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GEOMETRIES AND CHAMBER SYSTEMS

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1 INTRODUCTION

Chamber systems have been introduced by Ronan [12] and Tits [15] for the needs of the theory of universal 2-covers. Every geometry can be viewed as a chamber system and every chamber system admits a universal 2-cover (Ronan [12]). Thus, given a geometry Γ, we can consider the universal 2-cover $\widetilde{\mathcal{C}(\Gamma)}$ of the chamber system $\mathcal{C}(\Gamma)$ of Γ. If $\widetilde{\mathcal{C}(\Gamma)}$ is the chamber system of some geometry $\tilde{\Gamma}$, then $\tilde{\Gamma}$ is the universal 2-cover of Γ.

Unfortunately, things are not so easy as they look. It is likely that we always have $\widetilde{\mathcal{C}(\Gamma)} = \mathcal{C}(\tilde{\Gamma})$ for some geometry $\tilde{\Gamma}$. However, this conjecture has been proved only in particular cases (see §3.2.3). No proof is known for the general case.

Thus, either we renounce to consider universal 2-covers when we do not know in advance that they will correspond to some geometries; or we acknowledge chamber systems as respectable mathematical objects, deserving to be investigated of their own.

The first option cannot be taken serious. According to it, we should renounce to classify a family of flag-transitive geometries when the only method we can see is to determine the amalgamated product of rank 2 parabolics, which corresponds to a 2-simply connected chamber system (Section 6, Theorem 6.1), maybe not defined by any geometry.

The latter option only remains. Unfortunately, developing a rich general theory of chamber systems is not so easy. For instance, the so-called Direct Sum Theorem, which is the headstone of diagram geometry, fails to hold for chamber systems in general (§§4.2.3, 4.4).

Nevertheless, some theory has been developed and I am confident that more can be done. In particular, by considering what I call the cell-geometry of a chamber system (Section 5), problems on chamber systems appear to be equivalent to seemingly easier problems on certain quasi-thin geometries with string diagrams.

In sections 2, 3 and 4 I will survey some known facts on chamber systems, geometries and their relations. Most of what I will say has a natural group-theoretic translation. I shall give it in Section 6.

2 BASIC CONCEPTS

2.1 Chamber Systems

2.1.1 Some notation for equivalence relations

Chambers systems (to be defined in the next subsection) are families of equivalence relations satisfying certain properties. Thus, it will be useful to have stated some notation for equivalence relations.

Given a nonempty set C, we denote by Ω the largest equivalence relation on C, having C as its unique equivalence class. The identity relation $=$ will be denoted by $\mho$.

Given an equivalence relation Φ on C and an element $x \in C$, we denote the equivalence class of Φ containing x by $[x]\Phi$.

Given two equivalence relations Φ and Ψ on C, $\Phi \vee \Psi$ is the least equivalence relation containing both Φ and Ψ, whereas $\Phi\Psi$ is their product, defined by the following clause: given any two elements $x, y \in C$, $(x, y) \in \Phi\Psi$ if and only if $x\Phi z$ and $z\Psi y$ for some $z \in C$.

If $\Phi \subseteq \Psi$, then Ψ/Φ denotes the quotient of Ψ by Φ, defined on C/Φ by the following clause: $([x]\Phi, [y]\Phi) \in (\Psi/\Phi)$ if and only if $x\Psi y$.

2.1.2 Definition of chamber systems

A *chamber system* over a finite set of *types* I is a pair $\mathcal{C} = (C, (\Phi_i)_{i\in I})$ where C is a nonempty set, whose elements are called *chambers*, and $(\Phi_i)_{i\in I}$ is a family of equivalence relations on C with the following properties:

(C1) $\bigvee_{i\in I} \Phi_i = \Omega$;
(C2) $\Phi_i \cap \Phi_j = \mho$ for any two distinct types $i, j \in I$;
(C3) $|[x]\Phi_i| \geq 2$ for every type $i \in I$ and every chamber $x \in C$.

The positive integer $n = |I|$ is called the *rank* of $\mathcal{C}$. The relation Φ_i is called the *i-adjacency* relation ($i \in I$). Two chambers are said to be *adjacent* if they are i-adjacent for some $i \in I$.

Remark. The above definition of chamber systems is more restrictive than other ones that can be found in the literature. We have chosen it because it is as close as possible to the definition of geometries we will state in §2.2.1.

2.1.3 Cells, Residues, Panels and Vertices

Let $\mathcal{C} = (C, (\Phi_i)_{i \in I})$ be a chamber system of rank $n = |I|$. We set $\Phi_\emptyset = \mho$ and $\Phi_J = \bigvee_{j \in J} \Phi_j$ for every nonempty subset J of I.

Given a chamber x and a subset J of I, the pair $([x]\Phi_J, J)$ is called a *cell* of *type* J. The numbers $|J|$ and $n-|J|$ are respectively the *rank* and the *corank* of $([x]\Phi_J, J)$. We often write $[x]\Phi_J$ for $([x]\Phi_J, J)$ (thus identifying a cell with its set of chambers) when this abbreviation does not cause any confusion. (Note that it might happen that $[x]\Phi_J = [x]\Phi_K$ with $J \neq K$.)

Trivially, if X is a cell of type $J \neq \emptyset$ and Φ_i^X is the restriction of Φ_i to X, then $\mathcal{C}_X = (X, (\Phi_j^X)_{j \in J})$ is a chamber system of rank $|J|$ over the set of types J and it is called a *residue* of $\mathcal{C}$ of *type* J.

The cells of rank 1 are called *panels.* Those of corank 1 are called *vertices.* We say that two vertices X, Y are *incident* if $X \cap Y \neq \emptyset$. We denote by $\Gamma(\mathcal{C})$ the graph defined on the set of vertices of $\mathcal{C}$ by taking the above defined incidence relation as adjacency relation. This graph is called the *incidence graph* of $\mathcal{C}$.

2.1.4 Chamber systems as coloured graphs

A chamber system $\mathcal{C}$ can also be viewed as a coloured graph: its edges are the pairs of distinct adjacent chambers and an edge $\{x, y\}$ gets the colour i if x and y are i-adjacent. Every edge has just one colour (property (C2)), every vertex belongs to at least one edge of each colour (C3) and the graph is connected (C1). For every colour i, the subgraph formed by the edges of that colour is the disjoint union of cliques (indeed the i-adjacency relation is an equivalence relation).

This way of looking at chamber systems is the most convenient one when considering morphisms and automorphisms (§2.7).

2.1.5 Chamber systems as spaces with parallelism

A chamber $\mathcal{C}$ system of rank n can also be viewed as an incidence structure, with chambers and panels as "points" and "lines" respectively and a "parallelism" between lines, two lines being parallel when they have the same type as panels of $\mathcal{C}$. We denote this structure by $\Pi_{\mathcal{C}}$. It satisfies the following properties:

(C'1) any two points are joined by some path of points and lines;
(C'2) every line has at least two points;
(C'3) dictinct lines never meet in more then one point;
(C'4) the parallelism is an equivalence relation with n classes;
(C'5) every point belongs to precisely one line of each parallel class.

Properties (C'1) and (C'2) rephrase (C1) and (C3), respectively. Properties (C'3) and (C'5) embody (C2). Property (C'5) forces each parallel class to partition the set of points. That is, it reminds us that for every type i, the i-adjacency relation is an equivalence relation on the set of chambers. Finally, (C'4) expresses the fact that $\mathcal{C}$ has rank n. The type set of $\mathcal{C}$ can be viewed as the "line at infinity" of this structure.

It is clear that, given any incidence structure Π with parallelism satisfying the above properties (C'1)-(C'5), there is a chamber system $\mathcal{C}$ such that $\Pi = \Pi_{\mathcal{C}}$ and $\mathcal{C}$ is uniquely determined by Π, modulo re-naming the types.

The point of view now offered on chamber systems will be further developed in Section 5.

