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ネットワークにおける辺の重要度の評価について

On the Importance of Each Edge Using Its Traffic along Shortest Paths in a Network

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1 Introduction

In our daily life, we may usually select some shortest path from A to B in order to travel from a place A to a place B in a road network. It implies that the traffic passing through each road interval based on some rule of selecting shortest paths is considered as a measure of the importance of each road interval for a road network.

We formalize a road network as a digraph, namely, directed graph $G=(V,E)$ with two specified *source* s and *sink* t, where each edge e has a positive real length $l(e)$(, namely $l(e)>0$). As there may be a large number of shortest paths between two vertices in a road network, a user will select one of them to suit his own convenience. To describe the user's preference among shortest paths, we define a distribution function α_v: $f(E_v^-)\longmapsto f(E_v^+)$ at each vertex v, where $f(e)$ denotes the traffic passing through an edge e with respect to source s and sink t, E_v^- (E_v^+) represents the set of edges entering vertex v (leaving vertex v), and $f(E_v^-)$ $(f(E_v^+))$ represents an $|E_v^-|$ $(|E_v^+|)$ dimensional vector consisting of traffics $f(e)$'s passing through all edges e's in E_v^- (E_v^+). Furthermore, we assume that the distribution function α_v at each vertex v is computed in $O(\beta_v)$ time where β_v is a function of the input size, that α_v satisfies the conservation constraint(, namely, for each vertex v in a digraph G, the amount of the traffics entering vertex v is equal to the amount of the traffics leaving vertex v), and that the traffic $f(e)$ passing through each edge e is treated as a single commodity flow with respect to source s and sink t. Thus, we define the problem **IETSP** as follows.

Input: A digraph $G=(V,E)$ with source s and sink t where each edge e has a positive real length $l(e)(>0)$ and each vertex v has a distribution function α_v, and a required traffic $\mathcal{F}_{st}$ from source s to sink t.

Output: The traffic $f(e)$ passing through each edge e in order to move the required traffic $\mathcal{F}_{st}$ along only shortest paths from source s to sink t by using the distribution function α_v at each vertex v.

Related measures of the importance of each edge, e.g., the number of shortest paths passing through each edge [1, 4] and that of minimum spanning trees containing each edge [2], etc., in a network have been investigated.

In this paper, we propose a polynomial time algorithm of solving the problem **IETSP**, based on the property of the topological sort of vertices of an acyclic digraph.

2 An Algorithm for IETSP

The basic idea of the algorithm described in this section for solving problem **IETSP** is that we construct an acyclic subdigraph G_{st} from a digraph G by deleting *redundant* edges(, namely, edges not contained in any shortest path from s to t), and assign the traffic $f(e)$ passing through each edge e based on a topological sort of vertices in the acyclic subgraph G_{st}.

For a digraph $G = (V, E)$, let $d(u, v)(\forall u, v \in V)$ denote the length of the shortest path from u to v, where we assume that $d(u, u) = 0$ for any $u \in V$ and that if there is no path from u to v then $d(u, v) = \infty$. Let $L(\pi)$ denote the length of a path π from u to v, namely, the sum of lengths of edges in a path π. It is well-known that the lengths $d(u, v)$'s $(\forall u, v \in V)$ of shortest paths for all pairs of vertices over V are found in polynomial time (see e.g. [5]). Furthermore, we prove the following Lemma 1.

Lemma 1. In a digraph $G = (V, E)$ with source s and sink t, there is a shortest path from s to t containing an edge $(u, v) \in E$ if, and only if, $d(s, t) \neq \infty$ and $d(s, u) + l((u, v)) + d(v, t) = d(s, t)$ hold.

Proof. Necessity: Assume that G has a shortest path π from s to t containing edge (u, v), and let π : $\pi_{su}, (u, v), \pi_{vt}$, where π_{su} and π_{vt} are paths from s to u and from v to t, respectively. Clearly, $d(s, u) \neq \infty$ and $d(v, t) \neq \infty$ hold. As $d(s, t) = L(\pi_{st}) = L(\pi_{su}) + l((u, v)) + L(\pi_{vt})$ holds, we have $L(\pi_{su}) = d(s, u)$ and $L(\pi_{vt}) = d(v, t)$. Otherwise, $d(s, t) > L(\pi_{st})$ holds, contradicting the assumption.

