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Efficient Algorithms for Disjoint Paths in Hypercubes and Star Networks

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Abstract: Recently, graph parameters such as the number of disjoint paths between a pair of nodes (or between a node and a set of nodes and so on) and the length of these paths have been studied extensively due to their relations to (applications in) fault tolerance and transmission delay in communication networks. We give efficient algorithms for node disjoint path problems in hypercubes, star graphs, and incomplete star graphs which are defined to reduce the large gaps in the size of systems based on star graph topologies. Four disjoint path paradigms are discussed: (1) disjoint paths between a pair of nodes s and t, (2) disjoint paths from a node s to a set T of nodes, (3) disjoint paths from a set S of nodes to a set T of nodes, and (4) disjoint paths between node pairs (s_i, t_i), $1 \leq i \leq k$. We give algorithms for paradigms (3) and (4) in hypercubes and algorithms for all the paradigms in star graphs and incomplete star graphs. Our algorithms can find the maximum number of disjoint paths for these paradigms in optimal time. For an n-dimensional hypercube H_n, the length of the disjoint paths given by our algorithms is at most $2d(H_n) - 2 = 2n - 2$, where $d(G)$ is the diameter of the graph G. For an n-dimensional star graph G_n and an incomplete star graph $G_{n,m}$, the length of the disjoint paths constructed by our algorithms is at most $d(G_n) + c$ and $d(G_{n,m}) + c$, respectively, where c is a small constant.

Introduction

In the design and implementation of a large multiprocessor system, one of the key and most fundamental issues is the design of the interconnection network through which the processors can communicate efficiently. Taking the cost, reliability, fault tolerance, and transmission delay into consideration, a good interconnection network should be a sparse and homogeneous graph with connectivity as large as possible and diameter as small as possible. With the development of VLSI and fiber optics technologies, the size of multiprocessor system is increasing tremendously. The fault tolerance communication has become one of the central issues in today's multiprocessors system. Recently, some new parameters (will be given in the next section) which concerns a collection of disjoint paths (in this paper, disjoint path refers *node disjoint path* unless otherwisw mentioned) between a pair of nodes (or between a node and a set of nodes, and so on) have been introduced to evaluate the fault tolerance properties of interconnection networks [10, 18]. Much work has been done on finding the disjoint paths in general graphs as well as some special classes of interconnection networks [10, 9, 18, 4, 13]. In this paper, disjoint paths problems in hypercube, star graph, and incomplete star graph interconnection networks are considered.

Hypercubes and star graphs are interesting interconnection topologies for multiprocessor system [20, 2, 1, 19]. They possess rich recursive structure and symmetry properties as well as many desirable fault tolerance characteristics. In addition, with regard to the important properties of degree and diameter, the star graph is shown to be markedly superior. A problem with the star graph topology is that the number of nodes in a system must be the factorial of an integer. In practical terms, this is a severe restriction on the sizes of systems that can be built; there is a large gap between the numbers $(n-1)!$ and $n!$. This restriction can be overcome partially by introducing *incomplete star graph.* An incomplete star graph is a star graph missing certain of its substars. Since an n-dimensional star graph G_n has n $(n-1)$-dimensional substars, there are n possible n-dimensional incomplete star graphs $G_{n,m}$, $1 \leq m \leq n$, each of them is made of m $(n-1)$-dimensional substars and has $m(n-1)!$ nodes. [1] In this paper, we present efficient algorithms for hypercubes, star graphs, and incomplete star graphs for the following disjoint path paradigms:

1. Disjoint paths between a pair of distinct nodes s and t;

2. Disjoint paths between a node s and a set of distinct nodes $T = \{t_1, t_2, \ldots, t_k\}$;

3. Disjoint paths between a set of distinct nodes $S = \{s_1, s_2, \ldots, s_k\}$ and a set of distinct nodes $T = \{t_1, t_2, \ldots, t_k\}$; and