2.2 Geometries

2.2.1 Definition

A *geometry of rank* 1 is a set of size ≥ 2, namely a graph with no edges and at least two vertices. According to [10], a *geometry of rank* $n > 1$ is a pair (Γ, Θ) where Γ is a connected graph, Θ is an n-partition of Γ (called the *type-partition* of Γ) and, for every vertex x of Γ, the neighbourhood Γ_x of x, with the $(n-1)$-partition induced by Θ on it, is a geometry of rank $n-1$. The type-partition Θ is uniquely determined by the graph Γ ([10], Theorem 1.25). Thus, we will always write Γ for (Γ, Θ), even if this is an abuse.

Remark. More general definitions of geometries can be found in the literature. However, the definition we have stated is general enough to cover almost all interesting examples. Furthermore, if we chose a less restrictive definition, we should possibly renounce some of the few general theorems on geometries (such as the Direct Sum Theorem of Section 4, for instance).

2.2.2 Some terminology

Let Γ be a geometry of rank n. The adjacency relation of the graph Γ is called the *incidence relation* of the geometry Γ. The vertices and the cliques of the graph Γ are respectively called *elements* and *flags* of the geometry Γ. By convention, $\emptyset$ and the elements of Γ are flags, too. The *rank* (*corank*) of a flag F is its size $|F|$ (resp., $n-|F|$). A flag F is maximal if and only if it has rank n ([10], Lemma 1.7). Maximal flags are called *chambers* of Γ.

The *residue* Γ_F of a non-maximal flag F is the geometry of rank $n-|F|$ induced by Γ on the neighbourhood $\bigcap_{x\in F}\Gamma_x$ of F.

2.2.3 Types

Given a geometry Γ of rank n, let S be its set of elements and let Θ be its type partition. A surjective function $t : S \longrightarrow I$ having the classes of Θ as fibers is called a *type function* for Γ and I is said to be a *set of types* for Γ. A *geometry over a set of types* I is a pair (Γ, t) as above.

Let (Γ, t) be a geometry over the type set I. Given a flag F of Γ, the subsets $t(F)$ and $I - t(F)$ of I are respectively called the *type* and the *cotype* of F. Clearly, the cotype of a non-maximal flag F is a set of types for the residue Γ_F. We call it the *type* of Γ_F.

2.3 Geometric Chamber Systems

Given a geometry Γ over a set of types I, the chambers (i.e. maximal flags) of Γ form a chamber system $\mathcal{C}(\Gamma)$, two chambers of Γ being i-adjacent when they intersect in a flag of cotype i. Clearly, $\Gamma(\mathcal{C}(\Gamma)) \cong \Gamma$ for every geometry Γ. That is, we can always recover a geometry from its chamber system.

On the other hand, the incidence graph $\Gamma(\mathcal{C})$ of a chamber system $\mathcal{C}$ need not be a geometry, in general (see [8], Section 4). It might also happen that $\Gamma(\mathcal{C})$ is a geometry but $\mathcal{C}(\Gamma(\mathcal{C})) \ncong \mathcal{C}$ (that is, we cannot recover $\mathcal{C}$ from $\Gamma(\mathcal{C})$).

$\Gamma(\mathcal{C})$ is a geometry and $\mathcal{C} \cong \mathcal{C}(\Gamma(\mathcal{C}))$ if and only if the cells of $\mathcal{C}$ correspond to the cliques of the graph $\Gamma(\mathcal{C})$ (in particular, chambers correspond to maximal cliques), in such a way that a cell is the intersection of all vertices of $\mathcal{C}$ containing it and every clique of $\Gamma(\mathcal{C})$ is the set of the vertices of $\mathcal{C}$ containing some given cell. This happens if and only if the following hold (compare [3]; also [10], §12.5):

(G1) $\Phi_J = \bigcap_{j \notin J} \Phi_{I-\{j\}}$ for every $J \subseteq I$;
(G2) $\Phi_J \cap (\Phi_{I-\{i\}} \cdot \Phi_{I-\{j\}}) = (\Phi_J \cap \Phi_{I-\{i\}}) \cdot (\Phi_J \cap \Phi_{I-\{j\}})$ for any two distinct types $i, j \in I$ and every subset J of I containing both i and j.

If $\mathcal{C}$ belongs to a diagram having strings or isolated nodes as connected components (see §2.5), then (G1) implies (G2) (Meixner and Timmesfeld [5]).

If both (G1) and (G2) hold in $\mathcal{C}$, then we say that $\mathcal{C}$ is *geometric*. Trivally, the chamber system of a geometry is always geometric. Hence a chamber system is geometric if and only if it is the chamber system of some gometry.

Many examples of non-geometric chamber systems are known (see [8], Section 4 and remarks following Theorem 2.5; also [13], [4], [6], [17]; and [3]).

2.4 The Rank 2 Case

Trivially, all chamber systems of rank 2 are geometric. Hence we can always replace them with their geometries. Thus, we shall only speak of geometries here, recalling some notions on those of rank 2.

Let Γ be a geometry of rank 2. We take $\{1,2\}$ as its set of types. For $i = 1,2$, the i-diameter d_i of Γ is the maximal distance between two vertices of the graph Γ at least one of which has type i. All circuits of the graph Γ have even length. Thus Γ has even girth. The *gonality* g of the geometry Γ is half of the girth of the graph Γ.

If $g = d_1 = d_2 = m$, then Γ is called a *generalized m-gon* (a *generalized digon, triangle, quadrangle* if $m = 2,3,4$ respectively). Generalized triangles are precisely (possibly degenerate) projective planes. Generalized digons are just complete bipartite graphs with at least two elements in each class of the bipartition. Chamber systems of generalized digons are characterized by the relation $\Phi_1\Phi_2 = \Phi_2\Phi_1$.

If $g > 2$ then Γ is called a *semilinear space* (also *partially linear space* or *partial plane*). Note that $g > 2$ if and only if Γ satisfies (C'3) of §2.1.5. Therefore, the incidence structure $\Pi_{\mathcal{C}}$ defined in §2.1.5 is a semilinear space (provided that $\mathcal{C}$ has rank > 1).

2.5 Diagrams

2.5.1 Definition

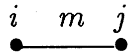
Let I be a set of types. A *diagram* $\mathbf{D}$ *over* I is a mapping defined from the set of unordered pairs of types $\{\{i,j\}\}_{i,j\in I, i\neq j}$ which assigns to every pair $\{i,j\}$ some class $\mathbf{D}_{i,j}$ of rank 2 geometries over $\{i,j\}$.

A geometry Γ (a chamber system $\mathcal{C}$) over I *belongs to the diagram* $\mathbf{D}$ over I if, for every pair of distinct types $i,j \in I$, the class $\mathbf{D}_{i,j}$ contains (the geometry of) every residue of Γ of type $\{i,j\}$. If generalized digons and other rank 2 geometries are hoarded up together in some class $\mathbf{D}_{i,j}$, then we also assume that at least one of the residues of Γ (of $\mathcal{C}$) of type $\{i,j\}$ is not a generalized digon. (Diagrams with such bad classes are never taken into consideration in "real life", but they might be, in principle.)

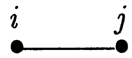
Clearly, a geometric chamber system belongs to a diagram $\mathbf{D}$ if and only if its geometry belongs to $\mathbf{D}$. This is not true in general for a non-geometric chamber system $\mathcal{C}$, even if $\Gamma(\mathcal{C})$ is a geometry.

2.5.2 Some conventions for diagrams

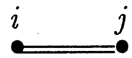
A diagram $\mathbf{D}$ over a type set I is usually depicted as a graph, drawing an edge between two types i, j if and only if $\mathbf{D}_{i,j}$ is not a class of generalized digons and labelling an edge $\{i,j\}$ by some symbol denoting the class $\mathbf{D}_{i,j}$. For instance, a label $m \geq 3$ on a stroke $\{i,j\}$ means that $\mathbf{D}_{i,j}$ is the class of generalized m-gons



Further conventions are used to make pictures easier to read. For instance, when $\mathbf{D}_{i,j}$ is the class of generalized triangles no label is put on the edge $\{i, j\}$



If $\mathbf{D}_{i,j}$ is the class of generalized quadrangles, then a double stroke the is used instead of the label 4



2.5.3 Diagrams as graph

Since a diagram can be viewed as a graph, we can extend to diagrams the terminology currently used for graphs, thus speaking of connected or diconnected diagrams, of the connected component of a diagram, of diagrams that are complete graph, or strings, ...

