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LOGICAL FORMULAS FOR PETRI NET ω-LANGUAGES

Hideki YAMASAKI
(山崎　秀記，　一橋大学)
Department of Mathematics, Hitotsubashi University
Kunitachi, Tokyo 186 Japan

Abstract

In this paper, we study Petri net ω-languages and logical formulas defining ω-languages. We consider some accepting conditions for Petri nets, and characterize the classes of Petri net ω-languages with these accepting conditions by logical formulas.

1 Preliminary

The set of integers {0, 1, -1, 2, -2, ...} is denoted by $\mathbf{Z}$, and the set of nonnegative integers is denoted by $\mathbf{N}$. For sets X and Y, Y^X denotes the set $\{f \mid f : X \to Y\}$ of all functions from X to Y. For a finite set $X = \{x_1, x_2, \ldots, x_n\}$, a function $f \in \mathbf{Z}^X$ is identified with the n-dimensional vector $\langle f(x_1), f(x_2), ..., f(x_n)\rangle$. Then for functions $f, g \in \mathbf{Z}^X$ and $z \in \mathbf{Z}$, the addition $f+g$, the scalar product zf, and the partial ordering $f \leq g$ are defined componentwise as usual.

Let Σ be an alphabet. We call a mapping $\alpha \in \Sigma^{\mathbf{N}}$ an *ω-word* over Σ, and write $\alpha = a_0a_1a_2\cdots$ where $a_n = \alpha(n)$ for each n. The set of all ω-words over Σ is denoted by Σ^ω, and that of all finite words over Σ is denoted by Σ^* as usual.

If $u = \alpha(0)\ldots\alpha(n)$ for some n, then u is called a *prefix* of α and we write $u < \alpha$. For $\alpha \in \Sigma^\omega$, we define $\downarrow\alpha = \{v \in \Sigma^* \mid v < \alpha\}$, $\underline{\alpha} = \{a \mid a = \alpha(n) \text{ for some } n\}$, and $\underline{\underline{\alpha}} = \{a \mid a = \alpha(n) \text{ for infinitely many } n\}$. For $L \subseteq \Sigma^\omega$, we define $\downarrow L = \bigcup_{\alpha\in L}\downarrow\alpha$.

For $K \subseteq \Sigma^*$ and $L \subseteq \Sigma^\omega$, we define $KL = \{u\alpha \mid u \in L \text{ and } \alpha \in K\}$ and $K^\omega = \{v_1v_2\ldots \mid v_1, v_2, \ldots \in K - \{\epsilon\}\}$, where $u\alpha$ is the ω-word obtained by concatenating u before α, and $v_1v_2\ldots$ is the ω-word obtained by concatenating $v_1, v_2, \ldots$ one after another.

We can consider Σ^ω a metric space with the distance d defined by:

$$d(\alpha,\beta) = \begin{cases} 0, & \text{if } \alpha = \beta \\ 2^{-k}, & \text{if } \alpha \neq \beta \text{ and} \\ & k = Min\{n \mid \alpha(n) \neq \beta(n)\}. \end{cases}$$

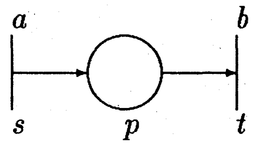
Then $L \subseteq \Sigma^\omega$ is a closed set if and only if $L = \{\alpha \mid \downarrow\alpha \subseteq \downarrow L\}$.

In this paper, when we mention a net or a *Petri net* N, we mean a marked λ-free labelled Petri net $N = (P, T, A, e, m_0, F)$, where P is a finite set of *places*, T a finite set of *transitions*, $A : T \to \mathbf{N}^P \times \mathbf{N}^P$, $e \in \Sigma^T$ a λ-free *labelling function*, $m_0 \in \mathbf{N}^P$ an *initial marking*, and $F \subseteq \mathbf{N}^P$ a finite set of *accepting markings*.

A *marking* m of a Petri net N is a function in $\mathbf{N}^P$, i.e., an assignment of tokens to the places. We say that the place p has $m(p)$ tokens at the marking m. For each transition t, $A(t) = \langle {}^\bullet A(t), A(t)^\bullet\rangle$ assigns a pair of functions ${}^\bullet A(t)$ and $A(t)^\bullet$ called the *input* and *output vector* of t, respectively.

