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Y_{555} and related topics

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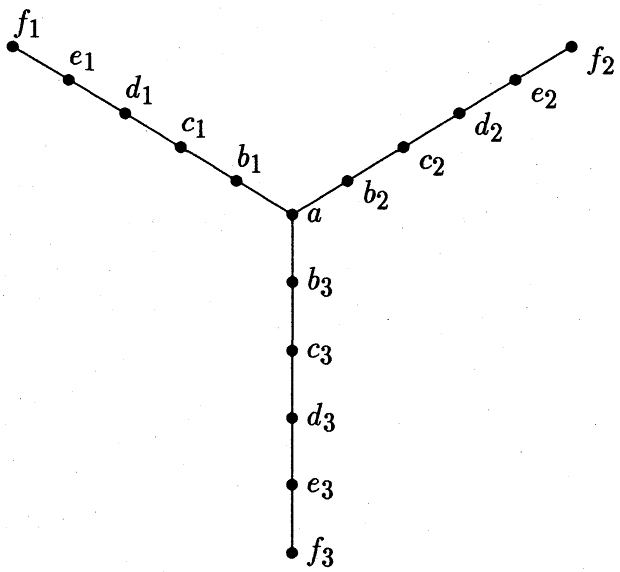
In this note, we will introduce some idea to study the Y_{555} group given in the paper

[CP] J.H.Conway and A.D.Pritchard : Hyperbolic reflections for the Bimonster and $3Fi_{24}$ in "Groups, Combinatorics and Geometry", Cambridge, 1992

and will report some observations together with some questions.

We denote the Monster simple group by $\mathbb{M}$, and the wreath product $\mathbb{M} \wr 2$ is called the Bimonster. The following diagram is called Y_{555}, and is regarded as a Coxeter diagram which gives 16 generators and some relations among them.



First we will collect some theorems on the presentation of the Bimonster.

Theorem A.

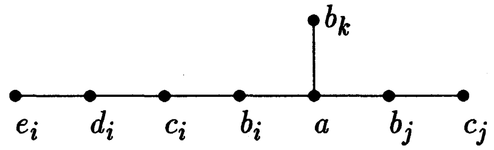
$$\mathbb{M} \wr 2 \cong < Y_{555}, (ab_1c_1ab_2c_2ab_3c_3)^{10} = 1 >$$

Theorem B. Suppose that the group G is a minimal group that possesses an S_5-subgroup S whose centralizer is isomorphic to S_{12} in which a 7 point stabilizer is conjugate to S.
Then $G \cong S_{17}$ or the Bimonster $\mathbb{M} \wr 2$.

Theorem C.

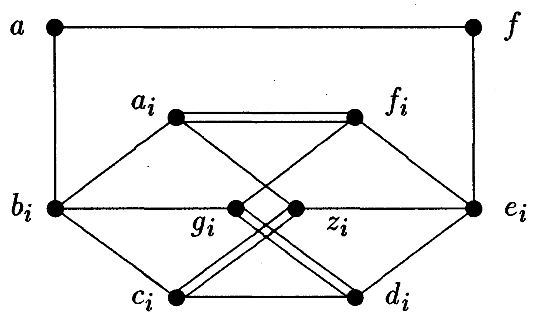
$$\mathbb{M} \wr 2 \cong < Y_{555}, f_i = f_{ij}(i, j = 1, 2, 3, i \neq j) >$$
$$f_{ij} = (ab_ic_id_ib_jc_jb_k)^9, \{i, j, k\} = \{1, 2, 3\}$$

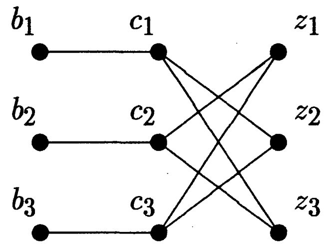
Remark. f_{ij} corresponds with the root of the highest height of the E_8-lattice:



We will call such a relation an E_8-*relation.*

Theorem D.(The 26 node theorem)
The bimonster $\mathbb{M} \wr 2$ contains 26 involutions, including the generators in Y_{555}, satisfying the Coxeter relations of the incidence graph of the projective plane of order 3.