4. Disjoint paths between mutually distinct node pairs (s_1, t_1), (s_2, t_2), ..., (s_k, t_k).

The above paradigms have attracted much attentions in both mathematical terms and interconnection network studies [4, 3, 10, 9, 18, 13]. By Menger's theorem, there are k disjoint paths in paradigms (1), (2), and (3) in a k-connected graph [16]. $(2k-1)$ connectivity is a necessary condition for a graph to have k disjoint paths in paradigm (4) [21]. In network studies, much effort has also been put on algorithms for finding the disjoint paths and the upper bound on the length of the paths [10, 9, 18, 4, 13]. It has been shown that finding the disjoint paths with the optimal length in paradigms (1) or (2) is $NP-$hard in general graphs [8, 11, 14]. For paradigm (4), even determining whether there exist k disjoint paths is an $NP-$complete problem [12]. Polynomial time algorithms for paradigm (4) have been found for very limited classes of graphs such as hypercubes [15] and star graphs [4]. To our best knowledge, not much work has been done for paradigm (3), though it has practical applications in random routing and so on. For n-dimensional hypercubes H_n (which is n-connected and has diameter $d(H_n) = n$), it was proved that n disjoint paths of length at most $d(H_n) + 1 = n + 1$ for paradigm (1) [19], n disjoint paths of length at most $n + 1$ for paradigm (2) [18], and $\lceil \frac{n}{2} \rceil$ disjoint path of length at most $2n - 2$ for paradigm (4) [15] can be found in $O(n^2)$, $O(n^2)$, and $O(n^3 \log n)$ time, respectively. For n-dimensional complete star graphs, (which is $(n-1)$-connected and has diameter $d(G_n) = \lfloor \frac{3(n-1)}{2} \rfloor$), it was shown that $n-1$ disjoint paths of length at most $d(G_n)+1$ for paradigm (1) can be found in $O(n^2)$ time [13], $n-1$ disjoint paths of length at most $5n-2$ for paradigm (2) can be found in $O(n^2)$ time [4], and $k \leq \lceil \frac{n-1}{2} \rceil$ disjoint paths of length at most $4(n-2)$ for paradigm (4) can be found in $O(n^4 \log n)$ time [4].

[1] An incomplete star graph of an arbitrary size can be defined by using incomplete substars. In this paper, however, we only use complete substars to made an incomplete star graph.

We give an algorithm which finds n disjoint paths of length at most $2n-2$ for paradigm (3) in n-dimensional hypercubes H_n in $O(n^2)$ optimal time. For paradigm (4) in H_n, we give an algorithm which finds $\lceil \frac{n}{2} \rceil$ disjoint paths of length at most $2n-2$ in $O(n^2)$ time. Our result improves significantly the previous result in time complexity. For n-dimensional star graphs G_n, we give four algorithms which find, in $O(n^2)$ optimal time, $n-1$ disjoint paths of length at most $d(G_n)+4$ for paradigm (1), $n-1$ disjoint paths of length at most $d(G_n)+4$ for paradigm (2), $n-1$ disjoint paths of length at most $d(G_n)+5$ for paradigm (3), and $\lceil \frac{n-1}{2} \rceil$ disjoint paths of length at most $d(G_n)+5$ for paradigm (4). Our algorithm for paradigm (2) in star graph G_n improves previous result significantly in the length of the constructed paths and our algorithm for paradigm (4) improves the previous result significantly in the length of the found path as well as in the run time. For n-dimensional incomplete star graph $G_{n,m}$, $1 \leq m \leq n-1$, we give four algorithms which find, in $O(n^2)$ optimal time, $n-2$ disjoint paths of length at most $d(G_{n,m})+4$, $n-2$ disjoint paths of length at most $d(G_{n,m})+4$, $n-2$ disjoint paths of length at most $d(G_{n,m})+5$, and $\lceil \frac{n-2}{2} \rceil$ disjoint paths of length at most $d(G_{n,m})+5$, for paradigms (1), (2), (3),and (4), respectively. To our best knowledge, our algorithms for paradigm (3) in hypercubes H_n and star graphs G_n and the algorithms for incomplete star graphs $G_{n,m}$ are the first algorithms for these problems. Compared with the result of [13], our algorithm for paradigm (1) in star graph G_n has some advantages in practical applications. Our algorithm can find a fault-free routing path of length at most $d(G_n)+4$ between s and t in linear time when at most $n-2$ arbitrary nodes (does not include s and t) failed in G_n. While it takes $O(n^2)$ time to solve the same problem by the result in [13]. Our algorithm for paradigm (1) in incomplete star graph $G_{n,m}$ in fact finds $\min\{deg(s), deg(t)\} \geq n-2$ disjoint paths. And thus it shows that $deg(s) = n-1$ and $deg(t) = n-1$ is a sufficient condition of having $n-1$ disjoint paths between s and t in $G_{n,m}$ (which is $(n-2)$-connected). For paradigm (2) in $G_{n,m}$, we give a necessary and sufficient condition of having $n-1$ disjoint paths between s and $n-1$ distinct nodes $\{t_1, t_2, \ldots, t_{n-1}\}$ in $G_{n,m}$ and give an algorithm which finds $n-1$ disjoint paths of length at most $d(G_{n,m})+6$ in $O(n^2)$ optimal time.