Example 1
Let $N = (\{p\}, \{s,t\}, A, e, \langle 0\rangle, \{\langle 2\rangle\})$, *where* $A(s) = \langle\langle 0\rangle, \langle 1\rangle\rangle$, $A(t) = \langle\langle 1\rangle, \langle 0\rangle\rangle$, $e(s) = a$

and $e(t) = b$. Then the Petri net N is illustrated as follows.



A transition t is *fireable* in a marking m if $m \geq {}^{\bullet}A(t)$, and if so, t may *be fired* at m resulting in the marking

$$m' = m - {}^{\bullet}A(t) + A(t)^{\bullet}.$$

In this case, we write $m[t\rangle$ or $m[t\rangle m'$. Intuitively, t removes ${}^{\bullet}A(t)(p)$ tokens from the place p, and distributes $A(t)^{\bullet}(p)$ tokens to p, when t fires.

The definitions and notations are extended to finite or infinite sequences of transitions. That is, $m[t_1t_2\ldots t_n\rangle$ or $m[t_1t_2\ldots t_n\rangle m'$ if $m[t_1\rangle m_1[t_2\rangle m_2 \ldots m_{n-1}\ [t_n\rangle m'$, and $m[\alpha\rangle$ if $m[\alpha(0)\rangle m_1[\alpha(1)\rangle m_2 \ldots$.

We define infinite behaviour of a Petri net N as the homomorphic image of infinite firing sequences by the λ-free labelling function e. For a Petri net $N = (P, T, A, e, m_0, F)$ and $\alpha \in T^{\omega}$, we define $N(\alpha) = m_0m_1m_2\ldots$ if $m[\alpha(0)\rangle m_1[\alpha(1)\rangle m_2 \ldots$. Let

$$\uparrow F = \{m' \mid m' \geq m \text{ for some } m \in F\}.$$

Then we consider the following five types of ω-languages accepted by N:

$$L_0(N) = \{e(\alpha) \mid m_0[\alpha\rangle\},$$

$$L_1(N) = \{e(\alpha) \mid \underline{N(\alpha)} \cap \uparrow F \neq \phi\},$$

$$L_2(N) = \{e(\alpha) \mid \underline{N(\alpha)} \subseteq \uparrow F\},$$

$$L_3(N) = \{e(\alpha) \mid \underline{\underline{N(\alpha)}} \cap \uparrow F \neq \phi\},$$

$$L_4(N) = \{e(\alpha) \mid \underline{\underline{N(\alpha)}} \subseteq \uparrow F\}.$$

We define $\mathbf{P}_i = \{L_i(N) \mid N \text{ is a Petri net over } \Sigma\}$ $(i = 0, \ldots, 4)$. The accepting conditions considered in [2, 3] are defined by F instead of $\uparrow F$.

Example 2 *For the Petri net N in the previous example, $L_0(N) = \{\alpha \mid \#_a(u) \geq \#_b(u)$ for any $u < \alpha\}$, $L_1(N) = L_0(N) - (ab)^{\omega}$, $L_2(N) = \phi$, $L_3(N) = L_0(N) - D(ab)^{\omega}$, and $L_4(N) = \{u \mid \#_a(u) = \#b(u)+2\}L_0(N) \cap L_0(N)$, where $\#_a(u)$ is the number of occurrence of the letter a in the string u, and D is the Dyck set over $\{a, b\}$.*

Let $M = (Q, \Sigma, \delta, s, F)$ be a *nondeterministic finite automaton* with the finite set Q of *states*, the *input alphabet* Σ, the *transition relation* $\delta \subseteq Q \times \Sigma \times Q$, the *initial state* s, and the set F of *accepting states*. Any $\alpha = \langle q_0, a_0, p_0\rangle\langle q_1, a_1, p_1\rangle\langle q_2, a_2, p_2\rangle \ldots \in \delta^{\omega}$ is called a run of M, if $q_0 = s$ and $p_i = q_{i+1}$ for any i. For a run α of M, we define $M(\alpha) = q_0q_1q_2q_3\ldots$ and $\Sigma(\alpha) = a_0a_1a_2\ldots$.

