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Resolvents of Dirac Operators

姫路工大理　楳田登美男 (Tomio Umeda)

1. Introduction

This note is based on the joint works [4, 5 and 6] with Chris Pladdy and Yoshimi Saitō, University of Alabama at Birmingham.

We consider the Dirac operator

$$H = -i\sum_{j=1}^{3}\alpha_j\frac{\partial}{\partial x_j} + \beta + Q(x), \tag{1.1}$$

which appears in the relativistic quantum mechanics. For the detailed definition of the Dirac operator (1.1) see §2. It is well-known that the liming absorption principle holds for the Dirac operator and, as a result, that the extended resolvents $R^{\pm}(\lambda)$ exist for any real value λ with $|\lambda| > 1$. The limiting absorption principle has a close connection with the spectral and scattering theory for the Dirac operator.

One of our purposes here is to investigate the asymptotic behavior of $R^{\pm}(\lambda)$ as $|\lambda| \to \infty$. Our results indicate that the extended resolvents of the Dirac operator decay much more slowly, in a certain sense, than those of Schrödinger operators. The other purpose is to derive radiation conditions for Dirac equations by which one can distinguish solutions obtained by the limiting absorption principle.

We now introduce the notation which will be used in this note. For $x \in \mathbf{R}^3$, we write

$$\langle x\rangle = \sqrt{1+|x|^2}. \tag{1.2}$$

For $s \in \mathbf{R}$, we define the weighted Hilbert spaces $L_{2,s}(\mathbf{R}^3)$ and $H_s^1(\mathbf{R}^3)$ by

$$L_{2,s}(\mathbf{R}^3) = \{f \ / \ \langle x\rangle^s f \in L_2(\mathbf{R}^3)\}, \tag{1.3}$$

and

$$H_s^1(\mathbf{R}^3) = \{f \ / \ \langle x\rangle^s \partial_x^\alpha f \in L_2(\mathbf{R}^3), |\alpha| \le 1\}, \tag{1.4}$$

where $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ is a multi-index, $|\alpha| = \alpha_1 + \alpha_2 + \alpha_3$, and

$$\partial_x^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \partial x_3^{\alpha_3}}. \tag{1.5}$$

The inner products and norms in $L_{2,s}(\mathbf{R}^3)$ and $H_s^1(\mathbf{R}^3)$ are given by

$$\begin{cases} (f,g)_s = \displaystyle\int_{\mathbf{R}^3} \langle x \rangle^{2s} f(x)\overline{g(x)}\, dx \\ \|f\|_s = \left[(f,f)_s\right]^{1/2} \end{cases} \tag{1.6}$$

and

$$\begin{cases} (f,g)_{1,s} = \displaystyle\int_{\mathbf{R}^3} \langle x \rangle^{2s} \left[\nabla f(x) \cdot \overline{\nabla g(x)} + f(x)\overline{g(x)}\right] dx \\ \|f\|_{1,s} = \left[(f,f)_{1,s}\right]^{1/2} \end{cases} \tag{1.7}$$

respectively. The spaces $\mathcal{L}_{2,s}$ and $\mathcal{H}_s^1$ are defined by

$$\begin{cases} \mathcal{L}_{2,s} = \left[L_{2,s}(\mathbf{R}^3)\right]^4 \\ \mathcal{H}_s^1 = \left[H_s^1(\mathbf{R}^3)\right]^4 \end{cases} \tag{1.8}$$

i.e., $\mathcal{L}_{2,s}$ and $\mathcal{H}_s^1$ are direct sums of the Hilbert spaces $L_{2,s}(\mathbf{R}^3)$ and $H_s^1(\mathbf{R}^3)$, respectively. The inner products and norms in $\mathcal{L}_{2,s}$ and $\mathcal{H}_s^1$ are also denoted by $(\ ,\ )_s$, $\|\ \|_s$ and $(\ ,\ )_{1,s}$, $\|\ \|_{1,s}$, respectively. When $s = 0$, we simply write

$$\begin{cases} \mathcal{L}_2 = \mathcal{L}_{2,0} \\ \mathcal{H}^1 = \mathcal{H}_0^1 \end{cases} \tag{1.9}$$

For a pair of Hilbert spaces X and Y, $\mathbf{B}(X, Y)$ denotes the Banach space of all bounded linear operators from X to Y, equipped with the operator norm

$$\|T\| = \sup_{x \in X \setminus \{0\}} \|Tx\|_Y / \|x\|_X, \tag{1.10}$$

where $\|\ \|_X$ and $\|\ \|_Y$ are the norms in X and Y.