If a diagram $\mathbf{D}$ splits into m connected components $\mathbf{D}_1$, $\mathbf{D}_2$,..., $\mathbf{D}_m$, then we write $\mathbf{D} = \mathbf{D}_1 + \mathbf{D}_2 + ... + \mathbf{D}_m$. We say that a diagram is *trivial* if it has no edges. For instance, the following is the trivial diagram of rank 3

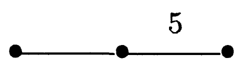
2.5.4 Coxeter diagrams

A diagram $\mathbf{D}$ over the type set I is called a *Coxeter diagram* if, for every choice of distinct types i, j, $\mathbf{D}_{i,j}$ is the class of generalized $m_{i,j}$-gons, for some $m_{i,j} = 2, 3, 4, ..., \infty$.

The following connected Coxeter diagrams of rank 3 are said to be of *spherical type.*

(A_3) 

(C_3) 

(H_3) 

Following Tits [15], we denote by $I_2(m)$ the diagram of rank 2 representing the class of generalized m-gons. We write A_2 for $I_2(3)$ and C_2 for $I_2(4)$. We denote the diagram of rank 1 by A_1. Thus, according to a convention stated in §2.5.3, $A_1 + A_1$ is a name for the class $I_2(2)$ of generalized digons, $A_1 + A_1 + A_1$ is the trivial diagram of rank 3 and $A_1 + I_2(m)$ is the following disconnected Coxeter diagram

$$\bullet \qquad \bullet\overset{(m)}{\rule[0.5ex]{3em}{0.4pt}}\bullet$$

The trivial diagram $A_1 + A_1 + A_1$ and the diagrams $A_1 + I_2(m)$ ($m < \infty$) are the disconnected Coxeter diagrams of rank 3 and *spherical type*. $A_1 + I_2(\infty)$ is the unique disconnected Coxeter diagram of rank 3 that is not of spehrical type.

2.6 Orders

A chamber system $\mathcal{C}$ admits *order* q at some type i if all panels of $\mathcal{C}$ of type i have size $q + 1$. If $\mathcal{C}$ admits the same order q at every type, then we say that it has *uniform order* q.

A chamber system $\mathcal{C}$ is said to be *thin* at a type i if it admits order 1 at i. It is *thin* if it has uniform order 1.

Similar definitions are stated for geometries.

2.7 Morphisms, Isomorphisms and Automorphisms

2.7.1 Automorphisms and morphisms of chamber systems

As we remarked in §2.1.4, a chamber is a coloured graph. An *automorphism* of a chamber system $\mathcal{C}$ is a colour-preserving automorphism of the coloured graph $\mathcal{C}$. We denote the automorphism group of $\mathcal{C}$ by $Aut(\mathcal{C})$. We say that $\mathcal{C}$ (a subgroup G of $Aut(\mathcal{C})$) is *transitive* if $Aut(\mathcal{C})$ (resp. G) is transitive on the set of chambers of $\mathcal{C}$.

A *morphism* (in particular, an *isomorphism*) of chamber systems over the same set of types is a colour-preserving morphism (isomorphism) of graphs.

2.7.2 Morphisms and automorphisms of geometries

Given two geometries Γ and Γ' over the same type set I with type functions t and t' respectively, a *morphism* from the geometry Γ to the geometry Γ' is a morphism of graphs $f : \Gamma \longrightarrow \Gamma'$ such that $t'f = t$ (that is, f *preserves types*).

Automorphisms of geometries can be defined without mentioning type functions at all. The type-partition Θ of a geometry Γ is uniquely determined by Γ, as we remarked in §2.2.1. Hence every automorphism f of the graph Γ induces a permutation τ_f of the classes of Θ. If τ_f is the identity, then we call f an *automorphism* of the geometry Γ. Otherwise f is called a *correlation* of Γ. We denote the automorphism group of the geometry Γ by $Aut(\Gamma)$.

Since $\Gamma(\mathcal{C}(\Gamma)) \cong \Gamma$ (see §2.3.2), a natural isomorphism exist between $Aut(\mathcal{C}(\Gamma))$ and $Aut(\Gamma)$.

2.8 The Functor $\mathcal{C}_I$

Given a finite nonempty set I, let $\mathbf{G}_I$ and $\mathbf{CS}_I$ be the categories of geometries and chamber systems respectively, over the type set I, with morphisms defined as in §2.7.2.

Given geometries Γ and Γ' over I, let $f : \Gamma \longrightarrow \Gamma'$ be a morphism. Since f preserves types, it maps chambers of Γ onto chambers of Γ' and it preserves i-adjacencies for every $i \in I$. Therefore it induces a morphism $\mathcal{C}(f)$ from $\mathcal{C}(\Gamma)$ to $\mathcal{C}(\Gamma')$.

The functor $\mathcal{C}_I : \mathbf{G}_I \longrightarrow \mathbf{CS}_I$ sending $\Gamma \in Obj(\mathbf{G}_I)$ to $\mathcal{C}(\Gamma) \in Obj(\mathbf{CS}_I)$ and $f \in Hom(\mathbf{G}_I)$ to $\mathcal{C}(f) \in Hom(\mathbf{CS}_I)$ is full and faithfull. However, it is not an equivalence of categories, since the class $\mathcal{C}_I(Obj(\mathbf{G}_I))$ of geometric chamber systems over I is a proper subclass of the class $Obj(\mathbf{CS}_I)$ of chamber systems over I.

However, $\mathcal{C}_I$ is an equivalence of categories between $\mathbf{G}_I$ and the category $\mathbf{GCS}_I$ of geometric chamber systems over I.

3 2-COVERS

3.1 Definition

Given two chamber systems $\mathcal{C}$ and $\mathcal{C}'$ of rank > 2 over the same set of types I and a morphism $f : \mathcal{C} \longrightarrow \mathcal{C}'$, we say that f is a *2-covering* if, for every cell X of $\mathcal{C}$ of rank 2, $f(X)$ is a cell of $\mathcal{C}'$ and f induces an isomorphism from the residue $\mathcal{C}_X$ of X in $\mathcal{C}$ to the residue $\mathcal{C}'_{f(X)}$ of $f(X)$ in $\mathcal{C}'$. If there is a 2-covering $f : \mathcal{C} \longrightarrow \mathcal{C}'$, then we say that $\mathcal{C}$ is a *2-cover* of $\mathcal{C}'$ and that $\mathcal{C}'$ is a *2-quotient* of $\mathcal{C}$. *2-coverings*, *2-covers* and *2-quotients* of geometries are defined in a similar way.

With the notation of §2.8, let $\mathbf{G}_{I,2}$ and $\mathbf{CS}_{I,2}$ be the subcategories of $\mathbf{G}_I$ and $\mathbf{CS}_{I,2}$ with the same objects as those categories but with 2-coverings as morphisms. The functor $\mathcal{C}_I$ induces a full and faithful functor $\mathcal{C}_{I,2}$ from $\mathbf{G}_{I,2}$ to $\mathbf{CS}_{I,2}$. If $\mathbf{GCS}_{I,2}$ is the category induced by $\mathbf{CS}_{I,2}$ on the class $Obj(\mathbf{GCS}_I)$ of geometric chamber systems over I, then $\mathcal{C}_{I,2}$ is an equivalence between the categories $\mathbf{G}_{I,2}$ and $\mathbf{GCS}_{I,2}$.

3.2 Universal 2-Covers

3.2.1 Universal 2-covers of chambers systems

Given a chamber system $\mathcal{C}$ of rank > 2 over a set of types I and a chamber system $\tilde{\mathcal{C}}$ over I, we say that $\tilde{\mathcal{C}}$ is the *universal 2-cover* of $\mathcal{C}$ if there is a 2-covering $f : \tilde{\mathcal{C}} \longrightarrow \mathcal{C}$ such that, for every 2-covering $g : \mathcal{C}' \longrightarrow \mathcal{C}$, there is just one 2-covering $h : \tilde{\mathcal{C}} \longrightarrow \mathcal{C}'$ with $hf = g$. A 2-covering $f : \tilde{\mathcal{C}} \longrightarrow \mathcal{C}$ as above is said to be *universal*.

