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Traveling wave solutions in an epidemic model

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1. Introduction

The epidemic model described by the equations

$$(1.1)\quad \begin{aligned} \frac{\partial u}{\partial t} &= d_1\frac{\partial^2 u}{\partial x^2} + \beta uv - \alpha u \\ \frac{\partial v}{\partial t} &= d_2\frac{\partial^2 v}{\partial x^2} - \beta uv \end{aligned}$$

is discussed by several authors (see [1], [2], and so on). Here we assume that the population consists of only two populations, infectives u(x,t) and susceptibles v(x,t) which interact. Now, however, u and v are functions of the space variable x as well as time. We model the spatial dispersal of u and v by simple diffusion with constant coefficient d_1 and d_2 respectively. We consider the transition from susceptibles to infectives to be proportional to $\beta u v$, where β is a constant parameter. This form means that βv is the number of susceptibles who catch the disease from each infective. The parameter β is a measure of the transmission efficiency of the disease from infectives to susceptibles. We assume that the infectives have a disease induced mortality rate αu; $\frac{1}{\alpha}$ is the life expectancy of an infective. Where the parameter β , α, d_1 and d_2 are all positive constant.

By an appropriate change of independent and dependent variables, we can get

$$(1.2)\quad \begin{aligned} \frac{\partial u}{\partial t} &= d_1\frac{\partial^2 u}{\partial x^2} + uv - \gamma u \\ \frac{\partial v}{\partial t} &= d_2\frac{\partial^2 v}{\partial x^2} - uv \end{aligned}$$

where the parameter $R = 1/\gamma$ is usually refered to as the basic reproductive rate of the disease.

In Section 2 we describe the analytical results of (1.2) for γ as the fixed constant. In Section 3 we show the numerical results of the initial value problems for (1.2) when γ is dependent on x.

2. Analytic results

In this section we state the results for (1.2) corresponding to the following four situations and give the typical numerical results. The proofs of those results are in the references.

1) $d_1 = d_2 = 0$

In this case (1.2) is the classic Kermack - McKendrick model. In [1], they proved that their model exhibits the threshold phenomenon, i. e., when the initial susceptible number v_0 satisfies $v_0 < \gamma$, then no epidemic can occur. On the other hand if $v_0 > \gamma$ then u(t) initially increases and we have an epidemic. The 'epidemic' means that $u(t) > u_0$ for some $t > 0$ (see, [2]), where u_0 is the initial infection number.

We summarize the results for the existence of the travelling wave solutions of system (1.2). These are solutions of the special form

(1.3) $\quad u(x,t) = f(z), \quad v(x,t) = g(z), \quad z = x + ct,$

where c is a wave speed, which we have to determine. The form of (1.3) represents a wave of constant shape travelling in the positive (resp. negative) x - direction if $c < 0$ (resp. $c > 0$).

2) $d_1 > 0, d_2 = 0$

This is the limmiting case of d_2 small. The condition that d_2 is small corresponds to the situation that the susceptibles disperse very slowly relative to the infectives. In this case we have the following Theorem.

Theorem ([Kallen, [3]]) Let $\gamma < 1$, then there exist a travelling wave solution of (1.2) for every $c \geq c_0 = \sqrt{d_1(1-\gamma)}$.

No travelling wave exist for $\gamma \geq 1$.

Fig.1 shows the numerical result of initial value problems of (1.2) with $d_1 = 1$, $d_2 = 0$, $\gamma = 0.3$, where we take the initial values of (1.2) as

$$
(1.4)\qquad
\begin{aligned}
u(x,0) &= \begin{cases} 0.3\,(1 - x^2) & 0 \le x \le 1 \\ 0 & x \notin [0,\ 1] \end{cases} \\
v(x,0) &= 1.
\end{aligned}
$$