2. Main results

We first consider the free Dirac operator

$$H_0 = -i \sum_{j=1}^{3} \alpha_j \frac{\partial}{\partial x_j} + \beta, \tag{2.1}$$

where $i = \sqrt{-1}$ and $x = (x_1, x_2, x_3) \in \mathbf{R}^3$. Here α_j, β are 4×4 Hermitian matrices satisfying the anticommutation relations

$$\alpha_j \alpha_k + \alpha_k \alpha_j = 2\delta_{jk} I \qquad (j, k = 1, 2, 3, 4) \tag{2.2}$$

with the convention $\alpha_4 = \beta$, δ_{jk} being Kronecker's delta and I being the 4×4 identity matrix. It is known that H_0 restricted on $[C_0^\infty(\mathbf{R}^3)]^4$ is essentially selfadjoint in $\mathcal{L}_2$ and its selfadjoint extension, which will be denoted by H_0 again, has the domain $\mathcal{H}^1$.

We make the following assumption on the potential.

Assumption(A).

(i) $Q(x) = (q_{jk}(x))$ is a 4×4 Hermitian matrix-valued C^1 function on $\mathbf{R}^3$;

(ii) There exist positive constants ϵ and K such that

$$\langle x \rangle^{1+\epsilon} |q_{jk}| + \sum_{\ell=1}^{3} |\frac{\partial q_{jk}}{\partial x_\ell}| \leq K \tag{2.3}$$

for $j, k = 1, 2, 3, 4$.

Under Assumption(A) the multiplication operator $Q = Q(x)\times$ is a bounded selfadjoint operator in $\mathcal{L}_2$. Hence, by the Kato-Rellich theorem (Kato[3], p.287), H restricted on $[C_0^\infty(\mathbf{R}^3)]^4$ is also essentially selfadjoint in $\mathcal{L}_2$ and its selfadjoint extension, which will be denoted by H again, has the same domain $\mathcal{H}^1$ as H_0. We write

$$R_0(z) = (H_0 - z)^{-1} \tag{2.4}$$

and

$$R(z) = (H - z)^{-1}. \tag{2.5}$$

As we mentioned in the Introduction, the limiting absorption principle holds for the Dirac operator H.

Theorem (Yamada [10]). *Suppose that Assumption(A) is satisfied and let $s > 1/2$. Then for $\lambda \in (-\infty, -1) \cup (1, \infty)$, there exist the extended resolvents $R^\pm(\lambda) \in \mathbf{B}(\mathcal{L}_{2,s}, \mathcal{H}^1_{-s})$ such that*

$$\operatorname*{s-lim}_{\eta \downarrow 0} R(\lambda \pm i\eta) = R^\pm(\lambda) \qquad \text{in } \mathcal{H}^1_{-s}. \tag{2.6}$$

For $f \in \mathcal{L}_{2,s}$, $R^{\pm}(\lambda)f$ *are* $\mathcal{L}_{2,-s}$*-valued, continuous functions on* $(-\infty,-1)\cup(1,\infty)$. *Moreover,* $u^{\pm}(\lambda, f) := R^{\pm}(\lambda)f$ *satisfy*

$$(-i\sum_{j=1}^{3}\alpha_j\frac{\partial}{\partial x_j}+\beta+Q(x)-\lambda)u=f. \tag{2.7}$$

We now state the main theorems.

Theorem 1. *Let* $s>1/2$. *Then*

$$\|R_0^{\pm}(\lambda)\|=O(1) \qquad (|\lambda|\to\infty), \tag{2.8}$$

where $\|R_0^{\pm}(\lambda)\|$ *denote the operator norms of* $R_0^{\pm}(\lambda)$ *in* $\mathbf{B}(\mathcal{L}_{2,s},\mathcal{L}_{2,-s})$.

As we shall see in the remark after Theorem 2, $\|R_0^{\pm}(\lambda)\|$ cannot be small no matter how $|\lambda|$ is large. In this sense the estimate in Theorem 1 is best possible. However, $R_0^{\pm}(\lambda)$ can become small as $|\lambda|\to\infty$ in a weaker sense than in Theorem 1.

Theorem 2. *Let* $s>1/2$. *Then* $R_0^{\pm}(\lambda)$ *converge strongly to* 0 *as* $|\lambda|\to\infty$, *i.e.,*

$$\lim_{|\lambda|\to\infty}R_0^{\pm}(\lambda)f=0 \qquad \text{in } \mathcal{L}_{2,-s} \tag{2.9}$$

for any $f\in\mathcal{L}_{2,s}$.

Remark. Yamada [11] proved the following: *Let* $s>1/2$. *Then there exists a sequence* $\{f_n\}$ *in* $\mathcal{L}_{2,s}$ *such that*

$$\sup_n\|f_n\|_s<\infty, \tag{2.10}$$

and

$$\lim_{n\to\infty}(R_0^{\pm}(n)f_n,f_n)_0\neq 0. \tag{2.11}$$

Since

$$|(R_0^{\pm}(n)f_n,f_n)_0|\le\|R_0^{\pm}(n)\|\,\|f_n\|_s{}^2, \tag{2.12}$$

Yamada's example implies that $\|R_0^{\pm}(\lambda)\|$ cannot converge to 0 as $\lambda \to \infty$.