Theorem 3.1 (Ronan [12]) *Every chamber system of rank $n > 2$ admits a universal 2-cover.*

It is clear that the universal 2-cover of a chamber system is uniquely determined up to isomorphisms. A chamber system of rank > 2 is said to be *2-simply connected* if it is its own universal 2-cover. It easily follows from the definition of universal 2-covers that the universal 2-cover of a chamber system is 2-simply connected. That is, a chamber system of rank > 2 is 2-simply connected if and only if it is the universal 2-cover of some chamber system.

3.2.2 Universal 2-covers and classification problems

Determining universal 2-covers is a crucial step in many classification problems. Aiming to a classification for some family $\mathbf{C}$ of chamber systems, we might organize our work in two stages:

(1) describe the universal 2-covers of the members of $\mathbf{C}$;

(2) investigate quotients of the objects determined at the previous stage.

For instance, the following celebrated theorem of Tits is the first step in stage (1) when $\mathbf{C}$ is a class of chamber systems with Coxeter diagrams.

Theorem 3.2 (Tits [15]) *Let $\mathcal{C}$ be a chamber system belonging to a Coxeter diagram $\mathbf{D}$ over a set of types I and assume that, for every subset J of I of size 3 such that the diagram $\mathbf{D}_J$ induced by $\mathbf{D}$ on J is spherical, every residue of $\mathcal{C}$ of type J is 2-covered by a building. Then the universal 2-cover of $\mathcal{C}$ is a building.*

The reader can see chapter 22 of [1] (sections 2 and 3) for a survey of classification theorems exploiting the above result. I am not going to insist on this matter here.

3.2.3 Universal 2-cover of geometries

Universal 2-covers and *2-simple connectedness* can be defined for geometries in the same way as for chambers systems. On the other hand, no analogue of Theorem 3.1 is known for geometries. We can only obtain the following from that theorem. Let $\widetilde{\mathcal{C}}$ be the universal 2-cover of the chamber system $\mathcal{C}(\Gamma)$ of a geometry Γ. If $\widetilde{\mathcal{C}}$ is geometric, then the universal 2-cover of Γ exists: the geometry $\Gamma(\widetilde{\mathcal{C}})$ of $\widetilde{\mathcal{C}}$ is indeed the universal 2-cover of Γ.

However, the above does not tell us that every geometry of rank > 2 admits a universal 2-cover. Proving that every geometry admits a universal 2-cover is almost the same as proving the following

Conjecture 1 *The universal 2-cover of any geometric chamber system is geometric.*

No proof has yet been found for Conjecture 1. On the other hand, no counterexample is known to it. Furthermore, some partial results have been achieved, proving that Conjecture 1 holds true in many important cases. For instance, buildings are geometric chamber systems. Therefore the universal 2-cover of a chamber system as in Theorem 3.2 is geometric. Namely, every geometry satifying the hypotheses of Theorem 3.2 admits a universal 2-cover.

The following propositions are special cases of a theorem on $(n-1)$-covers stated in in [10] (Theorem 12.39).

Proposition 3.3 *The universal 2-covers of any geometric chamber system of rank 3 is geometric.*

Proposition 3.4 *Let $\mathbf{C}$ be a geometric chamber system of rank $n \geq 4$. If all residues of $\mathbf{C}$ of rank $n-1$ are 2-simply connected, then the universal 2-cover of $\mathbf{C}$ is geometric.*

More results on universal 2-covers of geometric chamber systems will be given in §4.3.

Remark. Conjecture 1 is slightly stronger than the conjecture that every geometry admits a universal 2-cover. Indeed, even if every geometric chamber system which is universal in the category $\mathbf{CS}_{I,2}$ is universal in $\mathbf{GCS}_{I,2}$ too, it still might happen that some object of $\mathbf{GCS}_{I,2}$ is universal in $\mathbf{GCS}_{I,2}$ without being universal in $\mathbf{CS}_{I,2}$. That is, there might be 2-simply connected geometries whose chamber systems are not 2-simply connected. Actually, I do not believe this can happen. However, I do not know how to prove that it is impossible.

3.2.4 Non-geometric 2-simply connected chamber systems

There are 2-simply connected chamber systems that are not geometric. An example of this kind is given in [9], with trivial diagram of rank 3. It is finite, but not transitive. It is likely that many other finite examples like this exist (see §5.4 of this paper). I do not know if any of them might admit a transitive automorphism group.

Further examples, with diagram $A_1 + I_2(m)$, are mentioned by Tits in [15] (§6.1.5(b)). They are neither finite nor transitive.

The Wester chamber system ([17], [6]; also [8], 4.6) is a non-geometric 2-simply connected chamber system of rank 4, with affine diagram $\widetilde{B}_3$. It is finite and transitive. A few examples of higher rank 5 and 6 containing the Wester chamber system as a residue are described in [17] (see also [6]).

4 REDUCIBILITY

4.1 Truncations and Direct Products

4.1.1 Truncations of geometries

Given a geometry Γ over a set of types I and a proper nonempty subset J of I, the *truncation* of Γ over J (*J-truncation* of Γ, for short) is the geometry $tr_J(\Gamma)$ over the set of types J induced by Γ on the set of elements of Γ of type $j \in J$.

If $\mathcal{C} = (C, (\Phi_i)_{i \in I})$ is the chamber system of Γ, then the chamber system of $tr_J(\Gamma)$ can be recovered in $\mathcal{C}$ as follows: the quotient C/Φ_{I-J} corresponds to the set of chambers of $tr_J(\mathcal{C})$ and, for every $j \in J$, $\Phi_{(I-J)\cup\{j\}}/\Phi_{I-J}$ is the j-adjacency relation.

4.1.2 Truncations of chamber systems

The above construction can be done for any chamber system $\mathcal{C} = (C, (\Phi_i)_{i \in I})$. It gives us a chamber system provided that both the following hold:

(T1) $\Phi_{(I-J)\cup\{j\}} \cap \Phi_{(I-J)\cup\{k\}} = \Phi_{I-J}$ for any two distinct types $j, k \in J$;
(T2) all classes of $\Phi_{(I-J)\cup\{j\}}/\Phi_{I-J}$ have size ≥ 2, for all $j \in J$.

(Needless to say, both (T1) and (T2) hold if $\mathcal{C}$ is geometric.) If (T1) and (T2) hold, then the chamber system $(C/\Phi_{I-J}, (\Phi_{(I-J)\cup\{j\}}/\Phi_{I-J})_{j \in J})$ will be called the *truncation* of $\mathcal{C}$ over J (also *J-truncation* of $\mathcal{C}$, for short). We denote it by $tr_J(\mathcal{C})$.

4.1.3 Direct sums of geometries

Given two finite nonempty disjoint sets J and K, let Γ_1 and Γ_2 be two geometries over the sets of types J and K respectively, with no elements in common, the *direct sum* of Γ_1 and Γ_2 is the graph $\Gamma = \Gamma_1 \oplus \Gamma_2$ obtained by taking Γ_1 and Γ_2 together and joining every vertex of Γ_1 with every vertex of Γ_2 by a new edge. Γ is in fact a geometry over the set of types $I = J \cup K$. We have $\Gamma_1 \cong tr_J(\Gamma)$ and $\Gamma_2 \cong tr_K(\Gamma)$ and, for every flag F of Γ of type K (of type J), Γ_1 (resp., Γ_2) is isomorphic to Γ_F.

Let $\mathcal{C}$, $\mathcal{C}_1$, $\mathcal{C}_2$ be the chamber systems of Γ, Γ_1 and Γ_2 respectively. Then $\mathcal{C}_1$ and $\mathcal{C}_2$ are the truncations of $\mathcal{C}$ over J and K respectively and we have:

(R1) $\Phi_j\Phi_k = \Phi_k\Phi_j$ for all $j \in J$ and $k \in K$;
(R2) $\Phi_J \cap \Phi_K = \mho$.