Then we can also define the following five types of ω-languages accepted by M:

$$L_0(M) = \{\Sigma(\alpha) \mid \alpha \text{ is a run of } M\},$$

$$L_1(M) = \{\Sigma(\alpha) \mid \underline{M(\alpha)} \cap F \neq \phi\},$$

$$L_2(M) = \{\Sigma(\alpha) \mid \underline{M(\alpha)} \subseteq F\},$$

$$L_3(M) = \{\Sigma(\alpha) \mid \underline{\underline{M(\alpha)}} \cap F \neq \phi\},$$

$$L_4(M) = \{\Sigma(\alpha) \mid \underline{\underline{M(\alpha)}} \subseteq F\}.$$

We define $\mathbf{E}_i = \{L_i(M) \mid M$ is a nondeterministic finite automaton over $\Sigma\}$ $(i = 0, \ldots, 4)$.

2 Inclusion relations

In the case of ω-languages accepted by nondeterministic finite automata, it is known that $\mathbf{E}_0 = \mathbf{E}_2 \subset \mathbf{E}_1 = \mathbf{E}_4 \subset \mathbf{E}_3$ [4, 5, 7]. We show the similar results for the classes $\mathbf{P}_i$ of Petri net ω-languages.

As a tool of the proofs in this section, we define a new accepting condition for a Petri net, which is described by a language over transitions. Let $N = (P, T, A, e, m_0, \phi)$ and $R \subseteq T^\omega$. We define

$$L(N, R) = \{e(\alpha) \mid m_0[\alpha\rangle \text{ and } \alpha \in R\}.$$

In the proof of the following theorems, we use the following notations to simplify the description. For $f \in \mathbf{Z}^X$ and $g \in \mathbf{Z}^Y$ $f \oplus g$ denotes the function in $\mathbf{Z}^{X \cup Y}$, defined by

$$f \oplus g(x) = \begin{cases} f(x) + g(x), & \text{if } x \in X \cap Y \\ f(x), & \text{if } x \in X \\ g(x), & \text{if } x \in Y. \end{cases}$$

For $n \in \mathbf{N}$ and a set X, n^X denote the constant function in $\mathbf{N}^X$ such that $n^X(x) = n$ for any $x \in X$. If X is a singleton $\{x\}$, then we write n^x instead of $n^{\{x\}}$. Thus, for example, for $p_0 \in P$,

$$0^P \oplus 1^{p_0}(p) = \begin{cases} 1, & \text{if } p = p_0 \\ 0, & \text{if } p \neq p_0. \end{cases}$$

Theorem 1 *For any* $i = 0, \ldots, 4$, $\mathbf{P}_i = \{L(N, R) \mid N$ *is a Petri net and* $R \in \mathbf{E}_i\}$.

Proof. Let $N = (P, T, A, e, m_0, \phi)$ and $M = (Q, \Sigma, \delta, s, F)$ be a finite automaton such that $L_i(M) = R$. We define the Petri net $N' = (P \cup Q, \delta, A', e', m_0 \oplus 1^s \oplus 0^Q, \{0^P \oplus 1^q \oplus 0^Q \mid q \in F\})$, where $A'(\langle q, t, q' \rangle) = \langle {}^\bullet A(t) \oplus 1^q \oplus 0^Q, A(t)^\bullet \oplus 1^{q'} \oplus 0^Q \rangle$, and $e'(\langle q, t, q' \rangle) = e(t)$ for any $\langle q, t, q' \rangle$. Intuitively, N' is a product of N and M, and simulates N and M, simultaneously. Thus it is clear that $L(N, R) = L(N, L_i(M)) = L_i(N')$.

Let $N = (P, T, A, e, m_0, F)$ and $L = L_i(N)$. For each $t \in T$ and $m \in F$, we add new transition t_m to N, such that $m_1[t_m\rangle m_2$ if and only if $m_1 \geq m$ and $m_1[t_m\rangle m_2$. Since $m_1 \geq m \in F$ means $m_1 \in \uparrow F$, t_m works same as t, and can check whether the current marking is in $\uparrow F$ or not.

We construct $N' = (P, T \cup T_F, A', e', m_0, \phi)$, where $T_F = \{t_m \mid t \in T \text{ and } m \in F\}$, $A'(t) = A(t)$ and $e'(t) = e'(t_m) = e(t)$, for each $t \in T$ and $m \in F$. Moreover, $A'(t_m) = \langle {}^\bullet A'(t_m), A'(t_m)^\bullet \rangle$ with

$${}^\bullet A(t_m)(p) = Max({}^\bullet A(t)(p), m(p)),$$

$$A(t_m)^\bullet(p) = {}^\bullet A(t_m)(p) + A(t)^\bullet(p) - {}^\bullet A(t)(p),$$

for any $p \in P$.