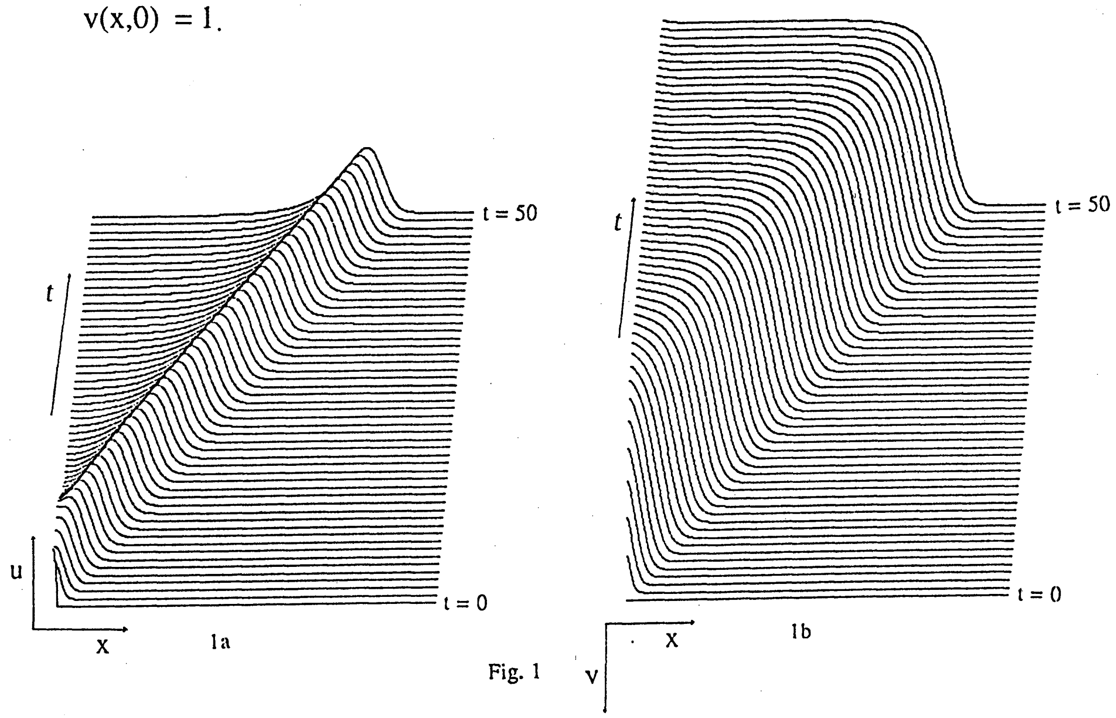


Fig. 1

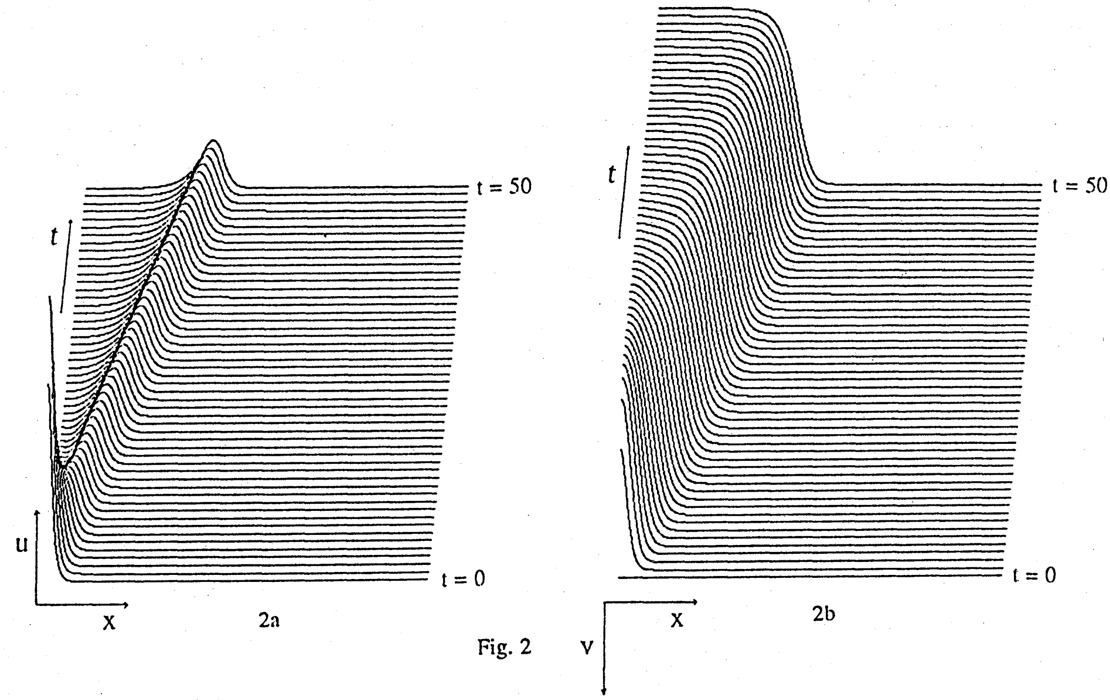
The initial values (u(x,0), v(x,0)) are even functions and the equations have the symmetry with respect to x, so that the original problem is reduced to solving (1.2) on the half space $x > 0$ with the Neumann zero conditions at $x = 0$ and the initial condition (1.4) for $x > 0$. In order to apply the Crank-Nicolson implicit scheme for this problem, we also pose the artificial boundary at $x = 100$, where the Dirichlet zero conditions are specified. All the numerical computations are carried out for this initial boundary value problem.