In the next section, we introduce some new graph parameters, give the definition of hypercubes, star graphs, and incomplete star graphs, and state our results.

Definitions and Results

The following two parameters for disjoint path paradigms (1) and (2) were defined to evaluate the fault tolerance and transmission delay in interconnection networks.

- Given two positive integers, w and l, define $A(w,l)$ to be the class of graphs G so that for any distinct nodes s and t in G, there exist w node disjoint paths of length at most l between s and t. For a k-connected graph G, the $k-$diameter, $d_k(G)$, is defined to be the minimum l so that $G \in A(k,l)$ [10].

- Similarly, let $B(w,l)$ be the class of graphs G so that for any s and distinct $t_1, \ldots, t_w$ in G, there exist w disjoint paths from s to $t_1, \ldots, t_w$ of length at most l, and for a k-connected graph G, the Rabin number, $r_k(G)$, is the minimum l so that $G \in B(k,l)$ [18].

Similar to the k-diameter and Rabin number, we define other two parameters for the disjoint path paradigms (3) and (4) in this paper.

- Let $C(w,l)$ be the class of graphs G so that for any distinct nodes $s_1,\ldots,s_w$ and $t_1,\ldots,t_w$ in G, there exist w disjoint paths of length at most l between the nodes of s_i's and the nodes of t_j's, $1 \le i,j \le w$, and for a k-connected graph G, the k-set-routing number, $s_k(G)$, is the minimum l so that $G \in C(k,l)$.

- Let $D(w,l)$ be the class of graphs G so that for w distinct node pairs $(s_1,t_1),\ldots,(s_w,t_w)$ in G, there exist w disjoint paths of length at most l between s_i and t_i, $1 \le i \le w$, and for a n-connected graph ($n \ge 2k-1$), the k-pairwise-routing number $p_k(G)$ is the minimum l so that $G \in D(k,l)$.

An n-dimensional hypercube is a graph H_n, where the nodes of H_n are in 1-1 correspondence with the n-bit binary sequences $(a_1 \ldots a_n)$, and two nodes $(a_1 \ldots a_n)$ and $(a'_1 \ldots a'_n)$ are connected by an edge if and only if these sequences differ in exactly one bit. There are 2^n nodes and $n2^{n-1}$ edges in an n-dimensional hypercube H_n. H_n has uniform node degree n, diameter $d(H_n) = n$, and H_n is n-connected. A hypercube is node and edge symmetric and it has a recursive structure (H_n is made of two copies of H_{n-1}).

Let $<n>= \{1,2,\ldots,n\}$. The nodes of an n-dimensional star graph G_n are in a 1-1 correspondence with the permutations $(p_1,p_2,\ldots,p_n)$ of $<n>$. Each node is identified by a permutation of $<n>$. Two nodes of G_n are connected by an edge if and only if the permutation of one node can be obtained from the other by interchanging the first symbol p_1 with the ith symbol p_i, $2 \le i \le n$. There are $n!$ nodes and $n! \times \frac{(n-1)}{2}$ edges in an n-dimensional star graph G_n. G_n has uniform node degree $n-1$, diameter $d(G_n) = \lfloor \frac{3(n-1)}{2} \rfloor$, and G_n is $(n-1)$-connected. G_n is node and edge symmetric. Star graphs have a highly recursive structure. G_n is made of n copies of G_{n-1}. Consider the partition of nodes of G_n into n mutually disjoint subsets $S_n(k)$, $1 \le k \le n$, where

$$S_n(k) = \{(p_1,p_2,\ldots,p_{n-1},k) | p_j \in <n> - \{k\} \text{ for } j \ne n, p_j \ne p_l \text{ for } j \ne l\}.$$

In G_n, the induced subgraphs of the set $S_n(k)$, $1 \le k \le n$, is each an $(n-1)$-dimensional star graph denoted as $G_n(k)$. Figure 1 gives H_4 and G_4.

