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Eigenvalue Problems of the Parameter Dependent System of Ordinary Differential Equations and Computer Aided Proof

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1 Introduction

Among the problems of stability and bifurcation of solutions of various equations of fluid dynamics, some of these can be reduced to an eigenvalue problem of system of ordinary differential equations including physical parameters. They are boundary value problems for linear systems of ODEs. It is, however, difficult to analyze how an eigenvalue of the system depends on parameters, since these systems are not self-adjoint and have variable coefficients.

In this article we propose a method to analyze this. Namely, taking as a concrete example the free boundary problem of viscous incompressible fluid flowing down an inclined plane, we study the stability of the stationary laminar flow when its Reynolds number changes. It can be expected that, at certain critical Reynolds number, this stationary solution becomes unstable and the Hopf bifurcation occurs. These will be proved by showing how the eigenvalue of this system behaves as parameters change. We explain how the above can be shown by a computer assisted proof. This method is an extended version of the ones employed in [1] and [2]. In [1] it was proved that, for the periodically forced dissipative systems of ODEs, there exist periodic solutions with the same period, double period, triple period and so on. The systems include Duffing equation as an example. In [2] it was shown that the autonomous systems including the Lorenz equation have a periodic solution. Therefore, our method is applicable to the system of nonlinear ODEs.

2 Stability of surface waves of viscous fluid flowing down an inclined plane

For the free boundary problem of viscous incompressible fluid flowing down an inclined plane, the existence of local in time solutions is obtained in [7], and, when the Reynolds number and the angle of inclination are small, the global existence theorem is proved in [6] for small initial data. As in [4] we consider two dimensional fluctuations of the steady laminar flow for simplicity. We use dimensionless variables employed in [4]. Let $\mathcal{R}$ be the Reynolds number of this laminar flow. α is the inclination angle. The shallow water parameter δ denotes a ratio between wave height and wave length. As $\mathcal{R}$ is increased, the problem of stability of the stationary solution arises. This stability analysis can be reduced to study the eigenvalue problem of the following linear equation written in terms of the stream function ψ.

$$\psi = 0, \quad \psi_y = 0, \quad \text{on } y = 0 \quad , \tag{2.1}$$

$$\begin{aligned} &\psi_{yyyy} + 2\delta^2\psi_{yyxx} + \delta^4\psi_{xxxx} \\ &\quad -\delta\mathcal{R}\left\{\psi_{yyt} + \delta^2\psi_{xxt} + 2\psi_x + \right. \\ &\quad \left.(2y - y^2)\left(\psi_{xyy} + \delta^2\psi_{xxx}\right)\right\} = 0, \quad \text{in } 0 < y < 1, \end{aligned} \tag{2.2}$$

$$\eta_t + \eta_x + \psi_x = 0, \quad \text{on} \quad y = 1, \tag{2.3}$$

$$\psi_{yy} - \delta^2\psi_{xx} = 2\eta, \quad \text{on} \quad y = 1, \tag{2.4}$$

$$\begin{aligned} &\psi_{yyy} - \delta\mathcal{R}\left(\psi_{yt} + \psi_{xy}\right) + 3\delta^2\psi_{xxy} \\ &\quad + 2\delta^3\mathcal{W}\csc\alpha\,\eta_{xxx} - 2\delta\cot\alpha\,\eta_x = 0, \quad \text{on} \quad y = 1. \end{aligned} \tag{2.5}$$

Here $\mathcal{W}$ is the Weber number. Since we are concerned with only linear disturbances periodic in the stream-wise direction and since the coefficients in (2.2) depend only on y, assuming the periodicity in x, we can consider ψ of the form

$$\psi = \phi(y)\exp(\, inx + \lambda t\,)\,. \tag{2.6}$$

The free surface position η can be recovered from (2.3) as

$$\eta = \frac{-in\phi(1)}{\lambda + in}\exp(\, inx + \lambda t\,)\,. \tag{2.7}$$

After substituting (2.6) and (2.7) into (2.1) – (2.5), we obtain the eigenvalue problem of the ODE for ϕ:

$$\phi(0) = 0, \quad \phi'(0) = 0, \quad \text{on } y = 0, \tag{2.8}$$

(2.9) $$\phi'''' - 2m^2\phi'' + m^4\phi$$
$$= im\mathcal{R}\left\{\left(2y - y^2 + \mu\right)\left(\phi'' - m^2\phi\right) + 2\phi\right\}, \quad \text{in } 0 < y < 1,$$

(2.10) $$\phi''(1) + m^2\phi(1) + \frac{2}{\mu + 1}\phi(1) = 0, \quad \text{on } y = 1,$$

(2.11) $$\phi'''(1) - im\mathcal{R}(\mu + 1)\phi'(1)$$
$$-3m^2\phi'(1) + \frac{2im^3}{\mu+1}\mathcal{W}\phi(1) + \frac{2im}{\mu+1}\cot\alpha\phi(1) = 0, \text{ on } y = 1.$$

Here we put $\mu = -\frac{\lambda}{in}$, $m = \delta n$. By this formulation, the original problem of stability is now reduced to investigate the behavior of the real part of the eigenvalue λ when the parameters $\mathcal{R}$ and m vary.