3) $d_1 = 0, d_2 > 0$

This is the limiting case of d_1 small. The condition that d_1 is small corresponds to the situation that infectives disperse very slowly relative to the susceptibles. In this case ,we have the following Theorem by using the typical shooting method as in [4].

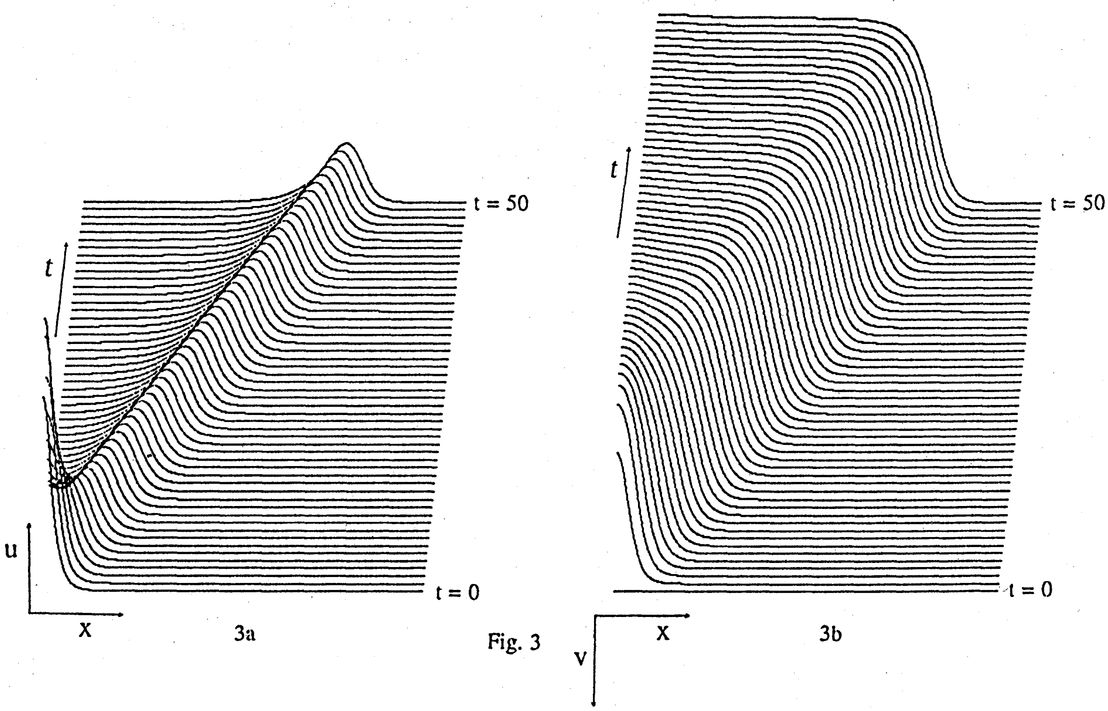
Theorem ([5]) For $\gamma < 1$, there exist a travelling wave solution of (1.2) for every $c > 0$.