In [CP], Conway and Pritchard defined the Monster roots, which are some vectors defined in the 16 dimensional space with the 19 coordinates

$$v = \begin{pmatrix} a & b & c & d & e & f & \\ g & h & i & j & k & l & t \\ m & n & o & p & q & r & \end{pmatrix}$$

with the quadratic form

$$a^2 + b^2 + ... + q^2 + r^2 - t^2$$

and the 3 relations

$$\begin{cases} a + b + c + d + e + f = t \\ g + h + i + j + k + l = t \\ m + n + o + p + q + r = t. \end{cases}$$

For a vector x, the reflection r_x is

$$r_x : y \longrightarrow y - \frac{2 < y, x >}{< x, x >} x,$$

where $< \cdot, \cdot >$ is the inner product.

The *fundamental Monster roots* are the vectors

$$a, b_i, c_i, d_i, e_i, f_i \ (i = 1, 2, 3)$$

given in Table 1. (In general, the term 'root' means a vector of squared length 2.) We denote by Π the set of the fundamental Monster roots. The reflections r_x $(x \in \Pi)$ satisfy the relation given by the diagram Y_{555}.

The (infinite) group G is defined by

$$G = < r_x \mid x \in \Pi > .$$

Then by Theorem A, there exists some normal subgroup N of G such that G/N is isomorphic to the bimonster $\mathbb{M} \wr 2$.

The *Monster roots* are the vectors in the G-orbit Π^G. We will define an equivalence relation between the Monster roots.

Definition.

$$x \doteq y \ (x, y \in \Pi^G) \Longleftrightarrow \bar{r}_x = \bar{r}_y \in G/N \cong \mathbb{M} \wr 2$$