For our present concern, the problem is to find $\mathcal{R} = \mathcal{R}_c$ at which λ becomes $\pm i\omega$ ($\omega \in \mathbf{R}$) for certain periodicity in x, m fixed, and ,further, to show

(2.12) $$\left.\frac{\partial \mathrm{Re}\lambda}{\partial \mathcal{R}}\right|_{\mathcal{R}=\mathcal{R}_c} > 0.$$

We carry out these in the following sections. By (2.12) and by the fact that the original evolution problem for the linearized system forms an sectorial operator, we see that a sufficient condition given in [5] for the occurence of the Hopf bifurcation holds. Hence, we see that the laminar flow becomes unstable for $\mathcal{R} > \mathcal{R}_c$ and the Hopf bifurcation occurs at $\mathcal{R} = \mathcal{R}_c$.

3 Criterion for existence of critical eigenvalue

To obtain the eigenvalue and the eigenfunction for (2.8) – (2.11), we consider the initial value problem for (2.9) for $y \geq 0$ and express its solution as

(3.1) $$\phi = a\phi_1(y) + b\phi_2(y), \quad y > 0,$$

where $\phi_j(y)$, $j = 1, 2$ satisfy (2.9) on $y > 0$ and the initial conditions

(3.2) $$\left\{\begin{array}{l} \phi_j(0) = 0, \quad \phi_j'(0) = 0, \quad j = 1, 2, \\ \phi_1''(0) = 1, \quad \phi_1'''(0) = 0, \\ \phi_2''(0) = 0, \quad \phi_2'''(0) = 1 \end{array}\right. .$$

a and b are constants to be determined. In order that the function (3.1) is the eigenfunction, (3.1) must satisfy the conditions (2.10) and (2.11). This condition is written as follows

(3.3) $$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = 0,$$

where the coefficients a_{ij} are explicitly given by $\phi_k(1)$, $\phi_k'(1)$, $\phi_k''(1)$, $\phi_k'''(1)$, $k = 1, 2$. In order that (3.1) is nontrivial, it is necessary that

$$\det A \equiv a_{11}a_{22} - a_{12}a_{21} = 0, \tag{3.4}$$

and (3.4) is sufficient for (3.1) to be the eigenfunction. Thus, we now come to search, for the fixed parameters α and m, the values of $\mathcal{R} = \mathcal{R}_c$, $\lambda = i\omega_c$ satisfying

$$\det A = 0.$$

We put

$$\det A = \mathcal{F}(\mathcal{R}, \lambda\,;\ \alpha, m)\,. \tag{3.5}$$

Noting that (3.4) can be rewritten as

$$\begin{aligned} \mathcal{F}(\mathcal{R}, \lambda) &= \mathcal{F}(\mathcal{R}_0, \lambda_0) \\ &\quad + \frac{\partial \mathcal{F}}{\partial \mathcal{R}}(\mathcal{R} - \mathcal{R}_0) + \frac{\partial \mathcal{F}}{\partial \lambda}(\lambda - \lambda_0) = 0, \end{aligned} \tag{3.6}$$

we can state our criterion for existence of the critical eigenvalue based on the simplified Newton method as below

Theorem *Suppose, for small $\varepsilon > 0$, there exist $\mathcal{R}_0$ and λ_0 such that*

$$\|\mathcal{F}(\mathcal{R}_0,\ \lambda_0)\| < \varepsilon. \tag{3.7}$$

Put

$$L_0 \equiv \left(\overline{\frac{\partial \mathcal{F}}{\partial \mathcal{R}}(\mathcal{R}_0, \lambda_0)}\ ,\ \overline{\frac{\partial \mathcal{F}}{\partial \lambda}(\mathcal{R}_0, \lambda_0)} \right). \tag{3.8}$$