Figs. 2-- 3 show the numerical result of the initial value problems of (1.2) as $d_1 = 0$, $d_2 = 1$ and $\gamma = 0.3$. In Fig. 2 $u(x,0) = 1/\cosh(x)$, $v(x,0) = 1$. In Fig. 3 $u(x,0)=1/\cosh(0.5x)$, v(x,0) = 1.



2a 2b

Fig. 2



3a 3b

Fig. 3

From Fig.2--3 we can see easily that the wave speed c is dependent on initial values of (1.2). By computation, we can also see that c is independent of the diffusion coefficient d_2.

4) $d_1 > 0, 0 < d < 2$, where $d = d_2 / d_1$

In this case we have the following Theorem.

Theorem ([6]) Let $\gamma < 1$ and $0 < d < 2$, then there exist a travelling wave solution of (1.2) for $c \geq c_0 = \sqrt{d_1(1-\gamma)}$.

No travelling wave exist for $\gamma \geq 1$.

Figs. 4, 5 and 6 show the numerical results of the initial value problems of (1.2) for (i) $d_1 = d_2 = 1$, $\gamma = 0.3$, (ii) $d_1 = 1$, $d_2 = 0.05$, $\gamma = 0.3$ and (iii) $d_1 = 0.2$, $d_2 = 1$, $\gamma = 0.3$, respectively, where the initial values are given by (1.4).