Based on Theorems 1 and 2, the Dirac operator with a small coupling constant can be handled; one can use the Neumann series expansion. Let

$$H_t = -i\sum_{j=1}^{3}\alpha_j\frac{\partial}{\partial x_j} + \beta + tQ(x), \tag{2.13}$$

where t is a real number. The extended resolvents of H_t will be denoted by $R_t^{\pm}(\lambda)$. Then we have the following

Theorem 3. *Suppose that $Q(x)$ satisfies Assumption(A) and $1/2 < s < (1+\epsilon)/2$. Let $R_t^{\pm}(\lambda)$ be the extended resolvents of H_t. Then for sufficiently small t*

(i) The operator norms of $R_t^{\pm}(\lambda)$ in $\mathbf{B}(\mathcal{L}_{2,s}, \mathcal{L}_{2,-s})$ are bounded as $|\lambda| \to \infty$;

(ii) $R_t^{\pm}(\lambda)$ converge strongly to 0 as $|\lambda| \to \infty$.

Finally we state a theorem which characterizes the extended resolvents $R^{\pm}(\lambda)$ through Dirac equations with radiation condition.

Theorem 4. *Suppose that $Q(x)$ satisfies Assumption(A) and $1/2 < s <$ $\min\{1,\ (1+\epsilon)/2\}$. Let $u^{\pm}(\lambda,\ f)$ be the same as in Theorem(Yamada). Then $u^{\pm}(\lambda,\ f)$ are unique solutions of the equation*

$$\begin{cases} (-i\sum_{j=1}^{3}\alpha_j\frac{\partial}{\partial x_j} + \beta + Q(x) - \lambda)u = f \\ u \in \mathcal{L}_{2,-s} \cap \mathcal{H}^1_{\mathrm{loc}} \\ (\partial_\ell \mp i\sqrt{\lambda^2 - 1}\widetilde{x_\ell})u \in \mathcal{L}_{2,s-1}, \quad \ell = 1,2,3 \quad \text{if} \quad \lambda > 1 \\ (\partial_\ell \pm i\sqrt{\lambda^2 - 1}\widetilde{x_\ell})u \in \mathcal{L}_{2,s-1}, \quad \ell = 1,2,3 \quad \text{if} \quad \lambda < -1 \end{cases}$$

respectively, where $\widetilde{x_\ell} = x_\ell/|x|$.

In the rest of this note, we shall give an outline of the proof of Theorem 1. The proofs of Theorems 2 and 3 can be found in [5]. Theorem 4 is the main portion of the the results in [6], where radiation condition for Dirac operators is discussed.

3. A Known result for Schrödinger operators

The limiting absorption principle for Schrödinger operators has been extensively studied. We will use a result due to Saitō [8, 9]. For this reason, we make a review. Let T denote the selfadjoint operator which is defined to be the closure of $-\Delta + V(x)$ restricted on $C_0^\infty(\mathbf{R}^n)$, where $V(x)$ is a real-valued function satisfying

$$(3.1) \qquad |V(x)| \le C\langle x\rangle^{-1-\epsilon}$$

for $C > 0$ and $\epsilon > 0$. Let $\Gamma(z) = (T - z)^{-1}$. Then it is well-known that the limiting absorption principle holds for T, that is, for any $\lambda > 0$, there correspond the extended resolvents $\Gamma^{\pm}(\lambda)$ in $\mathbf{B}(L_{2,s}(\mathbf{R}^n), L_{2,-s}(\mathbf{R}^n))$ such that

$$(3.2) \qquad \operatorname*{s-lim}_{\eta\downarrow 0} \Gamma(\lambda \pm i\eta)f = \Gamma^{\pm}(\lambda)f \qquad \text{in } L_{2,-s}$$

for any f in $L_{2,s}(\mathbf{R}^n)$. Furthermore, it is known that $\Gamma^{\pm}(\lambda)f$ are $L_{2,-s}(\mathbf{R}^n)$-valued continuous functions in $(0, \infty)$. (Saitō [8], Ikebe-Saitō [2] and Agmon [1].) As for asymptotic behaviors of $\Gamma^{\pm}(\lambda)$, we have

Theorem 3.1 (Saitō [8, 9]). *Let $\|\Gamma^{\pm}(\lambda)\|$ be the operator norms of $\Gamma^{\pm}(\lambda)$ in $\mathbf{B}(L_{2,s}(\mathbf{R}^n), L_{2,-s}(\mathbf{R}^n))$. Then*

$$(3.3) \qquad \|\Gamma^{\pm}(\lambda)\| = O(\lambda^{-1/2}) \qquad (\lambda \to \infty).$$