(Note that (R1) just says that all residues of type $\{j, k\}$ with $j \in J$ and $k \in K$ are generalized digons.) Therefore, the set C of chambers of $\mathcal{C}$ can be identified with the direct product of the sets of chambers of $\mathcal{C}_1$ and $\mathcal{C}_2$, representing $x \in C$ as $([x]\Phi_K, [x]\Phi_J)$. For every $j \in J$, the j-adjacency relation Φ_j of $\mathcal{C}$ corresponds to the pair of equivalence relations $((\Phi_{K\cup\{j\}})/\Phi_K, \mho_2)$, where $\mho_2$ is the identity relation on the set of chambers of $\mathcal{C}_2$. The k-adjacency relations with $k \in K$ can be represented in a similar way. The fact that $tr_J(\Gamma) \cong \Gamma_F$ for every flag F of Γ of type K can now be rephrased as follows: we have $tr_J(\mathcal{C}) \cong \mathcal{C}_X$, for every cell X of $\mathcal{C}$ of type J.

4.1.4 Direct products of chamber systems

The above suggest the following definition. Let $\mathcal{C}_1 = (C_1, (\Psi_j)_{j\in J})$ and $\mathcal{C}_2 = (C_2, (\Psi_k)_{k\in K})$ be any two chamber systems over mutually disjoint sets of types J and K. We define the *direct product* $\mathcal{C} = \mathcal{C}_1 \times \mathcal{C}_2$ of $\mathcal{C}_1$ and $\mathcal{C}_2$ by taking $I = J \cup K$ as set of types, $C = C_1 \times C_2$ as set of chambers and the pairs $(\Psi_j, \mho_2)$, $(\mho_1, \Psi_k)$ as adjacency relations ($j \in J$, $k \in K$ and $\mho_i$ is the identity relation on C_i, $i = 1, 2$).

Trivially, (R1) and (R2) of §4.1.3 hold in $\mathcal{C}$ for the partition $\{J, K\}$ of I. Conditions (T1) and (T2) of §4.1.2 also hold, $tr_J(\mathcal{C}) \cong \mathcal{C}_1 \cong \mathcal{C}_X$ for every cell X of $\mathcal{C}$ of type J and $tr_K(\mathcal{C}) \cong \mathcal{C}_2 \cong \mathcal{C}_Y$ for every cell Y of $\mathcal{C}$ of type K.

$\mathcal{C}_1 \times \mathcal{C}_2$ is geometric if and only if both $\mathcal{C}_1$ and $\mathcal{C}_2$ are geometric. If this is the case, then $\Gamma(\mathcal{C}_1 \times \mathcal{C}_2) = \Gamma(\mathcal{C}_1) \oplus \Gamma(\mathcal{C}_2)$.

Conversely, let $\mathcal{C}$ be a chamber system over a set of types I and let $\{J, K\}$ be a partition of I in two disjoint nonempty subsets. Assume that (R1) and (R2) hold in $\mathcal{C}$ for the partition $\{J, K\}$. Then (T1) and (T2) also hold ([10], 12.5.2). Thus we can consider the truncations of $\mathcal{C}$ over J and K. By (R1) and (R2), we have $\mathcal{C} \cong tr_J(\mathcal{C}) \times tr_K(\mathcal{C})$.

Needless to say, the above can be generalized to define products of any finite number of chamber systems.

Remark. It is not difficult to find a category in which direct products of chamber systems are precisely product objects. On the other hand, I do not know of any sensible category where direct sums of geometries are coproduct or product objects.

4.2 Reducibility

4.2.1 Definition

A chamber system (a geometry) is said to be *reducible* if it splits as the direct product (the direct sum) of some of its truncations. Otherwise, it is called *irreducible.*

Clearly, every reducible chamber system $\mathcal{C}$ (geometry Γ) splits as the direct product (sum) of a finite number of irreducible chamber systems (geometries) and that splitting is unique modulo permutations of the factors (summands). The factors (summands) of that splitting are called the *irreducible components* of $\mathcal{C}$ (of Γ). If $\mathcal{C}$ (resp. Γ) is irreducible, then we say that it is its own unique *irreducible component.*

4.2.2 The Direct Sum Theorem for geometries

The structure of a reducible chamber system is completely determined by its irreducible components. Thus, in many contexts we can safely restrict our interest to irreducible cases. In the geometric case, the irreducible components are easily recognized from the diagram, as stated by the following well known theorem.

Theorem 4.1 (Direct Sum Theorem) *Given a diagram* $\mathbf{D}$*, let* Γ *be a geometry belonging to* $\mathbf{D}$*. Then the irreducible components of* Γ *are the truncations of* Γ *over the connected components of* $\mathbf{D}$*.*

The reader can find an easy proof of this theorem in [10] (chapter 4, §4.2).

4.2.3 Completely reducible chamber systems

Unfortunately, the statement of the Direct Sum Theorem fails to hold for non-geometric chamber systems. Many counterexamples are given in [8], Section 4 (also [9]). Many of them are finite and transitive.

The reason of that failure is soon explained. (R1) is the only information we can get from the disconnectedness of a diagram, but (R1) is not sufficient to obtain splittings in direct products. We also need (R2) for that. However, (R2) does not hold in non-geometric chamber systems, in general. Thus, a chamber system $\mathcal{C}$ is certainly irreducible if it belongs to a connected diagram, but the converse is not true in general.

We say that a chamber sytem $\mathcal{C}$ with disconnected diagram graph $\mathbf{D}$ is *completely reducible* if $\mathcal{C}$ admits truncations over every connected component of $\mathbf{D}$ and these truncations are the irreducible components of $\mathcal{C}$.

Clearly, if $\mathbf{D}$ has just two connected components, then reducibility and complete reducibility are the same property. Also, a chamber system of rank 3 with a trivial diagram is completely reducible if and only if it is reducible.

4.3 Covers of Direct Products

Let $\mathcal{C}$ be a chamber system of rank $n \geq 3$ with type set I, let $\{I_1, I_2, ..., I_m\}$ be a partition of I such that $\mathcal{C}$ admits truncation over each of I_1, I_2,..., I_m and let $\mathcal{C} = \prod_{i=1}^m \mathcal{C}_i$, where $\mathcal{C}_i$ is the I_i-truncation of $\mathcal{C}$, for $i = 1, 2, ..., m$. If $|I_i| \geq 3$, then $\tilde{\mathcal{C}}_i$ will denote the universal 2-cover of $\mathcal{C}_i$. Otherwise, we set $\tilde{\mathcal{C}}_i = \mathcal{C}_i$. Let $\tilde{\mathcal{C}}$ be the universal 2-cover of $\mathcal{C}$. Then the following holds ([10], 12.5.2):

Theorem 4.2 $\tilde{\mathcal{C}} = \prod_{i=1}^m \tilde{\mathcal{C}}_i$.

The next corollaries easily follow from this theorem:

Corollary 4.3 *Assume that, for every $i = 1, 2, ..., m$, either $\mathcal{C}_i$ is 2-simply connected or it has rank ≤ 2. Then $\mathcal{C}$ is 2-simply connected.*

Corollary 4.4 *If all truncations $\mathcal{C}_1$, $\mathcal{C}_2$,..., $\mathcal{C}_m$ have rank ≤ 2, then $\mathcal{C}$ is 2-simply connected.*

A direct product of geometric chamber systems is geometric. Hence by Theorem 4.2 we also get the following

Corollary 4.5 *Assume that for all $i = 1, 2, ..., m$ the chamber system $\tilde{\mathcal{C}}_i$ is geometric. Then $\tilde{\mathcal{C}}$ is geometric.*

Assume furthermore that $\mathcal{C}$ is geometric and that I_1, I_2,..., I_m are the connected components of a diagram $\mathbf{D}$ for $\mathcal{C}$. Then $\mathcal{C} = \prod_{i=1}^m \mathcal{C}_i$ by the Direct Sum Theorem and $\tilde{\mathcal{C}} = \prod_{i=1}^m \tilde{\mathcal{C}}_i$ by Theorem 4.2. By Proposition 3.3 and Corollary 4.5 and recalling that direct products of geometric chamber systems are geometric, we get the following

Corollary 4.6 *Assume the above and assume furthermore that $|I_i| \leq 3$ for every $i = 1, 2, ..., m$. Then $\tilde{\mathcal{C}}$ is geometric.*

4.4 The Reducibility Problem

Let's turn back to Theorem 3.2. Given a chanber system $\mathcal{C}$ belonging to a Coxeter diagram $\mathbf{D}$ of rank ≥ 3, assume we want to know if $\mathcal{C}$ can be obtained as a 2-quotient of a building. According to Theorem 3.2 we should check if all rank 3 residues of $\mathcal{C}$ belonging to subdiagrams of spherical type are 2-quotients of buildings. In particular, we should check this for disconnected subdiagrams. Buildings are geometries. Hence, by the Direct Sum Theorem, a chamber system belonging to a disconnected Coxeter diagram with all components of rank ≤ 2 is a building if and only if it is completely reducible. Thus, $\mathcal{C}$ is 2-covered by a building only if, for every triple of types J such that $\mathbf{D}$ induces a disconnected diagram on J, all residues of $\mathcal{C}$ of type J have reducible universal 2-covers.

