\} \subset L^+(X)$, $\{g_n\} \subset L^+(X)$, $\{h_n\} \subset L^+(Y)$, $\{k_n\} \subset L^+(Y)$ such that

$$f_n + g_n \longrightarrow a, \; - Kf_n + h_n \longrightarrow b, \; Kf_n + k_n \longrightarrow c \text{ pointwise as } n \longrightarrow \infty.$$

For every $x \in X$, it is easily seen that the sequence $\{f_n(x)\}$ remains bounded. Namely there exists $d(x) \geq 0$ such that $f_n(x) \in [0, d(x)]$ for all n. Then $\{f_n\} \subset \Pi_{x \in X} [0, d(x)]$, and the latter set - a product of compact intervals - is compact in $L(X)$ according to Tychonoff's theorem. So there exists a subsequence, again denoted by $\{f_n\}$, which converges pointwise to f in $L(X)$. Then $f \in L^+(X)$. It is clear that $g_n \longrightarrow g \in L^+(X)$ such that $f + g = a$. Since G is locally finite, $Kf_n \longrightarrow Kf$, and hence $h_n \longrightarrow h \in L^+(Y)$ such that $- Kf + h = b$. Likewise $k_n \longrightarrow k \in L^+(Y)$ such that $Kf + k = c$. So $(a, b, c) \in C$, and C is closed.

The proof of Theorem 1 is now complete.

3. Alternative approach

We suppose now that the graph G is finitely connected, and we give for this case another proof of the sufficiency of (N1) which does not employ Farkas' Lemma, but relies instead on paths and cycles. We recall the following. A path P in G is a triplet $P = \{X(P), Y(P), p\}$, where

- $X(P) = \{x_0, x_1, \cdots, x_n\}$ is a finite nonempty sequences of node,
- $Y(P) = \{y_1, \cdots, y_n\}$ is a finite sequence of pairwise different arcs with $K(x_{i-1}, y_i)K(x_i, y_i) = - 1$ for $i = 1, \cdots, n$,
- $p \in L_0(Y)$(the path function) is given by

$$p(y_i) := K(x_i, y_i) \text{ for } i = 1, \cdots, n,$$
$$p(y) := 0 \text{ for } y \notin Y(P).$$

A path from $a \in X$ to $b \in X$ is a path with $x_0 = a$, $x_n = b$. A cycle is a path with $x_0 = x_n$. We denote by $\mathbb{P}_{a,b}$ the set of all paths from a to b, and we denote by $\mathbb{Z}$ the set of all cycles. Finite connectedness of G means that $\mathbb{P}_{a,b} \neq \emptyset$ for all a, b $\in$ X. Henceforth we identify a path P with its path function, and we write accordingly $p \in \mathbb{P}_{a,b}$, $p \in \mathbb{Z}$, $y \in Y(p)$, etc. The empty path $\tilde{p} = 0$ belongs to $\mathbb{Z}$ as well as to $\mathbb{P}_{a,a}$ for all $a \in X$. We fix now $a \in X$ and conduct the proof of the sufficiency of (N1) as follows. From (N1) we have $\ell(w) \leq \rho(w)$ on $L_0(Y)$, where ℓ is superlinear and ρ is sublinear. From the Sandwich Theorem [2; p.112] there exists a linear functional ξ on $L_0(Y)$ such that

(3) $\quad \ell(w) \leq \xi(w) \leq \rho(w) \quad$ on $L_0(Y)$.

For two paths $p_1, p_2 \in \mathbb{P}_{a,x}$ we obtain from (3), since $K^*(p_1 - p_2) = 0$, that $\xi(p_1 - p_2) \leq \rho(p_1 - p_2) = 0$. Likewise we obtain $\xi(p_2 - p_1) \leq 0$. Since ξ is linear, this implies $\xi(p_1) = \xi(p_2)$. From this follows $\xi(p) = 0$ for all $p \in \mathbb{Z}$ (since every cycle p can be represented as $p = p_1 - p_2$ with $p_1, p_2 \in \mathbb{P}_{a,x}$ for a suitably chosen $x \in X$). We can now define unambiguously $u \in L(X)$ by

