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CONTINUOUS AND SEMI-DISCRETE TRILINEAR EQUATIONS: INVESTIGATING THEIR INTEGRABILITY

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Abstract

We examine the integrability of a class of trilinear equations (due to Matsukidaira and collaborators) that can be written as a single determinant. The method used is a combination of the Painlevé-singularity analysis for the fully continuous equations and the singularity confinement for the discrete ones. This method can thus be applied to equations that are both discrete and continuous. Our main result is that while the fully continuous equation passes the integrability test this is not the case once part of the equation is discretized. We investigate the reductions of the semi-discrete equations that satisfy the confinement criterion for integrability obtaining the most general integrability candidates.

The bilinear formalism of Hirota has been a most valuable tool in the study of the integrability of nonlinear evolution equations [1]. It offers a purely algebraic way to deal with soliton equations, once they are transformed to a bilinear form through a dependent-variable transformation. Sato's theory has shown that the bilinear equations are just determinantal identities [2]. The construction of explicit solutions becomes thus algorithmic and straightforward. Recently,a new class of determinant identities for soliton equations was proposed [3]. They do not possess a Hirota bilinear form but, rather, a trilinear one. This trilinear extension originated in a study of the Broer-Kaup system. This 1+1-dim system possesses a bilinear form but its trilinear one looks more natural and, what is more important, can be extended to higher order, higher (up to 2+2) dimension systems. These 4-dimensional systems are expected to be integrable. They do possess a rich family of solutions (in the form of bi-directional Wronskians) and, although this is not a rigorous proof of integrability, it is a strong indication thereof.

By analogy to the bilinear case, the trilinear equations may serve as a starting point for the construction of (fully or semi-) discrete equations. This line of research was pursued in a series of papers where trilinear discrete equations together with their solutions (in the form of bi-directional Casorati determinants) were derived [4,5]. One of the important results of this investigation was the reduction of the trilinear equation to the 1C+1D-dimensional relativistic Toda lattice (RTL) [6]. (The notation 1C+1D indicates that the system has one continuous and one discrete dimension and will be used throughout this paper). As a matter of fact, a 2C+2D-dimensional generalization of the RTL was proposed and some of its special solutions were constructed [6]. The existence of this rich class of solutions and the analogy with the bilinear results of Hirota raise the all-important question of the integrability of these equations. The study of this integrability is the object of this paper.

Our investigation of integrability will be based on the two most successful integrability detectors: Painlevé analysis (for continuous systems), singularity confinement (for discrete systems) and their combination. We shall not present the two methods here; detailed presentations can be found in the literature [7,8]. To put it in a nutshell, both methods are based on the singularities that appear spontaneously depending on the initial conditions. The integrability requirement for continuous systems is the absence of critical singularities (Painlevé property) while for discrete systems integrability is related to confined movable singularities i.e. singularities that are confined to a finite region of the lattice. Only the lowest-order member of the hierarchy will be examined in each case. This assumption is not unreasonable in view of our findings: whenever nonintegrability is detected at the lowest order one expects the whole hierarchy to be nonintegrable. Conversely the fact that one integrability test is positive for the lowest-order member does not guarantee the integrable character even for the equation at hand (although, admittedly, it constitutes a strong indication of integrability).

Let us start with the fully continuous case. The simplest four dimensional trilinear equation writes:

$$\begin{vmatrix} \tau & \tau_t & \tau_x + \tau_{tt} \\ \tau_y & \tau_{ty} & \tau_{xy} + \tau_{tty} \\ \tau_z - \tau_{yy} & \tau_{tz} - \tau_{tyy} & \tau_{zx} + \tau_{ztt} - \tau_{xyy} - \tau_{ttyy} \end{vmatrix} = 0. \tag{1}$$

The Painlevé analysis of this equation is straightforward. On the "singularity manifold" $\phi(x, y, z, t) = 0$ the τ-function behaves as $\tau = \phi^n$ with $n = 0$ or 1. The case $n = 0$ corresponds to a regular solution, while $n = 1$ (a simple zero for τ) leads to a singular behaviour in the nonlinear variables. The resonance in this case are -1,0,1 and 2. Expanding τ up to order three in ϕ we find that the resonance conditions are automatically satisfied. Thus (1) passes the Painlevé test and is presumably integrable. From equation (1) we can obtain one with a much simpler form. We introduce a scaling of both dependent and independent variables such that $\tau_y << \tau_z$ and $\tau_t << \tau_x$. By going to the limit we obtain (after a row and column permutation) the form:

$$\begin{vmatrix} \tau_{ty} & \tau_y & \tau_{xy} \\ \tau_t & \tau & \tau_x \\ \tau_{tz} & \tau_z & \tau_{zx} \end{vmatrix} = 0. \tag{2}$$