An incomplete n-dimensional star graph $G_{n,m}$ is a graph that is made of m, $1 \le m \le n$, copies of $(n-1)$-dimensional star graphs. More precisely, for $1 \le m \le n$, the incomplete star graph $G_{n,m} = (S_{n,m}, E_{n,m})$ is defined as,

$$S_{n,m} = \{(p_1,p_2,\ldots,p_n) | p_n \in <m>, p_i \in <n> - \{p_n\} \text{ for } i \ne n, p_i \ne p_j \text{ for } i \ne j\}$$

and

$$\begin{aligned} E_{n,m} = \; & \{((p_1,p_2,\ldots,p_n),(p_i,p_2,\ldots,p_{i-1},p_1,p_{i+1},\ldots,p_n)) | \\ & (p_1,p_2,\ldots,p_n) \in S_{n,m}, (p_i,p_2,\ldots,p_{i-1},p_1,p_{i+1},\ldots,p_n) \in S_{n,m}, \text{ and } 2 \le i \le n\}. \end{aligned}$$

That is, $G_{n,m}$ is the graph which is made of the substars $G_n(1),\ldots,G_n(m)$ of G_n and the edges between every pair of these substars. $G_{n,1} = G_{n-1}$, $G_{n,n} = G_n$, and $G_{n,m}$ is $(n-2)$-connected for $1 \le m \le n-1$. There are $m(n-1)!$ nodes and $m(n-1)! \times \frac{n-2}{2} + (n-2)! \times \frac{m(m-1)}{2}$

edges in $G_{n,m}$. $G_{n,m}$ has diameter $d(G_{n,m}) = d(G_n)$, for $2 \le m \le n$. For a node $s = (p_1, p_2, \ldots, p_n) \in G_{n,m}$, $1 \le m \le n-1$, either $deg(s) = n-2$ or $deg(s) = n-1$.

Using the notations introduced at the beginning of this section, the previous results and our results for paradigms (1)-(4) are stated as follows.

Previous results:

$d_n(H_n) = d(H_n) + 1$ (time $O(n^2)$) [19],
$r_n(H_n) = d(H_n) + 1$ (time $O(n^2)$) [18],
$p_n(H_n) \le 2d(H_n) - 2$ (time $O(n^3 \log n)$) [15],
$d_{n-1}(G_n) = d(G_n) + 1$ (time $O(n^2)$) [13],
$r_{n-1}(G_n) \le 5n - 2$ (time $O(n^2)$) [4], and
$p_{n-1}(G_n) \le 4(n-2)$ (time $O(n^4 \log n)$) [4].

Results of this paper (details can be found in [5, 6, 17, 7]):

$s_n(H_n) \le 2n - 2$ (time $O(n^2)$ and no previous result known) [17],
$p_n(H_n) \le 2n - 2$ (time $O(n^2 \log n)$) [7],
$d_{n-1}(G_n) \le d(G_n) + 4$ (time $O(n^2)$ and advantages in practical applications) [5],
$r_{n-1}(G_n) \le d(G_n) + 4$ (time $O(n^2)$) [5],
$s_{n-1}(G_n) \le d(G_n) + 5$ (time $O(n^2)$ and no previous result known) [17],
$p_{n-1}(G_n) \le d(G_n) + 5$ (time $O(n^2)$) [5],
$d_{n-2}(G_{n,m}) \le d(G_{n,m}) + 4$ (time $O(n^2)$ and no previous result known) [6],
$r_{n-2}(G_{n,m}) \le d(G_{n,m}) + 4$ (time $O(n^2)$ and no previous result known) [6],
$s_{n-2}(G_{n,m}) \le d(G_{n,m}) +$ (time $O(n^2)$ and no previous result known) [6], and
$p_{n-2}(G_{n,m}) \le d(G_{n,m}) + 5$ (time $O(n^2)$ and no previous result known) [6].