(Table 1) The fundamental Monster roots and the 26 nodes

	$i=1$	$i=2$	$i=3$
a		$\begin{pmatrix} 0&0&0&0&0&1& \\ 0&0&0&0&0&1&1 \\ 0&0&0&0&0&1& \end{pmatrix}$	
b_i	$\begin{pmatrix} 0&0&0&0&1&\bar{1}& \\ 0&0&0&0&0&0&0 \\ 0&0&0&0&0&0& \end{pmatrix}$	$\begin{pmatrix} 0&0&0&0&0&0& \\ 0&0&0&0&1&\bar{1}&0 \\ 0&0&0&0&0&0& \end{pmatrix}$	$\begin{pmatrix} 0&0&0&0&0&0& \\ 0&0&0&0&0&0&0 \\ 0&0&0&0&1&\bar{1}& \end{pmatrix}$
c_i	$\begin{pmatrix} 0&0&0&1&\bar{1}&0& \\ 0&0&0&0&0&0&0 \\ 0&0&0&0&0&0& \end{pmatrix}$	$\begin{pmatrix} 0&0&0&0&0&0& \\ 0&0&0&1&\bar{1}&0&0 \\ 0&0&0&0&0&0& \end{pmatrix}$	$\begin{pmatrix} 0&0&0&0&0&0& \\ 0&0&0&0&0&0&0 \\ 0&0&0&1&\bar{1}&0& \end{pmatrix}$
d_i	$\begin{pmatrix} 0&0&1&\bar{1}&0&0& \\ 0&0&0&0&0&0&0 \\ 0&0&0&0&0&0& \end{pmatrix}$	$\begin{pmatrix} 0&0&0&0&0&0& \\ 0&0&1&\bar{1}&0&0&0 \\ 0&0&0&0&0&0& \end{pmatrix}$	$\begin{pmatrix} 0&0&0&0&0&0& \\ 0&0&0&0&0&0&0 \\ 0&0&1&\bar{1}&0&0& \end{pmatrix}$
e_i	$\begin{pmatrix} 0&1&\bar{1}&0&0&0& \\ 0&0&0&0&0&0&0 \\ 0&0&0&0&0&0& \end{pmatrix}$	$\begin{pmatrix} 0&0&0&0&0&0& \\ 0&1&\bar{1}&0&0&0&0 \\ 0&0&0&0&0&0& \end{pmatrix}$	$\begin{pmatrix} 0&0&0&0&0&0& \\ 0&0&0&0&0&0&0 \\ 0&1&\bar{1}&0&0&0& \end{pmatrix}$
f_i	$\begin{pmatrix} 1&\bar{1}&0&0&0&0& \\ 0&0&0&0&0&0&0 \\ 0&0&0&0&0&0& \end{pmatrix}$	$\begin{pmatrix} 0&0&0&0&0&0& \\ 1&\bar{1}&0&0&0&0&0 \\ 0&0&0&0&0&0& \end{pmatrix}$	$\begin{pmatrix} 0&0&0&0&0&0& \\ 0&0&0&0&0&0&0 \\ 1&\bar{1}&0&0&0&0& \end{pmatrix}$
a_i	$\begin{pmatrix} 0&0&0&0&0&1& \\ 1&0&0&0&0&0&1 \\ 1&0&0&0&0&0& \end{pmatrix}$	$\begin{pmatrix} 1&0&0&0&0&0& \\ 0&0&0&0&0&1&1 \\ 1&0&0&0&0&0& \end{pmatrix}$	$\begin{pmatrix} 1&0&0&0&0&0& \\ 1&0&0&0&0&0&1 \\ 0&0&0&0&0&1& \end{pmatrix}$
z_i	$\begin{pmatrix} 1&1&0&0&0&0& \\ 0&0&0&0&1&1&2 \\ 0&0&0&0&1&1& \end{pmatrix}$	$\begin{pmatrix} 0&0&0&0&1&1& \\ 1&1&0&0&0&0&2 \\ 0&0&0&0&1&1& \end{pmatrix}$	$\begin{pmatrix} 0&0&0&0&1&1& \\ 0&0&0&0&1&1&2 \\ 1&1&0&0&0&0& \end{pmatrix}$
g_i	$\begin{pmatrix} 2&0&0&0&0&1& \\ 0&0&0&1&1&1&3 \\ 0&0&0&1&1&1& \end{pmatrix}$	$\begin{pmatrix} 0&0&0&1&1&1& \\ 2&0&0&0&0&1&3 \\ 0&0&0&1&1&1& \end{pmatrix}$	$\begin{pmatrix} 0&0&0&1&1&1& \\ 0&0&0&1&1&1&3 \\ 2&0&0&0&0&1& \end{pmatrix}$
f		$\begin{pmatrix} 1&1&0&0&0&0& \\ 1&1&0&0&0&0&2 \\ 1&1&0&0&0&0& \end{pmatrix}$	

The following equivalence is a key of [CP].

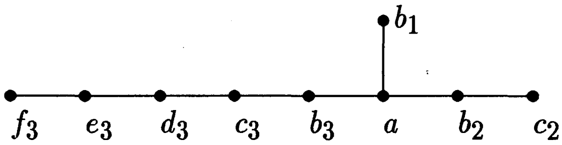
Proposition 1. (Theorem 2 of [CP])

$$v=\begin{pmatrix} 0&0&0&0&2&4& \\ 0&0&0&2&2&2&6 \\ 1&1&1&1&1&1& \end{pmatrix} \doteq \begin{pmatrix} 0&0&0&0&1&\bar{1}& \\ 0&0&0&0&0&0&0 \\ 0&0&0&0&0&0& \end{pmatrix} = b_1$$

Notice that the sum of the vectors

$$v+b_1=\begin{pmatrix} 0&0&0&0&3&3& \\ 0&0&0&2&2&2&6 \\ 1&1&1&1&1&1& \end{pmatrix} = f_3+2e_3+3d_3+4c_3+5b_3+6a+4b_2+2c_2+3b_1,$$

and this equals the primitive isotropic vector of the following $\tilde{E}_8$-lattice.