Then it is clear that $L_0(N) = L(N', T^\omega)$, $L_1(N) = L(N', T^* T_F T^\omega)$, $L_2(N) = L(N', T_F^\omega)$, $L_3(N) = L(N', (T^* T_F)^\omega)$, $L_4(N) = L(N', T^* T_F^\omega)$. □

Corollary 2 $\mathbf{P}_0 = \mathbf{P}_2 \subseteq \mathbf{P}_1 = \mathbf{P}_4 \subseteq \mathbf{P}_3$.

Proof. It is clear from the Theorem 1 and the results for $\mathbf{E}_i$'s. □

In the sequel, we only consider the case $i = 0, 1, 3$. To prove the strict inclusions between these classes, we prove the following topological properties of the classes $\mathbf{P}_0$ and $\mathbf{P}_1$.

Lemma 3 *For any Petri net* N, $L_0(N)$ *is a closed set, and* $L_1(N)$ *is a denumerable union of closed sets.*

Proof. Let $N = (P, T, A, e, m_0, F)$, and $\downarrow\alpha \subseteq \downarrow L_0(N)$. We will show that $\alpha \in L_0(N)$. Consider the set $C = \{w \mid e(w) < \alpha, \text{ and } m_0[w\rangle\}$ of all the fireable finite sequences generating the prefixes of α. Then C is infinite. By König's Lemma, there exists $\beta \in T^\omega$ such that $\downarrow\beta \subseteq C$. It means that $m_0[\beta\rangle$ and $e(\beta) = \alpha$. Hence $\alpha \in L$.

Let $N_m = (P, T, A, e, m, F)$ for $m \in \mathbf{N}^P$. Then, $L_1(N) = \bigcup\{e(w) L_0(N_m) \mid m_0[w\rangle m \in \uparrow F\}$, which is a denumerable union of closed sets. □

Then the next theorem follows from the topological characterizations of ω-regular languages [4, 5].

Theorem 4 $\mathbf{P}_0 = \mathbf{P}_2 \subset \mathbf{P}_1 = \mathbf{P}_4 \subset \mathbf{P}_3$.

Theorem 5 *The classes* $\mathbf{P}_i$ $(i = 0,1,3)$ *of Petri net* ω*-languages are closed under union, intersection, and projection.*

Proof. Let $N_j = (P_j, T_j, A_j, e_j, m_j, \phi)$ for $j = 1,2$. We define a Petri net N which can simulate N_1 and N_2 simultaneously, as follows. $N = (P_1 \cup P_2, T, A, e, m_1 \oplus m_2, \phi)$, where $T = \{\langle t_1, t_2\rangle \in T_1 \times T_2 \mid e_1(t_1) = e_2(t_2)\}$, $A'(\langle t_1, t_2\rangle) = \langle {}^\bullet A_1(t_1) \oplus {}^\bullet A_2(t_2), A_1(t_1)^\bullet \oplus A_2(t_2)^\bullet\rangle$, $e(\langle t_1, t_2\rangle) = e_1(t_1)$, for any $\langle t_1, t_2\rangle \in T$. For any $R_j \subseteq T_j^\omega$ $(j = 1,2)$, let $R_\cup = \{\alpha \in T^\omega \mid \alpha_1 \in R_1 \text{ or } \alpha_2 \in R_2\}$, $R_\cap = \{\alpha \in T^\omega \mid \alpha_1 \in R_1 \text{ and } \alpha_2 \in R_2\}$, where α_j is the ω-word over T_j obtained by concatenating j-th elements of $\alpha(i)$ for $i = 0,1,\ldots$. Then it is clear that $L(N_1, R_1) \cup L(N_2, R_2) = L(N, R_\cup)$, $L(N_1, R_1) \cap L(N_2, R_2) = L(N, R_\cap)$.

The closure under projection is clear from the definition. □

3 Normal form of Petri nets

We define a normal form of Petri nets and show that any Petri net can be transformed into a normal form Petri net.