This equation is particularly interesting: it possesses the Painlevé property being a limit of (1). However in the limiting process all singular behaviours disappear and (2) possesses only regular solutions. Putting $\tau = e^u$ we can write (2) as:

$$u_{xt} u_{yz} - u_{xy} u_{tz} = 0. \tag{3}$$

A reduction of (3) is just the Monge-Ampère equation $u_{xx} u_{zz} - u_{xz}^2 = 0$, a well-known integrable PDE. Putting $u = e^z v(x)$ we can further reduce it $v_{xx} v - v_x^2 = 0$, an equation that does not obviously possess any singular behaviour. We expect (3) to be integrable as well as its lower dimensional reductions, obtained by taking $t = z$ and/or $y = x$.

Introducing difference analogs for some (or all) of the differential operators in (2) allows one to derive semi- or fully discrete trilinear equations. The two-dimensional extension of RTL we referred to in the opening paragraph was obtained in this way:

$$\begin{vmatrix} \partial_x \tau_{m,n-1} & \tau_{m,n-1} & \tau_{m+1,n-1} \\ \partial_x \tau_{m,n} & \tau_{m,n} & \tau_{m+1,n} \\ \partial_x \partial_y \tau_{m,n} & \partial_y \tau_{m,n} & \partial_y \tau_{m+1,n} \end{vmatrix} = 0. \tag{4}$$

Are these equations integrable? In order to answer this question we will examine their singularity structure. Before proceeding further we introduce a convenient notation: $\tau \equiv \tau_{m,n}$, $\bar{\tau} \equiv \tau_{m,n+1}$, $\underline{\tau} \equiv \tau_{m,n-1}$, $\tilde{\tau} \equiv \tau_{m+1,n}$, $\underset{\sim}{\tau} \equiv \tau_{m-1,n}$. We start with the RTL equation, obtained from (4) assuming $\tilde{\tau} = \bar{\tau}$ and $x = y$:

$$\det M \equiv \begin{vmatrix} \underline{\tau}' & \underline{\tau} & \tau \\ \tau' & \tau & \bar{\tau} \\ \tau'' & \tau' & \bar{\tau}' \end{vmatrix} = 0 \tag{5}$$

where the prime represents the x-derivative. The equations for $\bar{\tau}$ read:

$$A\bar{\tau}' + B\bar{\tau} + C = 0 \tag{6}$$

with $A = \underline{\tau}'\tau - \underline{\tau}\tau'$, $B = \underline{\tau}\tau'' - \underline{\tau}'\tau'$, $C = \tau(\tau'^2 - \tau\tau'')$. As a differential equation for $\bar{\tau}$ (6) must possess the Painlevé property. From its expression it is clear that $\bar{\tau}$ will have a critical point whenever $A = 0$, at some point x_0, *unless* $A' + B = 0$ (in which case τ has a simple zero). The condition for $A = 0$ is just

$$\frac{\tau'}{\tau} = \frac{\underline{\tau}'}{\underline{\tau}} \tag{7},$$

at $x = x_0$. For $A' + B = 0$ we find: $\underline{\tau}''\underline{\tau} - \underline{\tau}'^2 = 0$. This appears as a constraint on $\underline{\tau}$ and could thus be not satisfied. However let us look closer at how we obtain $A = 0$ from the equation governing τ i.e. $\det \underline{M} = 0$. We find, substituting τ' from (7), that $\underline{\underline{\tau}}'$ cancels out and we are left with an expression that can be factorized as:

$$\det \underline{M} = (\underline{\tau}''\underline{\tau} - \underline{\tau}'^2)(\underline{\tau} - \tau\underline{\underline{\tau}}/\underline{\tau}) = 0 \tag{8}$$