Hence Proposition 1 is equivalent to

$$\begin{pmatrix} 0&0&0&0&3&3& \\ 0&0&0&2&2&2&6 \\ 0&2&1&1&1&1& \end{pmatrix} \doteq \begin{pmatrix} 0&0&0&0&0&0& \\ 0&0&0&0&0&0&0 \\ 1&\bar{1}&0&0&0&0& \end{pmatrix}.$$

This means $f_{32}=f_3$, one of the E_8-relations in Theorem C.

Now a simple question is *"What does happen on an E_6- or E_7-diagram in Y_{555} ?"*

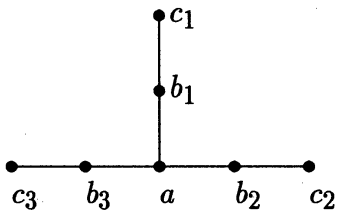
The following equivalence (we call it an *E_6-relation*) is proved in [CP].

Proposition 2.(Theorem 4 of [CP])

$$\begin{pmatrix} 0&0&0&1&2&2& \\ 0&0&0&1&2&2&5 \\ 0&0&0&1&2&2& \end{pmatrix} \doteq \begin{pmatrix} 0&0&0&2&1&1& \\ 0&0&0&2&1&1&4 \\ 0&0&0&2&1&1& \end{pmatrix}$$

The sum equals $3\times$(the primitive isotropic vector of $\tilde{E}_6$-lattice).

$$\begin{pmatrix} 0&0&0&3&3&3& \\ 0&0&0&3&3&3&9 \\ 0&0&0&3&3&3& \end{pmatrix}$$



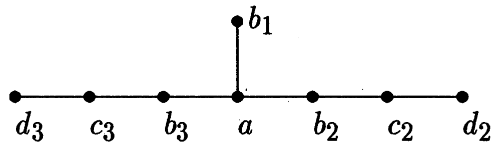
Similarly the following "*E_7-relation*" can be proved by using the relations in [CP].

Proposition 3.

$$\begin{pmatrix} 0&0&0&0&2&2& \\ 0&0&1&1&1&1&4 \\ 0&0&0&2&1&1& \end{pmatrix} \doteq \begin{pmatrix} 0&0&0&0&2&2& \\ 0&0&1&1&1&1&4 \\ 0&0&2&0&1&1& \end{pmatrix}$$

The sum equals $2\times$(the primitive isotropic vector of $\tilde{E}_7$-lattice).

$$\begin{pmatrix} 0&0&0&0&4&4& \\ 0&0&2&2&2&2&8 \\ 0&0&2&2&2&2& \end{pmatrix}$$



We will explain the above relations by using the orders of the products of some reflections.

Let $\tilde{X} = X \cup \{e\}$ be a subset of Π such that the diagram of $\tilde{X}$ is an affine diagram $\tilde{E}_n, n = 6, 7$ or 8, and the diagram of X is a spherical diagram E_n. Then we write $e = e(\tilde{X})$ and denote by $h = h(X)$ the root of the highest height of X.

Then the sum $e + h$ is a primitive isotropic vector. Moreover $< e, h >= -2$, and $r_e(h) = h + 2e$.

