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Construction of Field Theory Models in Four Dimensions

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Dedicated to Professor Noboru Nakanishi
on the occasion of his 60th birth day

1.Introduction

In the last 2 decades, we saw a rapid development of quantum field theory. Without any doubt Prof.Nakanishi is one of the few persons who presented us better understandigs of gauge fields on which modern field theory heavily depends. As is quite well known, the field theory with renormalization theory has been developed through Quantum Electrodynamics, Yukawa theory and the ϕ_4^4 theory. However our present-day conventional wisdom suggtests that only the asymptotically free theories can be non-trivial, which means ironicaly enough that QED, Yukawa and the ϕ_4^4 theories cannot exist without being trivial.

See [1, 2, 3, 4, 5, 6] for arguments which show that the ϕ^4 model in 4 dimensions is expected to be trivial.

Most of these arguments change if the signature of the coupling constant is reversed [7]: we show here that the four-dimensional ϕ^4 model with a negative coupling constant exists as a non-trivial theory. The functional integral is unstable in this case, and then we here complexify field variables $\{\phi(x); x \in Z^4\}$.

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2. Lattice Theory

We prepare the theory on the lattice space aZ^4, $a \equiv L^{-N}$ where L is a positive integer (≥ 2) and $N(>> 1)$ is an arbitralily large interger. Let $x \in Z^4$ and let $\hat{\phi}(ax)$ be the field on aZ^4. We replace $a\hat{\phi}(ax)$ by the field $\phi(x)$ on Z^4 by absorbing a which comes from the Riemannian sum. We start with the bare action v_0 at the distance scale $a = L^{-N}$, given by

$$\frac{1}{2}\Big(\sum_{|x-y|=1}(\phi(x)-\phi(y))^2+\sum_{x\in\Lambda} m_0^2\phi^2(x)\Big)+\lambda_0\sum_{x\in\Lambda}(\phi^4(x)-6G_0(x,x)\phi^2(x)) \quad (1)$$

which is set on Z^4 by the scaling mentioned above. Note that the coupling constant λ_0 is invariant and the mass is scaled: $m_0^2 = a^2\hat{m}_0^2$. In eq.(1) G_0 is the Greens's function of the free Hmiltonian of massless bosons on the Lattice Z^4 (see eq.(5) below.) The effective action v_n at the scale aL^n is defined by

$$\exp[-v_n(\Phi)] = \int \exp[-v_0(\phi)]\Pi_{x\in\Lambda_n}\delta[\Phi(x)-(C^n\phi)(x)]\Pi_{z\in\Lambda}d\phi(z), \quad (2)$$

where $\Lambda = \Lambda_0$ is a rectangular set of integer points in four dimensions:

$$\Lambda = \{(x_1,..,x_4); x_i = -\frac{L^K}{2}, -\frac{L^K}{2}+1,.., \frac{L^K}{2}-1\}. \quad (3)$$

$\Lambda_n = L^{-n}\Lambda \cap Z^4$ (K is an arbitralily large integer) and the block spin operator C is defined as an averaging operator with a scaling:

$$\phi(x) \rightarrow (C\phi)(x) = \frac{1}{L^3}\sum_{-L/2\leq z_\mu < L/2}\phi(Lx+z) \quad (4)$$

Then (2) is the integration over fluctuations around the fixed block spins $\Phi(x)$. The factor L^3 is chosen so that massless Gaussian measures are fixed points of C [5, 7]. If $\lambda_0 > 0$ is small, one can prove that this converges to a free system in the limit of $n \rightarrow \infty$. To obtain the continuum theory, we iterate the recursion formulae N times to obtain the theory at the unit distance scale. Then we let $N \rightarrow \infty$ keeping these quantities non-zero and finite. To do this, we have to choose $m_0^2 = m_0^{(N)2}$ carefully in a way of depending on $\lambda_0 = \lambda_0^N$.

Rotate $\phi(x) \in R$ by an angle $\alpha \in (\pi/8, \pi/4)$ in the complex plane : $\phi(x) \to e^{i\alpha}\phi(x)$. Then Re $e^{4i\alpha}\lambda_0 > 0$ and the integral (2) exists. Moreover Re $e^{2i\alpha} > 0$ means that the gaussian integrals $< P > \equiv \int P d\mu_1(e^{i\alpha}z)$ are well defined whenever P are polynomials of z.