However, irreducible chamber systems exist that belong to disconnected Coxeter diagrams of rank 3 (see [8], Section 4) and some of them are even 2-simply connected (see [9], for instance). It would be nice to get some control over this situation proving that pathological examples as those mentioned above are really exceptional; for instance, answering the following question in the negative:

Problem 1 *Is there any transitive* 2*-simply connected irreducible chamber system of rank* 3 *with disconnected Coxeter diagram ?*

In particular

Problem 2 *Is there any transitive* 2*-simply connected irreducible chamber system of rank* 3 *with trivial diagram ?*

Some partial results obtained in [8] (§5.1) seem to suggest that, if examples of that kind existed, they should have trivial diagrams. However, we are still from a solution of the above problems.

5 CELL-GEOMETRIES

5.1 Definition and Basic Properties

5.1.1 Cell-geometries and panel-spaces

Let $\mathcal{C}$ be a chamber system of rank $n > 1$ over the type set I. We can construct a geometry of rank n over the set of types $\{0, 1, 2, ..., n-1\}$ by taking as elements of type i the cells of $\mathcal{C}$ of rank i, and defining the incidence relation as follows: given two cells X, Y of type J and K respectively, we declare X and Y to be incident if either $X \subseteq Y$ and $J \subseteq K$ or $Y \subseteq X$ and $K \subseteq J$. It is easy to check that this is indeed a geometry. We call it the *cell-geometry* of $\mathcal{C}$, denoting it by $Gr_I(\mathcal{C})$.

The semilinear space $\Pi_{\mathcal{C}}$ considered in §2.1.5 is the $\{0, 1\}$-truncation of $Gr_I(\mathcal{C})$. We call it the *panel-space* of $\mathcal{C}$. As we remarked in §2.1.5, the panel-space of $\mathcal{C}$ uniquely determines $\mathcal{C}$. Hence $Gr_I(\mathcal{C})$ uniquely determines $\mathcal{C}$.

Note that, if $\mathcal{C}$ is geometric, then $Gr_I(\mathcal{C})$ is just the geometry of flags of $\Gamma = \Gamma(\mathcal{C})$, called the I-Grassmann geometry of Γ and denoted by $Gr_I(\Gamma)$ in [10]. The notation $Gr_I(\mathcal{C})$ is motivated by that.

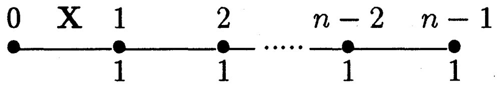
The residues of $Gr_I(\mathcal{C})$ of type $\{0, 1\}$ are the panel-spaces of the rank 2 residues of $\mathcal{C}$. They are linear spaces with even gonality (§2.4). Needless to say, the panel-space and the cell-geometry of a chamber system of rank 2 are the same thing.

Remark. Scharlau [14] has developed a general theory of "shadow geometries" of chamber systems (I should call them "Grassmann geometries", to be consistent with [10], Chapter 5). Cell-geometries as defined above are in fact examples of shadow geometries as in [14].

5.1.2 Diagram and orders of a cell-geometry

It is not difficult to prove that $Gr_I(\mathcal{C})$ belongs to the following diagram, where the label **X** on the first stroke of the diagram denotes a class of semilinear spaces with even gonality containing the panel-spaces of the rank 2 residues of $\mathcal{C}$

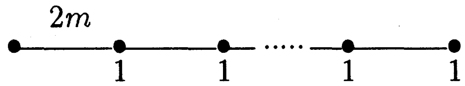
$(X.A_{n-1})$



The integers 0, 1,..., $n-1$ above the nodes of the diagram are the types. The number 1 below them is an order. That is, $Gr_I(\mathcal{C})$ is thin at all types $i > 0$.

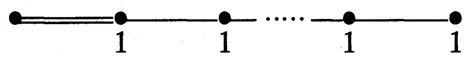
For instance, if $\mathcal{C}$ belongs to a Coxeter diagram with all edges labelled by m, then $Gr_I(\mathcal{C})$ belongs to the following Coxeter diagram:

$(I_2(2m).A_{n-1})$



In particular, if the diagram of $\mathcal{C}$ is trivial, then $Gr_I(\mathcal{C})$ is the dual of a thin-lined C_n-geometry

(C_n)



5.1.3 Interlude: parallelisms in geometries

Let Γ be a geometry of rank ≥ 2 and let 0 be one of the types of Γ. According to [2], a 0-*parallelism* of Γ is an equivalence relation $\|$ on the set of elements of Γ of type $\neq 0$ such that

(1) if $x \parallel y$, then x and y have the same type;

(2) given any two elements a, b of Γ of type 0 and elements $x, y \in \Gamma_a$ and $u, v \in \Gamma_b$, if $x \parallel u$, $y \parallel v$ and x is incident to y, then u is incident to v;

(3) for every element a of type 0 and every element x not of type 0, there is just one element y of Γ_a such that $y \parallel x$.

Let Γ admit a 0-parallelism. It easily follows from (1)-(3) that the residues of the elements of Γ of type 0 are mutually isomorphic. Any geometry isomorphic to them can be taken as the *geometry at infinity* of Γ (*line at infinity* of Γ when Γ has rank 2).

5.1.4 A parallelism in $Gr_I(\mathcal{C})$

Let now $\Gamma = Gr_I(\mathcal{C})$ for some chamber system $\mathcal{C}$ of rank $n \geq 2$ over the type set I. It is straightforward to check that the relation "having the same type" between cells of $\mathcal{C}$ is a 0-parallelism of $Gr_I(\mathcal{C})$. Let us denote it by $\|_C$. As geometry at infinity we take the geometry $\mathcal{P}(I)$ of the proper nonempty subsets of I.

The 0-parallelism $\|_C$ of $Gr_I(\mathcal{C})$ induces on the panel-space Π_C the parallelism considered in §2.1.5. If we take I as the line at infinity of Π_C, then $Gr_I(\mathcal{C})$ is just the parallel expansion of $\mathcal{P}(I)$ in Π_C, in the meaning of [2].

5.2 A Characterization of Cell-Geometries

5.2.1 Some terminology

Let Γ be a geometry of rank $n > 1$ belonging to the diagram $X.A_{n-1}$ of §5.1.2. The elements of Γ of type $n-1$, $n-2$ and $n-3$ (if $n \geq 3$) will be called *points*, *lines* and *planes* respectively, as if we were reading the diagram from right to left. We say that two points of Γ are *collinear* if they are incident to the same line. If $n \geq 3$, then we say that three points are *coplanar* if they are incident to the same plane.

The *collinearity graph* $\mathcal{G}(\Gamma)$ of Γ is the graph having the points of Γ as vertices and the collinearity relation as the adjacency relation. Note that, if Γ is the cell-geometry of a chamber system $\mathcal{C}$, then its collinearity graph is just the incidence graph $\Gamma(\mathcal{C})$ of $\mathcal{C}$.

If every line of Γ is incident to precisely two points (that is, Γ is thin at the type $n-1$), then we say that Γ is *thin-lined.* The diagram $X.A_{n-1}$ is such that, if Γ is thin-lined, then it is thin at all nodes types except possibly 0.