Sufficiency is obvious. □

By Lemma 1, we can delete all redundant edges with respect to s, t from a digraph G, based on the lengths $d(u, v)$'s of shortest paths in G, and obtain the following subdigraph $G_{st} = (V_{st}, E_{st})$ of G having no redundant edge.

$$
\begin{aligned}
E_{st} &= \{ \ (u, v) \in E \mid d(s, t) \neq \infty \ \textit{and} \\
&\qquad d(s, u) + l((u, v)) + d(v, t) = d(s, t) \ \}, \\
V_{st} &= \{ \ v \in V \mid (u, v) \ \textit{or} \ (v, u) \in E_{st} \}.
\end{aligned}
$$

It is clear that, for a digraph $G = (V, E)$ with source s and sink t, if $d(u, v)$'s $(\forall u, v \in V)$ are known, then G_{st} is obtained in $O(|E|)$ time. Furthermore, by the definition of G_{st}, for any edge (u, v) of the subdigraph G_{st} of G, there is a shortest path from s to t containing edge (u, v) in G. The following Lemma 2 is also proved.

Lemma 2. In a digraph $G = (V, E)$ with source s and sink t, each shortest path from s to t in G corresponds one-to-one to each path from source s to sink t in the subdigraph G_{st} obtained from G.

Proof. Let a shortest path from s to t in G be π_{st} :

$$s, (s, v_1), v_1, (v_1, v_2), v_2, \cdots, (v_k, t), t.$$

Then any subpath $\pi_{sv_i}(1 \le i \le k)$ of π_{sv} is a shortest path from s to v_i in G, as, otherwise, π_{st} is not a shortest path. Thus, by the definition of G_{st}, all edges on π_{st} must be in E_s.

On the other hand, let a path from s to t in G_{st} be π_{st} :

$$s, (s, v_1), v_1, (v_1, v_2), v_2, \cdots, (v_k, t), t.$$

By the definition of G_{st},

$$\begin{aligned} d(s, s) + l((s, v_1)) &= d(s, v_1), \\ d(s, v_1) + l((v_1, v_2)) &= d(s, v_2), \\ &\vdots \\ d(s, v_k) + l((v_k, t)) &= d(s, t) \end{aligned}$$

hold, where $d(s, s) = 0$. Hence, we have

$$L(\pi_{st}) = l((s, v_1)) + l((v_1, v_2)) + \cdots + l((v_k, t)) = d(s, t).$$

This means that π_{st} is a shortest path in G. □

Lemma 3. For a digraph $G = (V, E)$ with source s and sink t where each edge e has a positive real length $l(e)$, the subdigraph G_{st} obtained from G has no cycle, namely, is acyclic.

Proof. Assume that G_{st} has a cycle C. Let v be a vertex on C. Consider a shortest path π_{st} from s to t passing through vertex v, namely, $L(\pi_{st}) = d(s, t)$. Let π'_{st} be a path from s to t obtained by concatenating subpath π_{sv} of π_{st}, cycle C and subpath π_{vt} of π_{st}, where C is treated as a path from v to v. As $L(\pi'_{st}) = L(\pi_{sv}) + L(C) + L(\pi_{vt}) \le d(s, t) = L(\pi_{st})$ holds, we have $L(C) = 0$, which, however, contradicts the assumption that each edge e of G has a positive real length $l(e)(> 0)$. □