We explain our notation [5, 6]. Let $\phi_n(x) = (C\phi_{n-1})(x)$ be the block spin variable at distance scale L^n($\phi_0(x) = \phi(x)$).

Since

$$G_0(x,y) = (-\Delta)^{-1}(x,y) \sim (x-y)^{-2} \tag{5}$$

the correlation function of ϕ_n

$$G_n(x,y) = (C^n G_0 (C^+)^n)(x,y) \tag{6}$$

again satisfies

$$G_n(x,y) \sim (x-y)^{-2}. \tag{7}$$

This means that $\{\phi_n\}$ are very similar to the original $\{\phi = \phi_0\}$.We introduce two operators :the first one is Q which maps $f(x) \in R^{\Lambda_n \backslash L\Lambda_{n+1}}$ to $(Qf)(x) \in R^{\Lambda_n}$:

$$Q : f(x) \to (Qf)(x) = \begin{cases} f(x) & \text{if } x \notin LZ^4 \\ -\sum_{y \in B(x)} f(y) & \text{if } x \in LZ^4 \end{cases} \tag{8}$$

Then $C(Qf) = 0$ and Q is an operator which gets fluctuation fields from the field. The second one is the projection $R : R^{\Lambda_n} \to R^{\Lambda_n \backslash L\Lambda_{n+1}}$:

$$R : f(x) \to (Rf)(x) = \begin{cases} f(x) & \text{if } x \in \Lambda_n \backslash L\Lambda_{n+1} \\ 0 & \text{otherwise} \end{cases} \tag{9}$$

Then $C(QRf) = 0$ for any function $f(x)$ defined on Λ_n. Then $\{ \phi_n(x);\ x \in \Lambda_n \}$ is written in terms of spin variables $\{\phi_{n+1}\}$ of next distance scale and fluctuation fields $\{\xi_n(x);\ x \in \Lambda_n \backslash L\Lambda_{n+1}\}$:

$$\phi_n(x) = (A_n \phi_{n+1})(x) + (Q\xi_n)(x) \tag{10}$$

where $A_n : R^{\Lambda_{n+1}} \to R^{\Lambda_n}$ is given by

$$A_n(x,y) = (G_n C^+ G_{n+1}^{-1})(x,y) \tag{11}$$

and $\{\xi_n(x); x \in \Lambda_n \backslash L\Lambda_{n+1}\}$ are gaussian random variables of zero mean and covariance

$$\Gamma_n(x,y) = R(G_n - G_n C^+ G_{n+1} G_n^{-1}) R^+. \tag{12}$$

It is not difficult to see that

$$|A_n(Lx+\tilde{x},y)| \le C_1 \exp[-\beta|x-y|] \tag{13a}$$

$$|\Gamma_n(x,y)| \le C_2 \exp[-\beta|x-y|] \tag{13b}$$

where $|\tilde{x}| \le L/2$. For later convenience, we define

$$\mathcal{A}_n(x,y) = L^n A_0 A_1 \dots A_{n-1}(L^n x, y) \tag{14}$$

where $x \in L^{-n}\Lambda$ (lattice width=L^{-n}), $y \in \Lambda$ (lattice width=1) and introduce two variables ψ_n and z_n which are linear combinations of the original independent random variables $\{\phi_n\}$ and $\{\mathrm{z}_n\}$:

$$\psi_n(x) = (\mathcal{A}_n\phi_n)(x), z_n(x) = (\mathcal{A}_n Q \Gamma_n^{1/2}\mathrm{z}_n)(x), \tag{15}$$

where we put $\xi = \Gamma^{1/2}\mathrm{z}$ and $x \in L^{-n}\Lambda$. We may use notation $\int F(x)dx = L^{-4n}\sum_x F(x)$. Eq.(10) is simplified:

$$\psi_n(x) = \frac{1}{L}\psi_{n+1}(x/L) + z_n(x), \tag{16}$$

where $x \in L^{-n}\Lambda$. Their means are zero, and their covariances are respectively given by

$$\mathcal{G}_n(x,y) = (\mathcal{A}_n G_n \mathcal{A}_n^+)(x,y) \sim (x-y)^{-2} \tag{17a}$$

and by

$$\mathcal{T}_n(x,y) = (\mathcal{A}_n Q \Gamma_n Q^+ \mathcal{A}_n^+)(x,y) \sim e^{-\beta|x-y|}, \tag{17b}$$

where both x and $y \in L^{-n}\Lambda$. We also define

$$\begin{aligned}\mathcal{Q}_n(x,y) &= \sum_{k=0}^{n-1} L^{2(n-k)}\mathcal{T}_k(L^{n-k}x, L^{n-k}y) \\ &= L^{2n}G_0(L^n x, L^n y) - \mathcal{G}_n(x,y) \end{aligned} \tag{17c}$$

and

$$\mathcal{S}_n(x_1,x_2,x_3) = \mathcal{Q}_n(x_1,x_2)\mathcal{Q}_n(x_2,x_3). \tag{17d}$$