Thus either i) $\underline{\tau}''\underline{\tau} - \underline{\tau}'^2 = 0$, in which case the condition $A' + B = 0$ is satisfied and the Painlevé property holds, or ii) $\underline{\tau}^2 - \tau\underline{\underline{\tau}} = 0$ which might be a source of difficulties since $\underline{\underline{\tau}}$ appears explicitly. However this second case does not cause any problem either. In fact we start by expanding

$$\tau = (\underline{\tau}/\underline{\underline{\tau}})\underline{\tau} + (x - x_0)\varphi \tag{9a}$$

$$\tau' = (\underline{\tau}/\underline{\underline{\tau}})\underline{\tau}' + (x - x_0)\psi \tag{9b}$$

Differentiating (9a) and equating to (9b) gives a first relation between φ and ψ, while $\det \underline{M} = 0$ yields a second one and allows φ, ψ to be computed. We can thus calculate τ'' and obtain A, B, C explicitly. The net result is that *all* A, B, C have $(x - x_0)$ as a factor. Thus A vanishes because the entire equation (9) is multiplied by a vanishing factor and so this point does not correspond to a singularity. In conclusion the 1-dim RTL equation (5) passes the confined singularity test and should be integrable (and indeed it is!).

In order to investigate the integrability of the 2C+2D-dim RTL we shall limit ourselves to the 1D reduction obtained by taking $\tilde{\tau} = \bar{\tau}$ in (4):

$$\det M \equiv \begin{vmatrix} \underline{\tau}_x & \underline{\tau} & \tau \\ \tau_x & \tau & \bar{\tau} \\ \tau_{xy} & \tau_y & \bar{\tau}_y \end{vmatrix} = 0 \tag{10}$$

The equivalent to equation (6) is:

$$A\bar{\tau}_y + B\bar{\tau} + C = 0 \tag{11}$$

with $A = \underline{\tau}_x\tau - \underline{\tau}\tau_x$, $B = \tau_{xy}\underline{\tau} - \underline{\tau}_x\tau_y$. As previously, the condition for absence of critical points is $A_y + B = 0$ whenever $A = 0$. In terms of τ we have:

$$\frac{\tau_x}{\tau} = \frac{\underline{\tau}_{xy}}{\underline{\tau}_y} \quad \text{when} \quad \frac{\tau_x}{\tau} = \frac{\underline{\tau}_x}{\underline{\tau}} \tag{12}$$

Next we examine how one can have $A = 0$ from the equation $\det \underline{M} = 0$. The upshot is that τ_x does not appear in $\underline{M}$ (rather, τ_y appears) and thus the fine cancellation that we encountered in the 1C+1D case is not possible here. Thus the 2C+1D RTL does not pass the Painlevé test and we cannot expect it to be integrable. This is also an indication that the 2C+2D relativistic Toda lattice should not be integrable either.

Another interesting reduction of the 2C+2D-dim RTL is a 1C+2D one:

$$\det M \equiv \begin{vmatrix} \underline{\tau}' & \underline{\tau} & \underline{\tilde{\tau}} \\ \tau' & \tau & \tilde{\tau} \\ \tau'' & \tau' & \tilde{\tau}' \end{vmatrix} = 0 \tag{13}$$

and the question is, again, whether (13) is a candidate for integrability. We rewrite (13) as an equation for $\tilde{\tau}$:

$$A\tilde{\tau}' + B\tilde{\tau} + C = 0 \tag{14}$$

where A, B have exactly the same expressions as in the case of (6). Again, when A passes through zero we must have $A' + B = \underline{\tau}''\tau - \underline{\tau}'\tau' = 0$. But, $A = 0$ must result from the equation for τ', i.e. the one related to $\det \underset{\sim}{M} = 0$. This equation, as in the previous case, does not contain information on the useful quantity $\underline{\tau}''$ but rather on $\underset{\sim}{\tau}''$ and thus does not lead to any cancellation. So equation (14) possesses critical singularities and this 1C+2D reduction cannot be integrable either. We have seen above that one of the 1C+1C reductions, namely the RTL, is integrable. This reduction was obtained through the assumption $\tilde{\tau} = \bar{\tau}$. It is natural, then, to ask what happens if one considers the other reduction $\tilde{\tau} = \underline{\tau}$. In this case the equation becomes:

$$\det M \equiv \begin{vmatrix} \underline{\tau}' & \underline{\tau} & \underline{\underline{\tau}} \\ \tau' & \tau & \underline{\tau} \\ \tau'' & \tau' & \underline{\tau}' \end{vmatrix} = 0 \tag{15}$$