We say that a Petri net $N = (P,T,A,e,m_0,F)$ is in *normal form* if

1) there exists a place $p_0 \in P$ such that $m_0 = 1^{p_0} \oplus 0^P$,

2) there exists a place p_f such that $F = \{1^{p_f} \oplus 0^P\}$,

3) for any transition t fireable at markings in $\uparrow F$, ${}^\bullet A(t)(p_f) = 1$,

4) for any $p \in P$, and $t \in T$, ${}^\bullet A(t)(p) \leq 1$ and $A(t)^\bullet(p) \leq 1$, that is, each place p gets or lose at most one token at once.

Theorem 6 *For any Petri net* N*, we can construct a Petri net* N' *in normal form such that* $L_i(N) = L_i(N')$ *for any* i = *0, 1, 3.*

Proof. First we show that any Petri net $N = (P, T, A, e, m_0, F)$ can be transformed into a Petri net $N' = (P', T', A', m_0', e', F')$ which satisfies the conditions 1), 2) and 3). Let $P' = P \cup \{p_0, p_c, p_f\}$ and $T' = T \cup \{t' \mid m_0[t\rangle\} \cup \{t'' \mid t \in T\} \cup \{t_m \mid t \in T \text{ and } m \in N\}$. We define

$$\begin{aligned}
A'(t) &= \langle {}^\bullet A(t) \oplus 0^{p_0} \oplus 1^{p_c} \oplus 0^{p_f}, \\
&\qquad A(t)^\bullet \oplus 0^{p_0} \oplus 1^{p_c} \oplus 0^{p_f}\rangle \\
A'(t') &= \langle 0^P \oplus 1^{p_0} \oplus 0^{p_c} \oplus 0^{p_f}, \\
(m_0 &- {}^\bullet A(t) + A(t)^\bullet) \oplus 0^{p_0} \oplus 1^{p_c} \oplus 0^{p_f}\rangle, \\
A'(t'') &= \langle {}^\bullet A(t) \oplus 0^{p_0} \oplus 0^{p_c} \oplus 1^{p_f}, \\
&\qquad A(t)^\bullet \oplus 0^{p_0} \oplus 1^{p_c} \oplus 0^{p_f}\rangle, \\
A'(t_m) &= \langle {}^\bullet A(t_m) \oplus 0^{p_0} \oplus 1^{p_c} \oplus 0^{p_f}, \\
&\qquad A(t_m)^\bullet \oplus 0^{p_0} \oplus 0^{p_c} \oplus 1^{p_f}\rangle,
\end{aligned}$$

$$e'(t) = e'(t) = e'(t'') = e'(t_m) = e(t),$$

$$m_0' = 1^{p_0} \oplus 0^P, \text{ and } F = \{1^{p_f} \oplus 0^P\}.$$

Then the Petri net N' satisfies 1),2) and 3), and it is clear from the construction that $L_i(N) = L_i(N')$ for i = 0, 1, 3.

Next we show that we can decrease the number of places $q \in P'$ such that $Max\{{}^\bullet A'(t)(q), A'(t)^\bullet(q) \mid t \in T\} = n > 1$. Repeating the process, we can transform N' into a Petri net in normal form.

To construct $N''=(P'',T'',A'',e'',m_0'',F'')$, we replace q by n new places $q_1, q_2, \cdots, q_n$. For each transition t, let D_i $(1 \leq i \leq k_t)$ and E_j $(1 \leq j \leq l_t)$ be the enumerations of the subsets of $\{q_1, q_2, \cdots, q_n\}$ with ${}^\bullet A'(t)(q)$ and $A'(t)^\bullet(q)$ elements, respectively. Then we also replace the transition t by $n_t \times m_t$ transitions $t_{i,j}$ $(1 \leq i \leq k_t, 1 \leq j \leq l_t)$ such that,

$${}^\bullet A''(t_{i,j})(p) = \begin{cases} {}^\bullet A'(t)(p), & \text{if } p \neq q \\ 1, & \text{if } p \in D_i \\ 0, & \text{if } p \notin D_i \end{cases}$$

$$A''(t_{i,j})^\bullet(p) = \begin{cases} A'(t)^\bullet(p), & \text{if } p \neq q \\ 1, & \text{if } p \in E_j \\ 0, & \text{if } p \notin E_j \end{cases}$$

and $e''(t_{i,j}) = e(t)$.