Suppose further that, for small δ, there is a ρ_1 such that the estimate

$$\| D\mathcal{F}(\mathcal{R}\ ,\ \lambda) - L_0 \| < \delta \tag{3.9}$$

holds for any $(\mathcal{R}\ ,\ \lambda)$ such that

$$(\mathcal{R} - \mathcal{R}_0)^2 + |\lambda - \lambda_0|^2 < \rho_1^2\,.$$

For ε, ρ_1, δ and L_0 as above, if it holds that

$$\| L_0^{-1} \| \left(\frac{\varepsilon}{\rho_1} + \delta \right) \leq 1\,, \tag{3.10}$$

then there exist some $\mathcal{R}_c$ and λ_c in the ρ_1-neighborhood of $\mathcal{R}_0$ and λ_0 satisfying

$$\mathcal{F}(\ \mathcal{R}_c\ ,\ \lambda_c\ ) = 0\,. \tag{3.11}$$

To utilize this criterion to our problem, we only need to justify the following steps:

i) Find appropriate values $\mathcal{R}_0$ and λ_0, and estimate ε;
ii) At this pair of $\mathcal{R}_0$, λ_0, find an approximate derivative L_0 and estimate the norm $\| L_0^{-1} \|$;
iii) Estimate δ for which the estimate (3.9) holds in the ρ_1-neighborhood of $\mathcal{R}_0$ and λ_0;
iv) For these values in (*i, ii, iii*), prove that the criterion (3.10) holds.

Example I . We here cite a numerical example for the fixed parameters $\alpha = 0.5$ and $m = 0.5$. By the shooting method based on the fourth order Taylor difference scheme, ϕ and its derivatives are calculated. The number of mesh-points on the interval $0 < y \leq 1$ is $K = 1024 \times 32$. To obtain the zero $(\mathcal{R}_0, \lambda_0)$ of (3.5) we use the Newton scheme. We obtain numerically

$$(3.12) \qquad \begin{cases} \lambda_0 = 0.82985\,15563\,586 \times i \\ \mathcal{R}_0 = 5.26808\,55830\,985\,. \end{cases}$$

At this approximate zero, we obtain

$$(3.13) \qquad \begin{cases} \overline{|\det A|} < 0.2 \times 10^{-14} \\ \overline{\dfrac{\partial \det A}{\partial \mathcal{R}}} = -0.05529\,67703 - i \times 0.20463\,24459 \\ \overline{\dfrac{\partial \det A}{\partial \mu}} = 1.50558\,39103 - i \times 3.46028\,24006 \quad . \end{cases}$$

The notation with over line denotes the value obtained numerically. The error from the exact value can be derived by using the theory of pseudo trajectory, so we have

$$(3.14) \qquad \left|\det A(\mathcal{R}_0, \lambda_0) - \overline{\det A(\mathcal{R}_0, \lambda_0)}\right| < 0.428 \times 10^{-11}\,.$$

Thus we have

$$(3.15) \qquad \varepsilon = |\det A\,(\mathcal{R}_0, \lambda_0)| < 0.429 \times 10^{-11}\,.$$

The jacobian of $\mathcal{F}$ can also be estimated as

$$(3.16) \qquad \|D\mathcal{F}(\mathcal{R}, \lambda) - L_0\| < 0.4 \times 10^7 \times \rho_1 \quad \text{for } |\mathcal{R} - \mathcal{R}_0|^2 + |\lambda - \lambda_0|^2 < \rho_1^2\,.$$

In estimating these errors by using the theory of pseudo trajectory, we have to estimate the rounding error of numerical computation as well as the truncation error by discretization. The former must be performed on a computer by software for an interval arithmetic. The rounding error on each step of our difference scheme is expected to be less than 10^{-13} by the computation of double precision and expected to be less than 10^{-26} by the computation of quadruple precision. For the estimate below we need quadruple precision in order that $\rho_1 = 10^{-10}$ in the estimate in (3.7) – (3.11).

Thus, from (3.15), (3.16) and $\rho_1 = 10^{-10}$, and because of $\delta = 0.4 \times 10^7 \times 10^{-10}$,

$$\|L_0^{-1}\| \left(\frac{\varepsilon}{\rho_1} + \delta\right) = 10 \times \left(\frac{0.429 \times 10^{-11}}{10^{-10}} + 0.4 \times 10^7 \times 10^{-10}\right) < 0.5 , \tag{3.17}$$

our criterion holds. Hence, we see that there exist the exact eigenvalue $\lambda = i\omega_c$ and the Reynolds number $\mathcal{R} = \mathcal{R}_c$ in the ρ_1 -neighborhood of $(\mathcal{R}_0 , \rho_0)$ of (3.12).