This is a nice case where the singularity confinement criterion can be applied in a straightforward way. Equation (15) can be considered as an equation for $\underline{\underline{\tau}}$ in terms of $\tau, \underline{\tau}$ or, preferably, by upshifting it once, an equation for $\underline{\tau}$ in terms of $\bar{\tau}, \tau$:

$$\underline{\tau} = \frac{\tau^2\bar{\tau}'' - 2\tau\tau'\bar{\tau}' + \tau'^2\bar{\tau}}{\bar{\tau}'^2 - \bar{\tau}\bar{\tau}''} \tag{16}$$

Whenever $\bar{\tau}'^2 - \bar{\tau}\bar{\tau}''$ passes through zero, at some point x_0, $\underline{\tau}$ develops a pole. The question is now whether this pole disappears in the next steps. Downshifting (16) we find that $\underline{\underline{\tau}} \propto \underline{\tau}^2$ and thus the pole does not disappear at this step. From the structure of the equation it can be easily gathered that the divergence grows at each iteration and no confinement is observed. Thus (15) cannot be integrable, according to our criterion. Another, equivalent, way to study the singularity properties of (15) is to consider it as an equation for τ, in terms of $\underline{\tau}, \underline{\underline{\tau}}$. This equation is of second order and belongs to the class of equations studied by Painlevé and Gambier. The integrable cases in this classification are known and it can be shown (after some elementary transformations) that (15) does not belong to this subclass. So our conclusion at the present stage is that the fully continuous trilinear PDE is expected to be integrable on its full generality while its semidiscrete counterpart (4) is not. In fact only the 1C+1D relativistic Toda lattice reduction (5) passes the integrability test, while all the other reductions fail.

At this point, several extensions of our study suggest themselves. First, it is clear that (2) is not the most general single-determinant equation at that order (although it turns out that it is indeed the only one that can be expressed in terms of the trilinear equation introduced in [9]). A simple generalization of (2) is to introduce coefficients in the entries of (2). When all scalings are fixed, up to four of these coefficients remain free. So the generalization of (2) can be written in a symmetric form:

$$\begin{vmatrix} \alpha\tau_{ty} & \tau_y & \gamma\tau_{xy} \\ \tau_t & \tau & \tau_x \\ \beta\tau_{tz} & \tau_z & \delta\tau_{zx} \end{vmatrix} = 0 \tag{20}$$

Among the variety of equations with extra coefficients different from unity, there is one that is remarkable, namely a determinant:

$$\begin{vmatrix} \tau_{ty} & \tau_y & \tau_{xy} \\ \tau_t & 0 & \tau_x \\ \tau_{tz} & \tau_z & \tau_{zx} \end{vmatrix} = 0 \tag{21}$$

which is the limit $\alpha = \beta = \gamma = \delta \to \infty$. Just as in the case of (2) the solutions of this equation are also devoid of singular behaviour. The remarkable property of this equation is that it is invariant under the transformation $\tau \to e^{\tau}$. We expect (21) to be integrable just as (2). In fact, the 2-dimensional reduction of (21), obtained by putting $t = x$ and $z = y$,

$$2\tau_{xy}\tau_x\tau_y - \tau_{yy}\tau_x^2 - \tau_{xx}\tau_y^2 = 0 \tag{22}$$

can be reduced to quadratures. Putting $\tau_x \equiv f^{-1}$ and $\tau_y \equiv g^{-1}$ we obtain the compatibility condition $(f^{-1})_y = (g^{-1})_x$. Equation (22) writes: $\frac{f_y}{f}g - g_y + \frac{g_x}{g}f - f_x = 0$ and putting $u = f/g$ reduces it to the dispersionless Burgers equation $u_x - uu_y = 0$ that can be solved by characteristics and is known to possess shockwave type solutions. Once u is obtained one can construct the f and g by further integrations and finally obtain τ that is the solution of (22).

On the other hand the examination of semidiscrete analogs of (21), i.e. equations similar to (4) with a zero central element, lead to rather disappointing results. Among all the cases (lower dimensional reductions) analyzed we have not found a single one satisfying our integrability criterion. Thus we do not expect these systems to be integrable.

Another interesting direction is the integrability of fully discrete trilinear equations. In [5] discrete analogs of (2) were presented together with Casorati determinant classes of solutions. The indications discovered in the present paper of non-integrability of the discrete-continuous equations (5) cast a doubt on the integrability of these discrete trilinear systems. The question is worth investigating in detail and we plan to come back to it in some future publication.

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