5.2.2 The characterization theorem

Let $\mathcal{C}$ be a chamber system of rank n over a set of types I. The 0-parallelism $\|_C$ of $Gr_I(\mathcal{C})$ (see §5.1.4) induces an n-partition on the collinearity graph of $Gr_I(\mathcal{C})$, which is in fact the type-partition of the geometry $\Gamma(\mathcal{C})$. This property characterizes cell-geometries of chamber systems.

Theorem 5.1 *Let Γ be a geometry of rank $n \geq 2$ belonging to the diagram $X.A_{n-1}$ of §5.1.2, where* **X** *denotes a class of semilinear spaces with even gonality. Then the following are equivalent:*

(i) Γ is the cell-geometry of a chamber system;

(ii) the collinearity graph of Γ is n-partite and Γ is thn-lined.

This theorem is proved in [9] (Theorem 4.1). The proof is quite easy. We can explain it in a few words. Let Γ be thin-lined. Then an n-partition of $\mathcal{G}(\Gamma)$ uniquely determines a 0-parallelism $\|$ on Γ. The $\{0,1\}$-truncation of Γ endowed with the parallelism inherited from $||$ satifies (C'1)-(C'2) of §2.1.5. Hence it uniquely determines a chamber system $\mathcal{C}$ and Γ is just the cell-geometry of $\mathcal{C}$.

5.3 Universal Covers of Cell-Geometries

Using Theorem 5.1 and some results on Grassmann geometries from [7], the following can be proved (see [9], Corollary 4.2).

Lemma 5.2 *Let Γ be as in Theorem* 5.1, *with* $\mathbf{X}$ *denoting a class of panel-spaces of chamber systems of rank* 2. *Assume furthermore that Γ has rank $n > 2$ and that it is* 2*-simply connected. Then Γ is the cell-geometry of a* 2*-simply connected chamber system.*

The next theorem is a straightforward consequence of this lemma.

Theorem 5.3 *Given a chamber system $\mathcal{C}$ of rank $n > 2$ with type set I, let $\tilde{\mathcal{C}}$ be the universal* 2*-cover of $\mathcal{C}$. Then $Gr_I(\tilde{\mathcal{C}})$ is the universal* 2*-cover of $Gr_I(\mathcal{C})$.*

5.4 Cell-Geometries of Geometric Chamber Systems

5.4.1 The properties (LL), (IP), (TP) and (CP)

Let Γ belong to the diagram $X.A_{n-1}$ of §5.1.2, with $n \geq 3$. According to §5.2.1, the elements of type $n-1$, $n-2$, $n-3$ of Γ are called points, lines and planes respectively. We say that Γ *has a good system of lines* if its $\{n-2, n-1\}$-truncation is a semilinear space. This is the property usually called (LL) in the literature.

We say that (LL) *residually holds* in Γ if Γ_F has a good system of lines for every flag F of Γ of type $\{m, m+1, ..., n-1\}$, for every $m = 3, 4, ..., n$ (with the convention that $F = \emptyset$ if $m = n$; note that $\Gamma_\emptyset = \Gamma$).

By a theorem of [10] (Theorem 7.25), (LL) residually holds in Γ if and only if Γ satisfies the Intersection Property (IP) (the reader is referred to Chapter 6 of [10] for the statement and an analysis of this property).

We say that Γ satisfies the *Triangular Property* (TP) if any three mutually collinear points of Γ are coplanar in Γ. If for any set of pairwise collinear points of Γ there is an element of Γ incident to all of them, then we say that Γ satisfies the *Clique Property* (CP).

Assume that the Intersection Property (IP) holds in Γ (that is, (LL) residually holds in Γ). It is not difficult to prove that Γ satisfies the Clique Property (CP) if and only if the Triangular Property (TP) *residually holds* in Γ, that is (TP) holds in Γ_F for every flag F of Γ of type $\{m, m+1, ..., n-1\}$, for every $m = 3, 4, ..., n$ (with the convention that $F = \emptyset$ if $m = n$).

Note that, if (IP) holds in Γ and Γ is thin-lined, then (CP) says that the elements of Γ of type $i > 0$ are just the i-cliques of the collinearity graph of Γ.

5.4.2 A characterization of geometric chamber systems

We can now characterize geometric chamber systems by properties of their cell geometries.

Theorem 5.4 *Let $\mathcal{C}$ be a chamber system of rank $n \geq 3$. Then $\mathcal{C}$ is geometric if and only if both (LL) and (TP) residually hold in its cell-geometry.*

Sketch of the Proof. Property (LL) residually holds in $Gr_I(\mathcal{C})$ if and only if (G1) of §2.3 holds in $\mathcal{C}$. The Clique Property (CP) holds in $Gr_I(\mathcal{C})$ if and only if (G2) holds in $\mathcal{C}$. Moreover, (LL) residually holds in $Gr_I(\mathcal{C})$ if and only if $Gr_I(\mathcal{C})$ satisfies (IP). On the other hand, if (IP) holds in $Gr_I(\mathcal{C})$, then $Gr_I(\mathcal{C})$ satisfies (CP) if and only if (TP) residually holds in it. Hence $\mathcal{C}$ is geometric if and only of both (LL) and (TP) residualy hold in $Gr_I(\mathcal{C})$. □

Corollary 5.5 *A chamber system of rank* 3 *is geometric if and only if both (LL) and (TP) hold in its cell-geometry.*

(This is just a special case of the previous theorem.)

5.5 Back to Conjecture 1

By theorems 5.4 and 5.5, proving Conjecture 1 of §3.2.3 is the same as proving the following.

Conjecture 2 *Let Γ be as in Theorem* 5.1 *with $n > 2$ and let $\widetilde{\Gamma}$ be a* 2*-cover of Γ. Assume that both (LL) and (TP) residually hold in Γ. Then the same is true in $\widetilde{\Gamma}$.*

A proof of this conjecture is fairly easy in the rank 3 case. Thus we obtain a "geometric" proof of Proposition 3.3. Actually, this also shows that any 2-cover of a geometric chamber system of rank 3 is geometric (compare [10], Lemma 12.37). Conjecture 2 can also be proved in the following case, which includes the rank 3 case: a 2-covering $f : \widetilde{\Gamma} \longrightarrow \Gamma$ is given such that, for every point p of $\widetilde{\Gamma}$, an isomorphism from $\widetilde{\Gamma}_p$ to $\Gamma_{f(p)}$ is induced by f. Thus, we get Proposition 3.4.

5.6 Some Special Cases

Henceforth Γ is a thin-lined geometry belonging to the diagram $I_2(2m).A_{n-1}$ of §5.1.2, with $n \geq 3$.

5.6.1 The case where $m \geq 3$

Exploiting some results on Grassmann geometries from [7] and Theorem 5.1, the following has been proved in [9] (Corollary 4.3).

Proposition 5.6 *Let Γ be as above, with $m \geq 3$. Assume furthermore that Γ is 2-simply connected. Then Γ is the cell-geometry of a building belonging to a Coxeter diagram of rank n with all strokes labelled by m.*

5.6.2 Thin-lined C_n-geometries

When $m = 2$, $I_2(2m).A_{n-1}$ is the spherical diagram C_n (see §5.1.2) and Γ is called a (thin-lined) *C_n-geometry.* By generalizing an argument used by S. Rees in [11] for thin-lined C_3-geometries, it is possible to prove that the collinearity graph of a thin-lined C_n geometry is n-partite ([9], Lemma 5.1). Hence, by Theorem 5.1 we get the following:

Theorem 5.7 *Every thin-lined C_n geometry is the cell-geometry of a chamber system with trivial diagram.*

Therefore, and since the cell-geometry of a chamber system of rank n with trivial diagram is a thin-lined C_n-geometry, there is an obvious equivalence between the category of thin-lined C_n-geometries and the category of chamber systems of rank n with trivial diagram, with 2-coverings as morhisms in both of these categories.

By Theorem 5.6, a chamber system $\mathcal{C}$ of rank n with trivial diagram is geometric if and anly if both (LL) and (TP) residually hold in the corresponding thin-lined C_n-geometry. Trivally, a chamber system with trivial diagram is geometric if and only if it is completely reducible. On the other hand, a C_n-geometry is a polar space if an only if it satisfies (LL) residually ([10], Chapter 7, 7.4). Furthermore (TP) residually holds in every polar space ([10], Lemma 7.36). Therefore, finding a non completely reducible but 2-simply connected chamber system with trivial diagram is the same as finding a 2-simply connected thin-lined C_n-geometry that is not a polar space.