The E_6-relation is equivalent to $h + 2e \doteq e + 2h$ and

$$h + 2e \doteq e + 2h \Longleftrightarrow r_e(h) = r_h(e) \Longleftrightarrow |r_e r_h| = 3.$$

The E_7-relation is equivalent to $h + 2e \doteq h$ and we have

$$h + 2e \doteq h \Longleftrightarrow r_e(h) = h \Longleftrightarrow |r_e r_h| = 2.$$

The E_8-relation is also equivalent to $h \doteq e$ and this means

$$h \doteq e \Longleftrightarrow r_e = r_h \Longleftrightarrow |r_e r_h| = 1.$$

Remark. The numbers $3, 2, 1$ are the determinants of E_6, E_7, E_8-lattice. *Is there any mathematical background ?*

Theorem C shows that

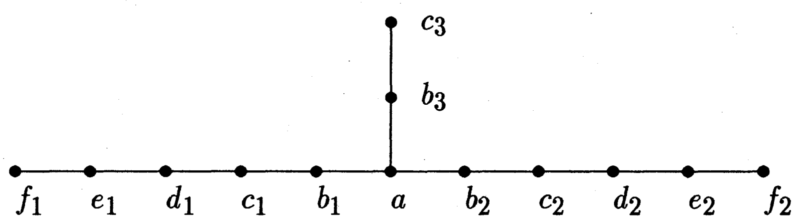
$$\mathbb{M} \wr 2 \cong < Y_{555}, E_8\text{-relations}(\forall E_8 \subset Y_{555}) > .$$

The following is a natural question.

Question.

$$< Y_{555}, E_7, E_6\text{-relations}(\forall E_7, E_6 \subset Y_{555}) >=?$$

Remark. The orthogonal group $O(11, 3)$ contains



satisfying the E_6-relation and $f_1 = f_{13}, f_2 = f_{23}$, (and $(ab_1c_1ab_2c_2ab_3c_3)^{30} = 1$).
It is known that

$$3.Fi_{24} \cong < Y_{552}, f_1 = f_{13} = f_{12}, f_2 = f_{23} = f_{21} > .$$

Finally we consider the affine diagrams $\tilde{X}$ contained in 26 node system given in Theorem D. The 26 vectors are listed in Table 1 ([CP]). By using them, we can easily calculate the orders $|r_{e(\tilde{X})}r_{h(X)}|$ in $G/N \cong \mathbb{M} \wr 2$.

The cases (1)-(3) are contained in Y_{555}, the set Π of fundamental Monster roots. We treated them in Propositions 1-3.

The cases (4)-(13) are not contained in Y_{555}. There is no diagram which gives a new relation.

An interesting fact is that for any X, Y of (4)-(13),

$$|r_{e(\tilde{X})}r_{h(X)}| \leq |r_{e(\tilde{Y})}r_{h(Y)}| \iff c(X) \geq c(Y)$$

where we denote by $c(X)$ the Coxeter number of X.

Our final question is *"Is there any mathematical background ?"*

(Table 2) The affine diagrams $\tilde{X}$ and the order $|r_{e(\tilde{X})} r_{h(X)}|$

(1)	$\tilde{E}_6$			: 3		E_6–relation
(2)	$\tilde{E}_7$			: 2		E_7–relation
(3)	$\tilde{E}_8$			: 1		E_8–relation
(4)		$\tilde{D}_4$		: 3	$\Longleftrightarrow$	(1)
(5)			$\tilde{A}_5$	: 3	$\Longleftrightarrow$	(1)
(6)		$\tilde{D}_5$		: 3	$\Longleftrightarrow$	(1)
(7)			$\tilde{A}_7$	: 2	$\Longleftrightarrow$	(2)
(8)		$\tilde{D}_6$		: 2	$\Longleftrightarrow$	(2)
(9)			$\tilde{A}_9$	: 2	$\Longleftrightarrow$	(2)
(10)	$\tilde{E}_6$			: 2	$\Longleftrightarrow$	(2)
(11)			$\tilde{A}_{11}$	: 1	$\Longleftrightarrow$	(3)
(12)		$\tilde{D}_8$		: 1	$\Longleftrightarrow$	(3)
(13)	$\tilde{E}_7$			: 1	$\Longleftrightarrow$	(3)