$\mathcal{T}_n$, $\mathcal{Q}_n$ and $\mathcal{S}_n$ have exponential decay property uniform in n and the difference between G_n and $\mathcal{G}_n$ is marginal. To calculate the renormalization recursion formulae (2), we separate $(\phi,(-\Delta)\phi) = \sum(\phi(x)-\phi(y))^2, (|x-y|=1)$ from $v_0 = v_0^{(N)}$ and represent it as

$$\Pi_n[\exp[-(\xi_n, \Gamma_n^{-1}\xi_n)/2]\Pi d\xi_n(x)] \equiv \Pi_n d\mu_{\Gamma_n}(\xi)$$

or as $\Pi_n d\mu_1(z_n)$, where $d\mu_1$ is the gaussian measure of zero mean and covariance 1 (see eq.(15)). Therefore our recursion formulae are

$$\exp[-v_{n+1}(\psi(.))] = \mathcal{N}^{-1}\int \exp[-v_n(\psi(./L)/L + z_n(.))]d\mu_1(z_n). \tag{18}$$

3. Renormalization Group Trajectory

We decompose the configurations of $\psi(x)$ into the set of small and smooth fields $\mathcal{K}_n(D) = \{\psi_n(x); |\psi_n(x)| < B|\lambda_n|^{-1/4}$, Holder type continuity of $\psi_n, x \in D\}$, and the set of complex large fields $\mathcal{D}_n(D) = \{\psi_n(x); |\mathrm{Im}e^{i\alpha}\psi_n(x)| < C|\lambda_n|^{-1/4}, x \in D\}$. In the region $\mathcal{K}_n$, v_n is obtained in a closed form by a perturbation theory (the convergent polymer expansion), while in the region $\mathcal{D}_n(D)$, we cannot use the perturbation and we use a probabilistic bound to show the contribution is very small.

Theorem Let the bare lattice action on $\Lambda = \Lambda^{(K)}$ be given by eq.(1) with the coupling constant $\lambda_0 < 0$ satifying

$$\frac{1}{\lambda_0} = \frac{1}{\lambda_{phys}} - \beta_2 N + c_3 \log(1 - \beta_2\lambda_{phys}N), \tag{19}$$

where $\lambda_{phys} < 0$ is the physical coupling constant, and $\beta_2(> 0)$ and c_3 are constants specified later. Assume $|\lambda_{phys}| << 1$. Then there exists $m_0^2 \in [-|\lambda_0|^{3/2}, |\lambda_0|^{3/2}]$ such that $\exp[-v_n(\psi)]$ exists for all $n < N$ and $\lim \exp[-v_{N-1}^{(N)}]$ exists. The series $\{\ v_n = v_n^{(N)}\}$ satisfy the following (i) and (ii):

(i) *Analyticity in the small field region*
There exist constants m_n^2 , λ_n, γ_n and η_n such that

$$\lambda_n \in [-c_-/(N + n_0 - n), -c_+/(N + n_0 - n)], \tag{20a}$$

$$m_n^2 \in [-c_1|\lambda_n|^{3/2}, c_1|\lambda_n|^{3/2}], \tag{20b}$$

$$\gamma_n \in [8\lambda_n^2 - O(\lambda_n^{2+\varepsilon}), 8\lambda_n^2 + O(\lambda_n^{2+\varepsilon})], \tag{20c}$$

$$\eta_n \in [96\lambda_n^3 - O(\lambda_n^{3+\varepsilon}), 96\lambda_n^3 + O(\lambda_n^{3+\varepsilon})]. \tag{20d}$$

where $c_\pm$ and n_0 are positive constants. Then v_n is analytic in $\mathcal{K}_n$ and admits the following expansion there:

$$v_n = \frac{1}{2}m_n^2 \int dx\psi_n^2(x) - 6\lambda_n \int dx\mathcal{G}_n(x,x)\psi_n^2(x)$$
$$+\lambda_n \int dx\psi_n^4(x) + \gamma_n \int dxdy\mathcal{Q}_n(x,y)\psi_n^3(x)\psi_n^3(y)$$
$$+\eta_n \int dx_1dx_2dx_3\mathcal{S}_n(x_1,x_2,x_3)\psi_n^3(x_1)\psi_n^2(x_2)\psi_n^3(x_3)$$
$$+(\text{irrelevant terms}) + \tilde{v}_n(\psi), \tag{21}$$

where $\partial^k\tilde{v}_n/\partial\psi_n^k|_{\psi=0} = 0$ for $k = 1,...,8$, $|\tilde{v}_n| \le const|\lambda_n|^{3/2}$.

(ii) *Uniform Boundedness of the Gibbs Factor*
The Gibbs factor $\exp[-v_n(\psi)]$ is anlytic in $\mathcal{D}$, and satisfies

$$|\exp[-v_n]| < \exp[-|\lambda_n|^{1/2}|\psi_n|^2 + |\lambda_n||Im\psi_n|^4 + D] \tag{22}$$

with a uniform constant D.

The non-triviality of the model follows from this. Some remarks: (1) For simplicity we neglected the wave function renormalization which comes from the quadratic terms in v_n. (2) Since $\{\psi_n(x)\}$ are extended to $L^{-n}\Lambda$ and are related to each other, one cannot pick out $\psi_n(x)$ and discuss $\psi_n(x)$ only. One would rather defines $\mathcal{K}_n$ and $\mathcal{D}_n$ in a much refined way so that the cluster expansion can be inductively used [5, 6, 7, 8].

4. Proof of the Theorem

We here discuss small field region only and show how the renormalization group flow is determined.

If $|\psi_n(x)| < B|\lambda_n|^{-1/4}$, we have the expansion (21). So set $\psi_n(x) = \psi_{n+1}(x/L)/L + z_n(x)$, and substitute it into the right hand side of eq.(21): $v_n = v_n^0 + \delta v_n$, where v_n^0 is the 0th term which does not contain z at all and δv_n is the remainder $(\delta v_n(z=0) = 0)$.

We use

$$v_{n+1} = v_n^{(0)} - \log[\int \exp[-\delta v_n]d\mu_1(z_n)]$$
$$= v_n^{(0)} + \{<\delta v_n> -(2!)^{-1} <\delta v_n, \delta v_n> +(3!)^{-1} <\delta v_n, \delta v_n, \delta v_n>$$
$$+ \text{ remainder } \}, \tag{23}$$

where

$$< . > \equiv \int_{e^{i\alpha}\mathbf{R}} (.) d\mu_1(z), < \delta v, \delta v > \equiv < \delta v \delta v > - < \delta v >^2, \tag{24}$$

and so on. Even if $|\psi_{n+1}|$ are small, $\psi_{n+1}/L + z_n$ can be large and thus the expansion (21) fails there. In this case we use the bound (22). For simplicity we skip all these difficulties. After some calculation, we see that the following recursion relations control the flow: $v_n \to v_{n+1}$, except for the minor terms which are very small or have kernels which decrease fast.

$$\begin{array}{ll}
(\mathrm{R.1}) & m_{n+1}^2 = L^2 m_n^2 - \alpha_1 \lambda_n^2 + O(\lambda_n^3, m_n^2 \lambda_n), \\
(\mathrm{R.2}) & \lambda_{n+1} = \lambda_n - \beta_2 \lambda_n^2 - \beta_3 \lambda_n^3 - \beta_4 m_n^2 \lambda_n + O(\lambda_n^4, m_n^2 \lambda_n^2) \\
(\mathrm{R.3}) & \gamma_{n+1} = 8\lambda_{n+1}^2 + O(\lambda_{n+1}^3, m_n^2 \lambda_n), \\
(\mathrm{R.4}) & \eta_{n+1} = 96\lambda_{n+1}^3 + O(\lambda_{n+1}^4, m_n^2 \lambda_{n+1}^2), \\
(\mathrm{R.5}) & \tilde{v}_{n+1} = O((|\lambda_n| + |m_n^2|)^4).
\end{array}$$

Here $\beta_2 > 0$ comes from the one-loop diagram which is very important feature of the present system. (R.2 implies $\lambda_n \to 0$ if $\lambda_0 > 0$ and the other way around if $\lambda_0 < 0$.)