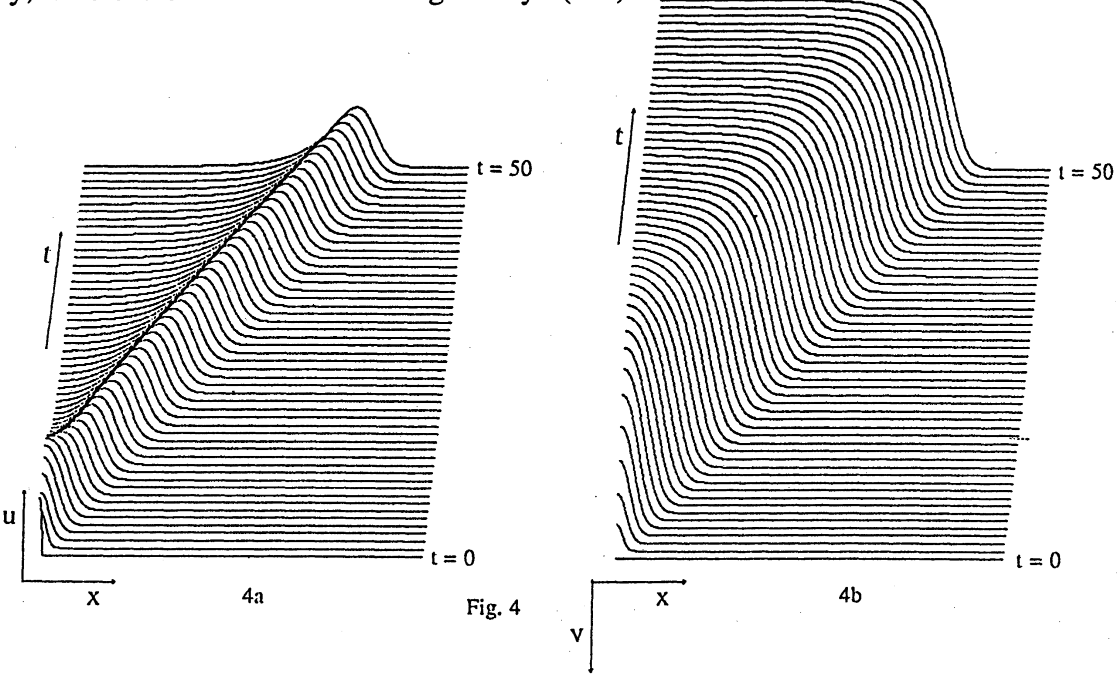


Fig. 4

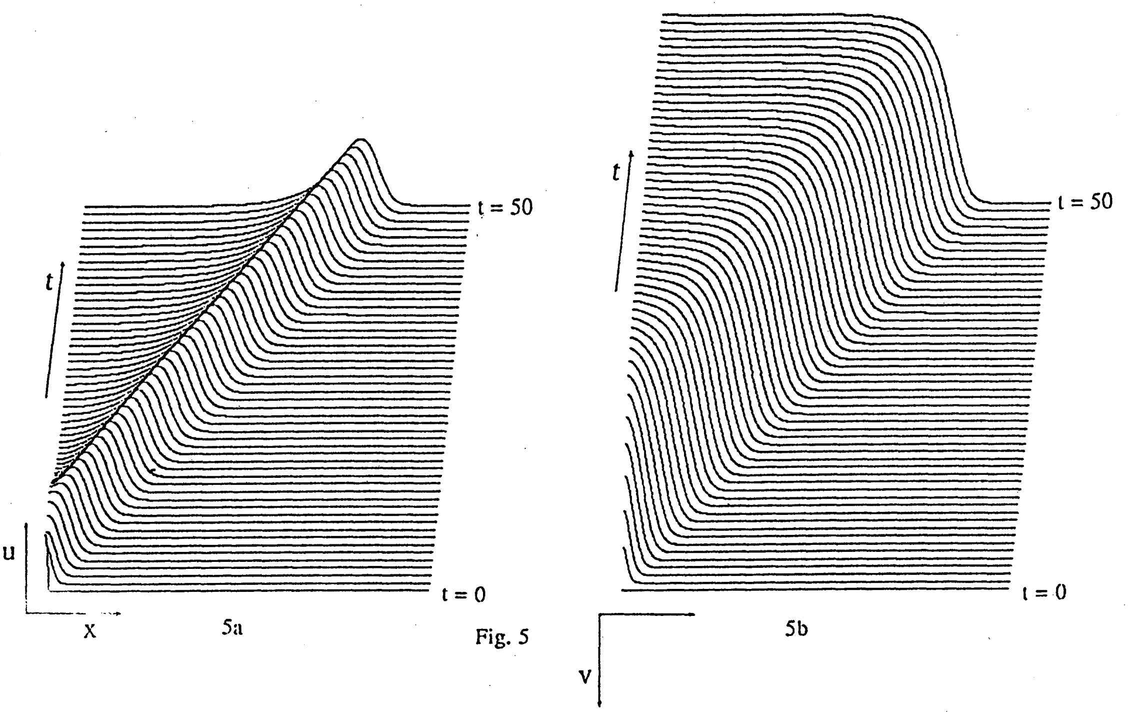


Fig. 5

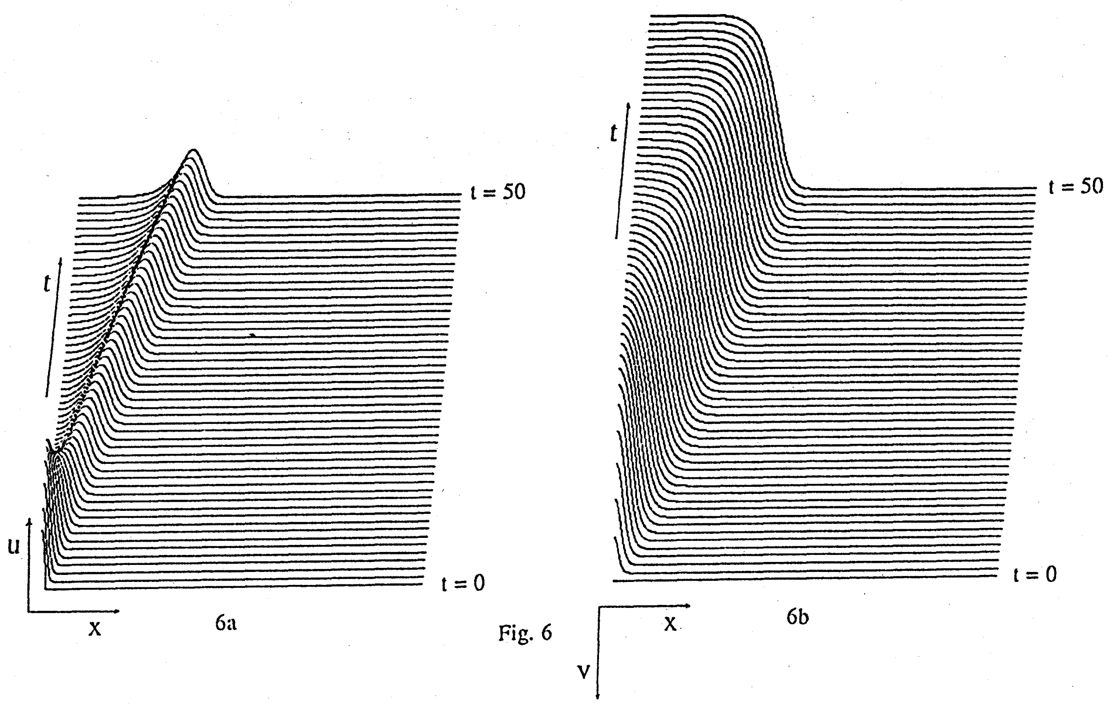


Fig. 6

Figs. 4--6 numerically indicate that the wave speed c is only dependent on d_1.

3. Numerical results where γ is dependent on the x

The results of Section 2 suggest that an epidemic occurs when $\gamma < 1$, and that no epidemic can occur when $\gamma \geq 1$. Therefore, the question is whether we can prevent the epidemic or at least decrease the speed of an epidemic by change the functional form of $\gamma(x)$.

This is a difficult question to answer analytically for the space dependent γ, so that we investigate it numerically. Here we take $x \in [0, 100]$ and use the Crank-Nicalson implicit scheme to the initial value problems of (1.2). All the numerical computation is carried out for $d_1 = 1$, $d_2 = 0.5$ and the initial values are given by (1.4).

Now we describe the numerical results of initial value problems of (1.2) for the various kinds of γ.