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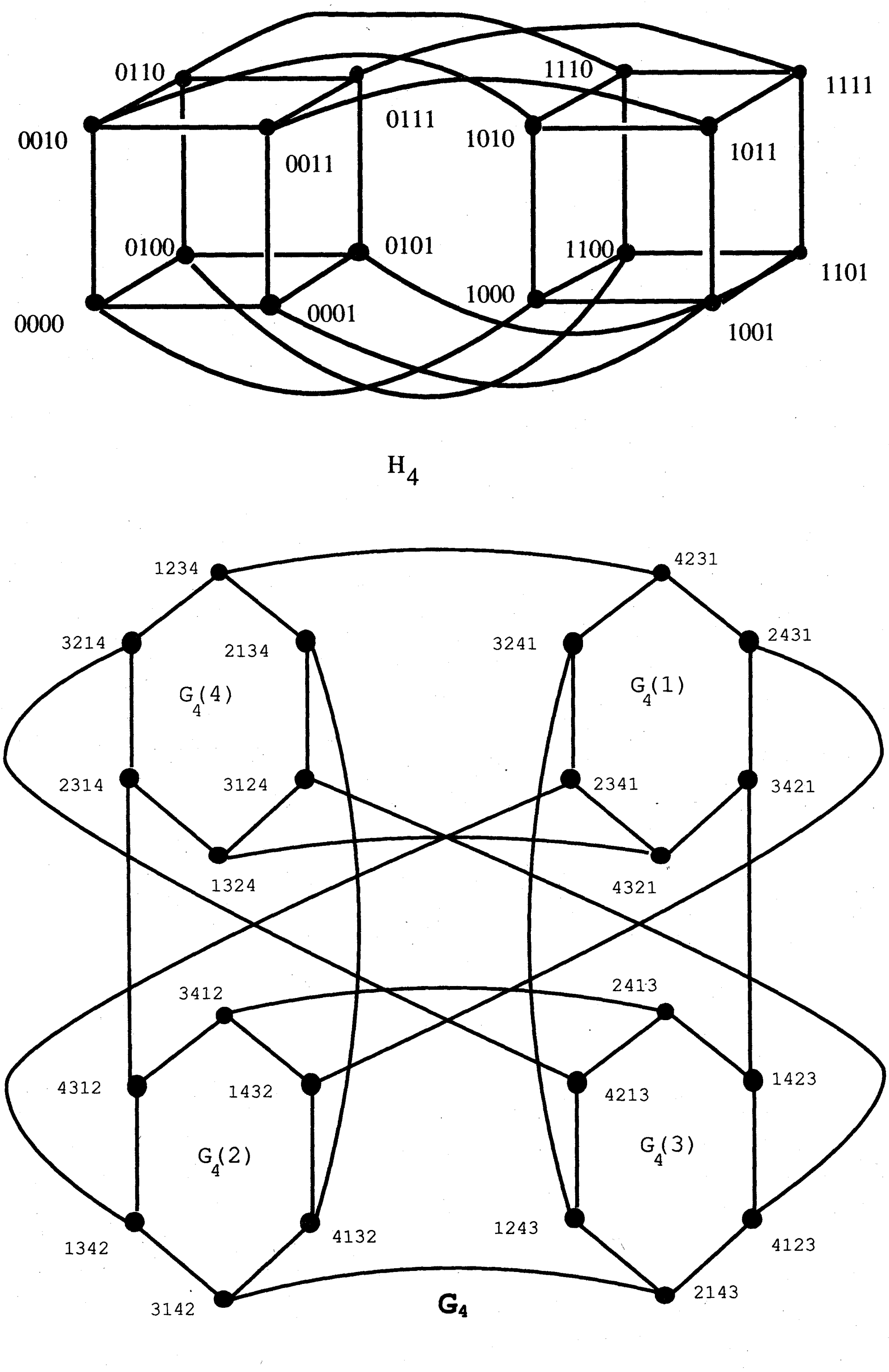

Figure 1: Hypercube H_4 and Star Graph G_4