4 Behavior of eigenvalue at critical Reynolds number

We finally show how to study the behavior of the eigenvalue $\lambda = \lambda(\mathcal{R}; m, \alpha)$ in the neighborhood of $\mathcal{R} = \mathcal{R}_c$. For notational convenience we write the equation (2.9) and the boundary conditions (2.8), (2.10) and (2.11) as

$$L\phi = 0 \quad \text{and} \quad B\phi = 0 \tag{4.1}$$

respectively. Let L^* and B^* be the formal adjoint operator of L and the adjoint boundary conditions respectively. It is known that, if (4.1) has a nontrivial solution, then the adjoint problem

$$L^*\psi = 0 \quad \text{and} \quad B^*\psi = 0 \tag{4.2}$$

also has a nontrivial solution. Let ψ be the nontrivial solution of (4.2) corresponding to λ_c and $\mathcal{R}_c$. Differentiating (2.9) in $\mathcal{R}$ yields

$$L\frac{\partial\phi}{\partial\mathcal{R}} = a[\phi]\frac{\partial\lambda}{\partial\mathcal{R}} + b[\phi] \tag{4.3}$$

where

$$a[\phi] = -\frac{m}{n}\mathcal{R}(\phi'' - m^2\phi) \tag{4.4}$$

and

$$b[\phi] \;=\; im\left\{\left(2y - y^2 - \frac{\lambda}{in}\right) \;+\; 2\phi\right\}. \tag{4.5}$$

Taking the $L^2(0,1)$-inner product of (4.3) with the solution ψ of the problem adjoint to (4.1), we obtain

$$\left(a[\phi]\frac{\partial\lambda}{\partial\mathcal{R}} \;+\; b[\phi]\;,\;\psi\right)_{L^2} \;=\; 0\,. \tag{4.6}$$

From this we have

$$\left.\frac{\partial\lambda}{\partial\mathcal{R}}\right|_{\mathcal{R}=\mathcal{R}_c} \;=\; -\frac{(b[\phi],\psi)_{L^2}}{(a[\phi],\psi)_{L^2}}\,. \tag{4.7}$$

We calculate this integration in double precision and obtain numerically

$$\begin{aligned} &\left.\overline{\frac{\partial\lambda}{\partial\mathcal{R}}}\right|_{\mathcal{R}=\mathcal{R}_0} \\ &= \;0.01753\;58825 \;+\; i\times 0.02193\;88106\,. \end{aligned} \tag{4.8}$$

For this numerical integration we use the trapezoidal rule, so the error can be estimated by

$$\begin{aligned} &\left|\phi(y) - \left(\overline{\phi}_k + \frac{\overline{\phi}_{k+1}-\overline{\phi}_k}{\Delta y}(y - k\Delta y)\right)\right| \\ &\le \max_y|\phi''(y)|\,(\Delta y)^2 + \max_k\left|\phi(k\Delta y) - \overline{\phi}_k\right| \\ &\qquad\qquad \text{for}\quad k\Delta y \le y \le (k+1)\Delta y \end{aligned} \tag{4.9}$$

and by the theory of pseudo trajectory. Thus we can conclude that

$$\left.\frac{\partial\,\mathrm{Re}\,\lambda}{\partial\mathcal{R}}\right|_{\mathcal{R}=\mathcal{R}_c} \;>\; 0\,. \tag{4.10}$$

This shows that the stationary solution is stable for $\mathcal{R} < \mathcal{R}_c$ and becomes unstable for $\mathcal{R} > \mathcal{R}_c$ and that the Hopf bifurcation occurs at $\mathcal{R} = \mathcal{R}_c$.

Example II. We here cite another example. Take $\alpha = 0.5$ and $m = 1.0$. The number of mesh-points is $K = 1024 \times 32$. We compute by quadruple precision.

$$\left\{\begin{array}{lcl} \lambda_0 &=& 1.33019\;05847\;491\;\times\; i \\ \mathcal{R}_0 &=& 21.37190\;01766\;506\,. \end{array}\right. \tag{4.11}$$

At this approximate zero, we obtain

$$
(4.12) \qquad \begin{cases} \overline{|\det A|} < 1.0 \times 10^{-20} \\[1ex] \overline{\dfrac{\partial \det A}{\partial \mathcal{R}}} = -0.41370\,54211 - i \times 0.10135\,29912 \\[2ex] \overline{\dfrac{\partial \det A}{\partial \mu}} = -40.78671\,79530 - i \times 26.09613\,53005 \end{cases}
$$

$$
(4.13) \qquad \overline{\frac{\partial \lambda}{\partial \mathcal{R}}}\bigg|_{\mathcal{R}=\mathcal{R}_0} = 0.00284\,15752 + i \times 0.00832\,50456
$$

$$
(4.14) \qquad \| L_0{}^{-1} \| < 7.26820 .
$$

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