Note that on the Petri net N'', the tokens in q on N' are distributed to the places $q_1, q_2, \cdots, q_n$, and the arcs from or to q in N' are also distributed to these places.

It is easy to see that the $L_i(N') = L_i(N'')$ for $i = 0, 1, 3$. □

4 Characterizations by logical formulas

We define the monadic second-order theory K over an alphabet Σ for natural numbers, which is introduced by Parigot and Pelz [2, 3]. K has two sorts of variables, number variables $x, y, \ldots$ ranging over $\mathbf{N}$, and set variables $X, Y, \ldots$ ranging over the power set of $\mathbf{N}$. K also has set constants P_a for each $a \in \Sigma$.

The *terms* of K are expressions of form n or $x + n$, where x is a number variable and n is a constant in $\mathbf{N}$. The *atomic formulas* of K are expressions of form $u \leq t$, $t \in W$ or $V \preceq W$, where u, t are terms and V, W are set variables or P_a for some $a \in \Sigma$. Here, $\leq$ and $\in$ are usual 'less than or equal to' and 'belong to' relations, and $V \preceq W$ is true if and only if there exists a one to one function $f : W \to V$ such that $f(x) \leq x$ for any $x \in W$.

The formulas of K, called K-*formulas*, are defined as usual. That is, $\varphi \wedge \psi, \varphi \vee \psi, \neg\varphi, \forall x\varphi, \exists x\varphi, \boldsymbol{\forall} X\varphi, \boldsymbol{\exists} X\varphi$ are formulas for any formula or atomic formula φ, ψ, number variable x and set variable X. We use boldface quantifier symbols $\boldsymbol{\forall}$ and $\boldsymbol{\exists}$ for set variables to distinguish from those for number variables.

Note that the K-formulas not containing the symbol $\preceq$ is the S1S-*formulas* considered in Büchi [1].

We say that an ω-word $\alpha \in \Sigma^\omega$ satisfies K-sentence (i.e., formulas without free variables) ψ, if ψ is true under the interpretation $P_a = \{n \mid \alpha(n) = a\}$. Then, K-sentence ψ define the set $L(\psi)$ of all ω-words satisfying ψ. For a set of K-formulas Δ, we define that $\mathbf{L}(\Delta) = \{L(\psi) \mid \psi \in \Delta\}$, the class of ω-languages defined by the sentences in Δ.

For a language R over quantifier symbols $\{\forall, \exists, \boldsymbol{\forall}, \boldsymbol{\exists}\}$, $[R]$ denotes the set of S1S-formulas of the prenex normal form

$$\Xi_1\xi_1\Xi_2\xi_2 \cdots \Xi_n\xi_n \ \psi(\xi_1, \xi_2, \cdots, \xi_n),$$

where $\Xi_1\Xi_2\cdots\Xi_n$ is a string in R, and ψ is a quantifier-free formula.

On the relation between S1S-formulas and ω-regular languages, we have shown the following theorem [6].

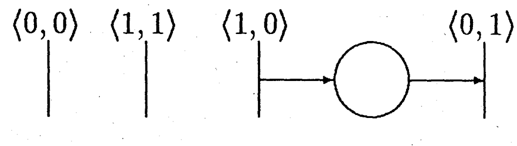
Theorem 7 $\mathbf{E}_0 = \mathbf{L}([\exists^*\boldsymbol{\forall}])$, $\mathbf{E}_1 = \mathbf{L}([\exists^*\exists\boldsymbol{\forall}])$, $\mathbf{E}_3 = \mathbf{L}([\exists^*\boldsymbol{\forall}\exists])$.

For any $\alpha \in (\Sigma_1 \times \Sigma_2 \times \ldots \times \Sigma_n)^\omega$, α_i is defined to be the ω-words obtained by concatenating the i-th elements of $\alpha(j)$ for $j = 0, 1, 2, \ldots$. We say that $\alpha \in (\{0,1\}^{n+k} \times \Sigma)^\omega$ satisfies the formula $\psi(X_1, \ldots, X_n, x_1, \ldots, x_k)$, if $\alpha_{n+1} \in \Sigma^\omega$ satisfies $\psi(C_1, \ldots, C_n, d_1, \ldots, d_k)$, where $C_i = \{j \mid \alpha_i(j) = 1\}$ for $i = 1, \ldots, n$ and $\alpha_{n+i}(j) = 1$ if and only if $j = d_i$ for $i = 1, \ldots, k$. We write $L(\psi) = \{\alpha \mid \alpha \text{ satisfies } \psi\}$.