Fig. 7 and 8 shows the numerical result of initial value problems of (1.2) for (i) $\gamma = 0.3$ if $x \in [0, 40]$, $\gamma = 1$ if $x \notin [0, 40]$ and (ii) $\gamma = 1$ if $x \in [10, 20]$, $\gamma = 0.3$ if $x \notin [10, 20]$, respectivelly.

From Fig. 7 and Fig. 8 we can see that an epidemic dies out as $t \to \infty$ where the values of γ satisfies $\gamma \geq 1$.

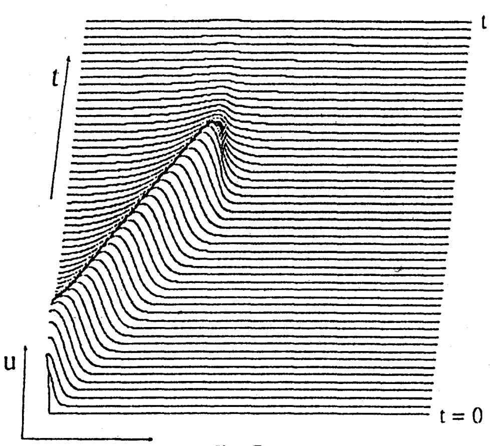


Fig. 7

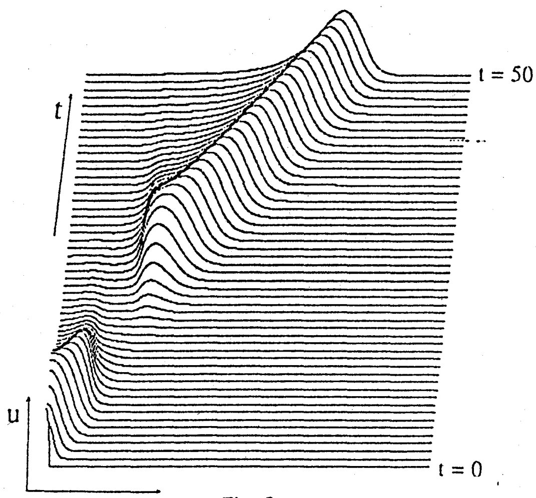


Fig. 8

In the following we assume $\gamma(x)$ to be given by the stepwise periodic function as follows

$$\gamma(x) = \varepsilon_1 < 1 \quad \text{for } x_{2m} < x < x_{2m+1}$$

$$\gamma(x) = \varepsilon_2 \geq 1 \quad \text{for } x_{2m+1} < x < x_{2m+2} \qquad m = 0, 1, 2, \ldots\ldots$$

where $x_0 = 0$ and $x_{i+1} = x_i + l_i$, with l_i denoting the width of the i-th patch (i is any integer). And we put that $l_{2m} = l_1(\text{const.})$, $l_{2m+1} = l_2\,(\text{const.})$ for all m so that $\gamma(x)$ is the periodic step function as shown in Fig. 9, where we set $L = l_1 + l_2$.