The parameters γ_n and η_n are completely determined. [λ_{n+1} and m_{n+1}^2 of course have feedback from γ_n and η_n which appear in R.1 and R.2 as $O(\lambda_n^3)$. Thus as will be seen, their effects are well controlled.]

Consider the flow of $\zeta_n =^t (m_n^2, \lambda_n)$ defined by R.1 and R.2 or by

$$\zeta_{n+1} = \begin{pmatrix} L^2 & -\alpha_1 \lambda \\ -\beta_4 \lambda & 1 - \beta_2 \lambda - \beta_3 \lambda^2 \end{pmatrix} \zeta_n$$

$$\lambda = \lambda_n (\text{ or } = \lambda_{n_0} \text{ for some } n_0 \le n)$$

and insist that $\{ |\zeta_n| \}$ stay as $O(1)$ for $n < N$. Then we find that $m_n^2 = [\alpha_1/(L^2 - 1)]\lambda_n^2 + O(\lambda_n^3)$. Thus the third and fourth terms in the right hand side of R.2 are replaced by $-\tilde{\beta}_3 \lambda_n^3$. Deviding both sides of R.2 by $\lambda_n \lambda_{n+1}$, we get:

$$\begin{aligned}
\frac{1}{\lambda_n} - \frac{1}{\lambda_{n+1}} &= -\beta_2 \frac{\lambda_n}{\lambda_{n+1}} - \tilde{\beta}_3 \frac{\lambda_n^2}{\lambda_{n+1}} + O(\lambda_n^2) \\
&= -\beta_2 - (\beta_2^2 + \tilde{\beta}_3)\lambda_n + O(\lambda_n^2).
\end{aligned}$$

Assume that λ_N is the observed physical coupling constant $\lambda_{phys} < 0$, ($|\lambda_{phys}| << 1$). Thus $\lambda_0 = \lambda_0^{(N)}$ should satisfy

$$\frac{1}{\lambda_0} - \frac{1}{\lambda_{phys}} = -\beta_2 N - (\beta_2^2 + \tilde{\beta}_3) \sum_{k=0}^{N-1} [\frac{1}{\lambda_{phys}} - k\beta_2]^{-1} + O(1)$$

$$= -\beta_2 N + \frac{\beta_2^2 + \tilde{\beta}_3}{\beta_2} \log[1 - \lambda_{phys}\beta_2 N] + O(1).$$

This is the relation (20).

A remaining delicate problem is how to choose the mass counter term, since a tiny change in the mas counter term yields a large deviation in the renormalized mass. See R.1. The reader will also see that R.1 implies that the self-energy diverges like a^{-2}. The Sinai-Bleher method [5, 6, 7, 8] is used to consider this problem. Assume m_n^2 changes in I_n containing the origin. Then as a function of m_n^2 , m_{n+1}^2 is continuous and $I_{n+1} \equiv \mathrm{Range}(m_{n+1}^2)$ contains I_n. Thus we can choose m_0^2 as our requirements hold.

5. Discussions

In the final part of this note, we argue how to obtain the renormalized divergence free n-point functions [8]. Let $x, y \in aZ^4$. We have replaced $a\hat{\phi}(x)$ by the field $\phi(x/a)$ on Z^4. Therefore changing the theory on aZ^4 to that on Z^4, we have:

$$G_a(x,y) = L^{2N} G_1(L^N x, L^N y) = \tilde{\mathcal{G}}_{1<}(x,y) + \sum_{n=1}^{N} L^{2n} \tilde{\mathcal{T}}_n(L^n x, L^n y) \qquad (25)$$

where $\tilde{\mathcal{G}}_{1<}$ is the two point function of ψ_n with the Gibbs measure given by $e^{-\mathcal{H}_N}$, living on the unit lattice space, and thus is free from any diveregnces. $\tilde{\mathcal{T}}_n$ is the two-point correlation function of the z_n variables with small corrections from the interaction, and decays exponentially in $|x - y|$ uniformly in n. This is in fact approximately equal to the original $\mathcal{T}_n$, see eq.(16) and eq.(17b,c). Then this converges absolutely, and we have renormalized and finite Schwinger (Green's) functions. It is justified to say that the divergences in the theory are cancelled by the asymptotic freedom of the theory and yield a non-trivial field theory.

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