$$u(x) := \xi(p) \quad \text{for some } p \in \mathbb{P}_{a,x}.$$

This implies in particular $u(a) = 0$, since $\tilde{p} = 0 \in \mathbb{P}_{a,a}$. We show first that u fulfills (2). Let y be any arc, and set $x_1 := x^-(y)$, $x_2 := x^+(y)$. Denote by p_0 the path from x_1 to x_2 consisting only of the arc y. There exists $p_1 \in \mathbb{P}_{a,x_1}$ such that $u(x_1) = \xi(p_1)$. Then $p_1 + p_0 = p_2 + \bar{p}$, where $p_2 \in \mathbb{P}_{a,x_2}$ and $\bar{p} \in \mathbb{Z}$. Consequently

$$\begin{aligned} u(x_2) &= \xi(p_2) = \xi(p_2 + \bar{p}) = \xi(p_1 + p_0) = u(x_1) + \xi(p_0) \\ &\geq u(x_1) + \ell(p_0) \qquad \text{[from (3)]} \\ &= u(x_1) + A(y). \end{aligned}$$

Thus $u(x_2) - u(x_1) \geq A(y)$. Likewise we obtain $u(x_1) - u(x_2) \geq$

$- B(y)$. Since $(Ku)(y) = u(x_2) - u(x_1)$, we have $A \le du \le B$, and u fulfills (2).

Now we verify that u fulfills (1). Let $p \in \mathbb{P}_{a,x}$. By (3),

$$u(x) = \xi(p) \le \rho(p) = \beta(x).$$

Assume for the moment that $\beta(a) = 0$. Then

$$- u(x) = - \xi(p) = \xi(- p) \le \rho(- p) = \beta(a) = 0.$$

Thus $0 \le u \le \beta$, and u fulfills (1). If $\beta(a) > 0$ we proceed as follows. We form an extended graph $G^* := \{X^*, Y^*, K^*\}$ by adding to G a new node a^* and a new arc y^* which has a^* as the initial node and a as the terminal node. We define the additional data $\beta(a^*) := 0$, $A(y^*) := 0$, $B(y^*) := \beta(a)$. We denote ℓ^* and ρ^* the corresponding extensions of ℓ and ρ to G^*. Then, for $w \in L_0(Y^*)$

$$\ell^*(w) = \ell(w) - B(y^*)(w(y^*))^- = \ell(w) - \beta(a)(- w(y^*))^+,$$

$$\rho^*(w) = \rho(w) - \beta(a)k^+ + \beta(a)(k + w(y^*))^+,$$

where $k := (K^*w)(a)$. From the subadditivity of the function $(\cdot)^+$ follows $\ell^*(w) - \rho^*(w) \le \ell(w) - \rho(w)$. Thus, the validity of (N1) carries over from G to G^*. On G^*, since $\beta(a^*) = 0$, the previous reasoning applies, and we obtain a feasible potential u^* on $X^* = X \cup \{a^*\}$. The restriction of u^* to X satisfies the original requirements.

4. A particular case

We consider the case where the potential u is only requested to have property (2). We can prove the following result by using Lemma 1:

Theorem 2. *Let* G *be finitely connected. Then there exists* $u \in L(X)$ *satisfying* (2) *if, and only if, the following condition* *holds:*

(N2) $\ell(w) \leq 0$ *for all* $w \in L_0(Y)$ *with* $K^*w = 0$.

If p is a cycle, then $K^*p = 0$. Hence (N2) implies the following condition:

(N3) $\ell(p) \leq 0$ for all $p \in \mathbb{Z}$.

For finite graphs it is shown in [1; p. 157] that (N3) is necessary and sufficient for the existence of a potential $u \in L(X)$ which satisfies (2). Below we extend this result to finitely connected graphs and — what is more important — give a constructive proof for the existence of a feasible potential.

Let us agree to call two paths or cycles p_1 and p_2 parallel if

$$p_1(y) = p_2(y) \quad \text{for all } y \in Y(p_1) \cap Y(p_2).$$

If p_1 and p_2 are parallel, then $\ell(p_1 + p_2) = \ell(p_1) + \ell(p_2)$. It follows from (N3) that for fixed $a \in X$

(4) $\ell(p_1) + \ell(-p_2) \leq 0$

for all $p_1, p_2 \in \mathbb{P}_{a,x}$ and for all $x \in X$. In fact, $p_1 - p_2$ can be represented as a sum of parallel cycles $\bar{p}_i$ and therefore

$$\ell(p_1) + \ell(-p_2) \leq \ell(p_1 - p_2) = \ell(\Sigma_i \bar{p}_i) = \Sigma_i \ell(\bar{p}_i) \leq 0.$$

Theorem 3. *Let* G *be finitely connected. Then condition* (N3) *is necessary and sufficient for the existence of a function* u $\in L(X)$ *which satisfies* (2).