In this case, the invasion condition of an epidemic on the (l_1, l_2) - space is shown in Fig. 10, where $d_1 = 1$, $d_2 = 0.5$, $\varepsilon_1 = 0.3$ and $\varepsilon_2 = 1.5$, Invasion succeeds in the hatched regions and invasion fails in the unhatched region.

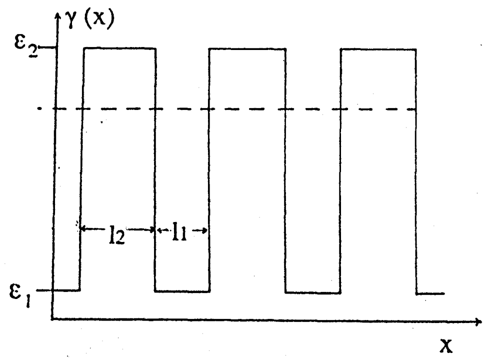


Fig.9

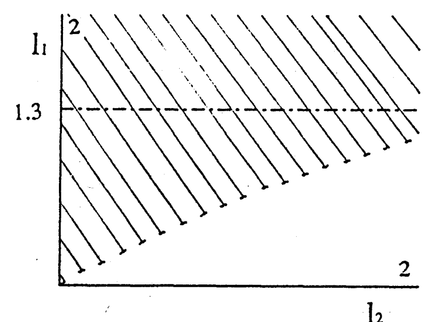


Fig.10

Figs. 11 and 12 are the numerical results of initial value problems of (1.2) for (i) $d_1 = 1$, $d_2 = 0.5$, $l_1 = 1$, $l_2 = 2$, $\varepsilon_1 = 0.3$, $\varepsilon_2 = 1.5$,where these values of parameters are in the unhatched regoins and (ii) $d_1 = 1$, $d_2 = 0.5$, $l_1 = 3$, $l_2 = 6$, $\varepsilon_1 = 1$, $\varepsilon_2 = 1.5$, where the values of parameters are in the hatched regoins.

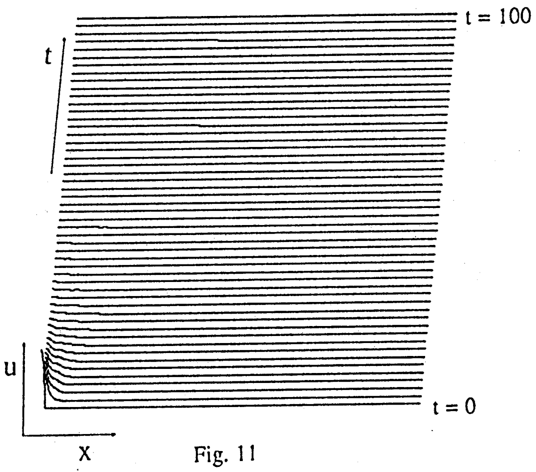


Fig. 11

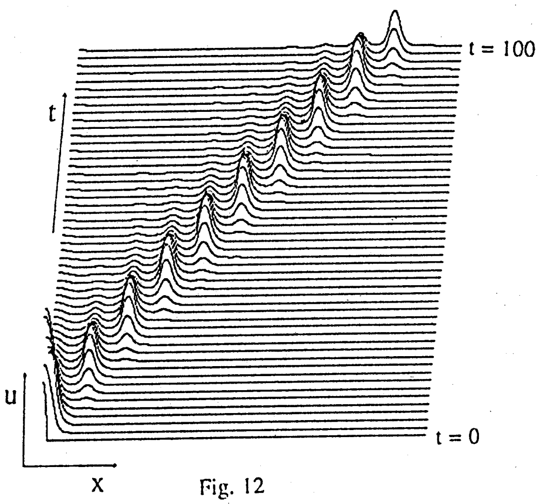


Fig. 12

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