Note that for the subdigraph G_{st} obtained from a digraph $G = (V, E)$, the in-degree of source s is zero and out-degree of sink t is zero. A topological sort of vertices [3] in the acyclic subdigraph G_{st} must start from source s and end at sink t, namely,

$$v_1(= s), v_2, v_3, \cdots, v_{|V_{st}|-1}, v_{|V_{st}|}(= t).$$

Lemma 4[3]. For a digraph $G = (V, E)$ with source s and sink t, any topological sort of vertices: $v_1(= s), v_2, \cdots, v_{|V_{st}|-1}, v_{|V_{st}|}(= t)$ in the subdigraph G_{st} obtained from G satisfies
(i) For any $i(1 \le i \le |V_{st}|)$, the tail v_j of any edge with head v_i is to the left of v_i, namely, for any edge (v_j, v_i) of E, $j < i$ holds, and
(ii) Any path from $v_1(= s)$ to v_i can contain only some vertices of $(s =)v_1, v_2, v_3, \cdots, v_i$. □

Based on the above discussions, we describe the following algorithm for problem **IETSP**.

Algorithm IETSP

Input: A digraph $G = (V, E)$ with source s and sink t where each edge e has a positive real length $l(e)(> 0)$, a distribution function α_v at each vertex v and a required traffic $\mathcal{F}_{st}$ from source s to sink t.

Output: The traffic $f(e)$ passing through each edge e of E in order to assign the traffic $\mathcal{F}_{st}$ along only shortest paths from source s to sink t by the given distribution function α_v at each vertex v.

Begin

A1. For each edge (u, v) of E, $f((u, v)) := 0$.

A2. Compute the lengths $d(u, v)$'s of shortest paths for all vertex pairs $u, v(\in V)$ by applying the algorithm shown in [5].

A3. Construct $G_{st} = (V_{st}, E_{st})$ by deleting redundant edges based on the known values $d(u, v)$'s obtained in **A2**.

A4. Obtain a topological sort of vertices $v_1(= s), v_2, \cdots, v_{|V_{st}|-1}, v_{|V_{st}|}(= t)$ in G_{st} by executing the algorithm shown in e.g., [3].

A5. For $i = 1$ to $|V_{st}|$, obtain $f(E^+_{v_i})$ by computing $f_{v_i}(E^-_{v_i})$, and output $f(E^+_{v_i})$.

End. □

The correctness of **Algorithm IETSP** is easily derived by the above lemmas. Now, we analyze the time complexity of **Algorithm IETSP**. The time complexity of executing **A1**, **A3**, and **A4** in **Algorithm IETSP** is $O(|E|)$ by the above lemmas. **A2** is executed in $O(|V|^3(\log\log|V|/\log|V|)^{1/2})$ time by the algorithm shown in [5]. The time complexity of executing **A5** is $O(|V| \max_{v\in V} O(\beta_v))$ as $f_{v_i}(E^-_{v_i})$ is computed in $O(\beta_v)$ time where β_v is a function of the input size. By the above analysis of the time complexity, we obtain the following Theorem 1.

Theorem 1. Given a digraph $G = (V, E)$ with source s and sink t where each edge e has a positive real length $l(e)(> 0)$, and each vertex v has a distribution function α_v, and a required traffic $\mathcal{F}_{st}$ from source s to sink t. Then, we can compute the traffic $f(e)$ passing each edge e in $O(\max\{|V| \max_{v\in V} O(\beta_v), |V|^3(\log\log|V|/\log|V|)^{1/2}\})$ time by **Algorithm**

IETSP, in order to move the required traffic $\mathcal{F}_{st}$ along only shortest paths from s to t by the distribution function α_v at each vertex v. $\square$

It is clear that if a distribution function α_v at each vertex v is computed in polynomial time for the input size, **Algorithm IETSP** is polynomial. Note that the assumption that a distribution function α_v at each vertex v is computable in polynomial time, is not strong.

3 Conclusion

Based on the property of the topological sort of vertices of an acyclic digraph G_{st} obtained by removing redundant edges from G, we obtain a polynomial time algorithm for solving the problem **IETSP**. It is easy to see that the results obtained in this paper also hold if a digraph G contains no cycle of a negative length or a zero length.

It is obvious that the problem with respect to all vertex pairs similar to the problem **IETSP** with respect to one vertex pair $\{s, t\}$ can be solved in polynomial time by applying **Algorithm IETSP** $|V|^2$ times.

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