Actually, there is at least one thin-lined C_3 geometry with these properties, as it is shown in [9]. Hence there is at least one chamber systems with trivial diagram that is 2-simply connected but not completely reducible. However, the automorphism group of that C_3-geometry is not transitive on the set of planes of the geometry. Namely, the corresponding chamber system is not transitive.

We can rephrase Problem 2 of §4.4 as follows:

Problem 3 *Let Γ be a 2-simply connected thin-lined C_3-geometry. Is it possible that $Aut(\Gamma)$ is transitive on the set of planes of Γ without Γ being a polar space ?*

By a result of S. Rees [11], this is in fact a problem on certain systems of latin squares.

6 COVERS AND AMALGAMS

6.1 Parabolic Systems

6.1.1 From chamber systems to parabolic systems

Given a transitive chamber system $\mathcal{C} = (C, (\Phi_i)_{i \in I})$, let G be a transitive subgroup of $Aut(\mathcal{C})$. Given a chamber $c \in C$, let B be the stabilizer of c in G and, for every $i \in I$, let P_i be the stabilizer in G of the panel $[c]\Phi_i$. The following hold:

(P1) $G = \langle P_i \rangle_{i \in I}$;
(P2) $P_i \cap P_j = B$ for any two distinct types $i, j \in I$;
(P3) $B \neq P_i$ for all $i \in I$;
(P4) $\bigcap_{g \in G} B^g = 1$.

Property (P1) follows from (C1) of §2.1.2 and from the transitivity of G. Properties (P2) and (P3) respectively correspond to (C2) and (C3) of §2.1.2. Property (P4) holds because G, being an automorphism group of $\mathcal{C}$, acts faithfully on the set of chambers of $\mathcal{C}$.

We denote the family $(P_i)_{i \in I}$ by $\mathcal{P}_c(G, \mathcal{C})$ and we call it the *parabolic system* defined by $\mathcal{C}$ in G at c. Note that, if d is another chamber of $\mathcal{C}$, then $\mathcal{P}_d(G, \mathcal{C})$ and $\mathcal{P}_c(G, \mathcal{G})$ are conjugated in G. Thus, as far as we are interested in $\mathcal{P}_c(G, \mathcal{C})$ modulo conjugation, we can write $\mathcal{P}(G, \mathcal{C})$ for $\mathcal{P}_c(G, \mathcal{G})$, dropping the subscript c.

Let us state some more notation, to be used later. Given $J \subseteq I$, we set $P_J = \langle P_j \rangle_{j \in J}$ for short, with the convention that $P_\emptyset = B$. Thus, P_J is the stabilizer in G of the cell $[c]\Phi_J$. In particular, $P_I = G$ (see (P1)). We also write $P_{i,j}$ for $P_{\{i,j\}}$.

Remark. The expression "parabolic system" is currently used in a rather more restrictive meaning in the literature, assuming that $\mathcal{C}$ belongs to a Coxeter diagram **D** and that, for any two types i, j joined in **D**, the residues of $\mathcal{C}$ of type $\{i, j\}$ are classical finite thick generalized polygons and $P_{i,j}$ acts on the cells it stabilizes as a Lie type group appropriate to that cell, with a few exceptions. Somebody uses the expression "amalgam" to mean what I have called a parabolic system. I find this a bit misleading: it reminds me of amalgamated products, which are related with the simply connected case (see §6.2). All considering, I prefer to give the expression "parabolic system" the broad meaning I have stated above. After all, the subgroups P_J of G are usually called *parabolic subgroups* of G in the literature, thus ...

6.1.2 From parabolic systems to chamber systems

Conversely, let B and $\mathcal{P} = (P_i)_{i\in I}$ be a subgroup of a group G and a finite family of subgroups of G satisfying properties (P1)-(P4) of the previous paragraph. We call $\mathcal{P}$ a *parabolic system* in G, of *rank* $n = |I|$.

We can construct a chamber system $\mathcal{C}(\mathcal{P})$ as follows. Take the right cosets in G of B as chambers and for every $i \in I$ define the i-adjacency relation Φ_i by declaring that fB and gB are i-adjacent when $g^{-1}f \in P_i$, for $f, g \in G$.

The group G, acting on the right cosets of B by left multiplication, is a transitive subgroup of $Aut(\mathcal{C}(\mathcal{P}))$ and we have $\mathcal{P}(G, \mathcal{C}(\mathcal{P})) = \mathcal{P}$.

On the other hand, if $\mathcal{C}$ is a transitive chamber system and G is a transitive subgroup of $Aut(\mathcal{C})$, then $\mathcal{C}(\mathcal{P}(G, \mathcal{C})) \cong \mathcal{C}$.

Thus, transitive chamber systems and parabolic systems are basically the same things. Properties (G1) and (G2) of §2.3, (T1) and (T2) of §4.1.2 and (R1) and (R2) of §4.1.3 can easily be translated into the language of parabolic systems: just substitute the letter P for the letter Φ everywhere in those properties.

6.2 Universal Covers and Amalgamated Products

Let $\mathcal{P} = (P_i)_{i\in I}$ be a parabolic system in a group G and let $\tilde{G}$ be the amalgamated product of the subgroups $P_{i,j}$ $(i, j \in I, i \neq j)$, with amalgamation over the subgroups P_i $(i \in I)$.

For every $i \in I$, the subgroup P_i of G lifts to a subgroup $\tilde{P}_i$ of $\tilde{G}$ and $\tilde{\mathcal{P}} = (\tilde{P}_i)_{i\in I}$ is a parabolic system in $\tilde{G}$. We call it the *universal* 2-*amalgam* of $\mathcal{P}$.

Theorem 6.1 *The chamber system $\mathcal{C}(\tilde{\mathcal{P}})$ of the universal 2-amalgam $\tilde{\mathcal{P}}$ of $\mathcal{P}$ is the universal 2-cover of the chamber system $\mathcal{C}(\mathcal{P})$ of $\mathcal{P}$.*

(Tits [16]; also [10], Theorem 12.28).

6.3 Revisiting Conjecture 1 and Problems 1 and 2

By Theorem 6.1, Conjecture 1 of §3.2.3 can be rephrased as follows for transitive chamber systems:

Conjecture 3 *Let $\mathcal{P} = (P_i)_{i\in I}$ be a parabolic system satisfying the following:*

$(G'1)$ $P_J = \bigcap_{j\notin J} P_{I-\{j\}}$ for every $J \subseteq I$;

$(G'2)$ $P_J \cap (P_{I-\{i\}} P_{I-\{j\}}) = (P_J \cap P_{I-\{i\}})(P_J \cap P_{I-\{j\}})$ for any two distinct types $i, j \in I$ and every subset J of I containing both i and j.

Then these properties also hold in the universal 2-amalgam of $\mathcal{P}$.

Since the universal 2-cover of a geometric chamber system of rank 3 is geometric (Proposition 3.3), the above conjecture holds true when $|I|= 3$. Problems 1 and 2 of §4.4 sound as follows:

Problem 4 *Let (P_1, P_2, P_3) be a parabolic system of rank 3 in a group G and assume that $P_1P_i = P_iP_1$ for $i = 2,3$. Let $\tilde{G}$ be the amalgamated product of the subgroups $P_{1,2}$, $P_{2,3}$ and $P_{3,1}$ with amalgamation over the subgroups P_1, P_2, P_3. Is it possible that $\tilde{G}$ is not embeddable into $P_1 \times P_{2,3}$?*

Problem 5 *Let (P_1, P_2, P_3) be a parabolic system of rank 3 in a group G and assume that $P_iP_j = P_jP_i$ for $i,j = 1,2,3$. Let $\tilde{G}$ be the amalgamated product of the subgroups $P_{1,2}$, $P_{2,3}$ and $P_{3,1}$ with amalgamation over the subgroups P_1, P_2, P_3. Is it possible that $\tilde{G}$ is not embeddable into $P_1 \times P_2 \times P_3$?*

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