Now, we show the main theorem. Let Δ be a set of formulas and $\overline{\Delta}$ be the smallest set of formulas constructed from the atomic formulas $V \preceq W$ and formulas in Δ using $\wedge$, $\vee$, $\exists$ and $\boldsymbol{\exists}$.

Theorem 8 *If* $\mathbf{L}(\Delta) = \mathbf{E}_i$ *then* $\mathbf{L}(\overline{\Delta}) = \mathbf{P}_i$ *for* $i = 0, 1, 3$.

Proof. $\mathbf{E}_i \subseteq \mathbf{P}_i$ from Theorem 1. Moreover the ω-language $L(X_1 \preceq X_2) \subseteq (\{0,1\}^2)^\omega$ is accepted by the following Petri net without the accepting condition.



Since each class of Petri net ω-languages is closed under union, intersection and projection, we have shown that the half part of the theorem.

Let $N = (P, T, A, e, 0^P \oplus 1^{p_0}, \{0^P \oplus 1^{p_f}\})$ be a Petri net in normal form. Note that in normal form Petri net, each place p can get or lose at most one token at once. To describe the infinite behaviour of the net N, we use the following set variables,

- X_t to represent the time t fires,
- E_p to represent the time p gets a new token,
- S_p to represent the time p loses one token,

for each $t \in T$ and $p \in P$. Let

$$\psi_1(x) = \bigvee_{t \in T} ((x \in X_t) \wedge (x \in P_{e(t)}) \wedge (\bigwedge_{t' \neq t} \neg(x \in X_{t'})))$$

which means that there exists a unique transition t that fires at time x.

$$\psi_2(x) = \bigwedge_{p \in P} ((x \in S_p) \leftrightarrow \bigvee_{{}^\bullet A(t)(p) = 1} (x \in X_t))$$

which means that a place p loses a token at time x if and only if a transition t with ${}^\bullet A(t)(p) = 1$ fires at the same time x.

$$\psi_3(x) = \bigwedge_{p \in P} ((x + 1 \in E_p) \leftrightarrow \bigvee_{A(t)^\bullet(p) = 1} (x \in X_t))$$

which means that a place p gets a token at time $x + 1$ if and only if a transition t with $A(t)^\bullet(p) = 1$ fires at time x.

$$\psi_4 = (0 \in E_{p_0}) \wedge (\bigwedge_{p \neq p_0} \neg(0 \in E_p))$$

which represents the condition for the initial marking.

$$\varphi_0 = true,$$

$$\varphi_1(y) = (y \in E_{p_f}),$$

$$\varphi_3(x, y) = (x \leq y) \wedge (y \in E_{p_f}),$$

here φ_i is a formula to represent the accepting condition of type i, $(i = 0, 1, 3)$. Finally,

$$\psi_5 = \bigwedge_{p \in P} (E_p \preceq S_p)$$

which means that each place p can lose only tokens which got previously.

Then, $L_0(N)$ is defined by

$$\exists_{t \in T} X_t \exists_{p \in P} E_p \exists_{p \in P} S_p (\forall x \; (\psi_1(x) \wedge \psi_2(x) \wedge \psi_3(x) \wedge \psi_4 \wedge \varphi_0) \wedge \psi_5).$$

$L_1(N)$ is defined by

$$\exists_{t \in T} X_t \exists_{p \in P} E_p \exists_{p \in P} S_p (\exists y \forall x \; (\psi_1(x) \wedge \psi_2(x) \wedge \psi_3(x) \wedge \psi_4 \wedge \varphi_1(y)) \wedge \psi_5).$$

$L_3(N)$ is defined by

$$\exists_{t \in T} X_t \exists_{p \in P} E_p \exists_{p \in P} S_p (\forall x \exists y \; (\psi_1(x) \wedge \psi_2(x) \wedge \psi_3(x) \wedge \psi_4 \wedge \varphi_3(y)) \wedge \psi_5).$$

Note that all $\psi_1, \ldots, \psi_4$ and φ_i $(i = 0, 1, 3)$ are S1S-formulas with no quantifiers. From Theorem 7, this completes the proof. □

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