Proof. Assume that (N3) holds. Fix $a \in X$. For every $x \in X$ we define

$$u(x) := \sup\{\ell(p);\ p \in \mathbb{P}_{a,x}\}.$$

From (4) follows $u(x) < \infty$ for all $x \in X$. Moreover we have $u(a) = 0$, since $\ell(p) \leq 0$ for all $p \in \mathbb{P}_{a,a}$ with equality holding $p = 0 \in \mathbb{P}_{a,a}$. We show that u fulfills (2). Let y be any arc, and set

$x_1 := x^-(y)$, $x_2 := x^+(y)$. Let p_0 be the path from x_1 to x_2 consisting only of the arc y. Let $\varepsilon > 0$. There exists $p_1 \in \mathbb{P}_{a,x_1}$ such that $\ell(p_1) > u(x_1) - \varepsilon$. Then $p_1 + p_0 = p_2 + \bar{p}$, where $p_2 \in \mathbb{P}_{a,x_2}$, $\bar{p} \in \mathbb{Z}$, and p_2 and $\bar{p}$ are parallel. So

$$\ell(p_2) + \ell(\bar{p}) = \ell(p_2 + \bar{p}) = \ell(p_1 + p_0) \geq \ell(p_1) + \ell(p_0),$$

and therefore

$$u(x_2) \geq \ell(p_2) \geq \ell(p_2) + \ell(\bar{p}) \geq \ell(p_1) + \ell(p_0)$$
$$> u(x_1) - \varepsilon + \ell(p_0) = u(x_1) - \varepsilon + A(y).$$

Since $\varepsilon > 0$ is arbitrary, $u(x_2) - u(x_1) \geq A(y)$. Similarly we obtain $u(x_1) - u(x_2) \geq -B(y)$, and u satisfies our requirement.

5. Generalization

We consider now discrete potential problems in somewhat at greater generality. Let X and Y be countable sets, and Ψ be a real-valued function defined on $X \times Y$ such that

- for each $x \in X$, $\Psi(x, \cdot)$ has finite support on Y;
- for each $y \in Y$, $\Psi(\cdot, y)$ has finite support on X.

The notations $L(X)$, $L(Y)$ etc. have the same meaning as before. We define the discrete derivative d and discrete Laplacian Δ through

$$(du)(y) := \textstyle\sum_{x \in X} \Psi(x, y)u(x),$$
$$(\Delta u)(x) := \textstyle\sum_{y \in Y} \Psi(x, y)[(du)(y)].$$

In case $\Psi(x, y) = -r(y)^{-1}K(x, y)$ with a positive function r on Y, du and Δu are studied in [4].

Given α, $\beta \in L(X)$, A, $B \in L(Y)$, λ, $\mu \in L(X)$ such that $\alpha \leq \beta$, $A \leq B$, $\lambda \leq \mu$, we consider the following generalized feasible potential problem:

(GFPP) Find $u \in L(X)$(potential function) satisfying the constraints:

(5) $\alpha \leq u \leq \beta$ on X,

(6) $A \leq du \leq B$ on Y,

(7) $\lambda \leq \Delta u \leq \mu$ on X.

We obtain a feasibility condition for this problem by Lemma 1. To formulate this we introduce the following notations. Define the mapping d^* from $L_0(Y)$ to $L_0(X)$ by

$$(d^*w)(x) := \sum_{y \in Y} \Psi(x, y)w(y).$$

Then $\langle du, w \rangle = \langle u, d^*w \rangle$ for all $u \in L(X)$ and $w \in L_0(Y)$. If $u \in L_0(X)$, then $\Delta u \in L_0(X)$. Let Δ^* be the restriction of Δ to $L_0(X)$. Then we have

$$\langle \Delta u, f \rangle = \langle u, \Delta^* f \rangle \text{ for all } u \in L(X) \text{ and } f \in L_0(X).$$

By the same reasoning as in Theorem 1, we can prove

Theorem 4. (GFPP) *has a solution if, and only if, the following condition is fulfilled:*

(N4) $0 \leq \langle \beta, h^+ \rangle - \langle \alpha, h^- \rangle + \langle B, w^+ \rangle - \langle A, w^- \rangle + \langle \mu, g^+ \rangle - \langle \lambda, g^- \rangle$

for all $w \in L_0(Y)$, $g \in L_0(X)$, $h \in L_0(X)$ *satisfying* $h + d^*w + \Delta^*g = 0$.

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