More precisely, Saitō proved

Theorem 3.2 (Saitō [8, 9]). *Let $s > 1/2$. Then for any $a > 0$ there exists a positive constant $C > 0$ such that*

$$(3.4) \qquad \|\Gamma(\kappa^2)\| \le C/|\kappa|$$

for all κ with $|\mathrm{Re}\,\kappa| > a$ and $\mathrm{Im}\,\kappa > 0$, where $\|\Gamma(\kappa^2)\|$ is the operator norm of $\Gamma(\kappa^2)$ in $\mathbf{B}(L_{2,s}(\mathbf{R}^n), L_{2,-s}(\mathbf{R}^n))$.

4. Pseudo-differential operators

The proof of Theorem 1 is based on the resolvent estimate for the Schrödinger operator (Theorem 3.2) as well as the theory of pseudo-differential operators. In connection with the limiting absorption principle, pseudo-differential operators which are bounded in $L_{2,s}$ are important.

Lemma 4.1. *Let $p(x,\xi) \in S^0_{0,0}$. Then for any $s > 0$ there exist a positive constant $C\,(=C_s)$ and a positive integer $\ell\,(=\ell_s)$ such that*

$$\|p(x,D)f\|_s \leq C|p|^{(0)}_\ell \|f\|_s \qquad (f \in \mathcal{S}(\mathbf{R}^3)). \tag{4.1}$$

The proof of Lemma 4.1 is based on the Calderón-Vaillancourt theorem and some techniques in the theory of pseudo-differential operators. We need to extend Lemma 4.1 to a system of pseudo-differential operators. For a 4×4 matrix-valued symbol $P(x,\xi) = \big(p_{jk}(x,\xi)\big)_{1\leq j,k\leq 4}$, we write $P(x,D) = \big(p_{jk}(x,D)\big)_{1\leq j,k\leq 4}$. If $p_{jk}(x,\xi) \in S^0_{0,0}$, $1 \leq j,k \leq 4$, we define

$$|P|^{(0)}_\ell = \Big\{\sum_{j,k=1}^{4} (|p_{jk}|^{(0)}_\ell)^2\Big\}^{1/2} \tag{4.2}$$

for $\ell = 0,1,2,\ldots$, where $|p_{jk}|^{(0)}_\ell$ are the semi-norms in the class $S^0_{0,0}$. We then have a natural extension of Lemma 4.1.

Lemma 4.2. *Let $p_{jk}(x,\xi) \in S^0_{0,0}$ for j, $k = 1,2,3,4$. Then for any $s > 0$ there exist a positive constant $C\,(=C_s)$ and a positive integer $\ell\,(=\ell_s)$ such that*

$$\|P(x,D)f\|_s \leq C\,|P|^{(0)}_\ell\,\|f\|_s \tag{4.3}$$

for $f \in [\mathcal{S}(\mathbf{R}^3)]^4$.

5. Outline of the proof of Theorem 1

In view of Theorem(Yamada), we see that Theorem 5.1 below implies Theorem 1. We first make a few remarks on the free Dirac operator H_0. For $f \in [\mathcal{S}(\mathbf{R}^3)]^4$,

$$H_0 f = \mathcal{F}^{-1}\hat{L}_0(\xi)\mathcal{F}f, \tag{5.1}$$

where

$$\hat{L}_0(\xi) = \sum_{j=1}^{3} \alpha_j \xi_j + \beta. \tag{5.2}$$

It is easy to see that

$$\left(\hat{L}_0(\xi)\right)^2 = \langle \xi \rangle^2 I. \tag{5.3}$$

Using (5.3), we get

$$R_0(z) f = \mathcal{F}^{-1} \left[\frac{\hat{L}_0(\xi) + z}{\langle \xi \rangle^2 - z^2} \right] \mathcal{F} f \tag{5.4}$$

for $f \in [\mathcal{S}(\mathbf{R}^3)]^4$ and $z \in \mathbf{C} \backslash \mathbf{R}$.

Theorem 5.1. *Suppose that* $s > 1/2$. *Then*

$$\sup \left\{ \|R_0(\lambda \pm i\eta)\| \,/\, 2 \leq |\lambda|, 0 < \eta < 1 \right\} < \infty, \tag{5.5}$$

where $\|R_0(\lambda \pm i\eta)\|$ *denotes the operator norm of* $R_0(\lambda \pm i\eta)$ *in* $\mathbf{B}(\mathcal{L}_{2,s}, \mathcal{L}_{2,-s})$.

Outline of the proof. Set

$$J = \left\{ z \in \mathbf{C} \,/\, 2 \leq |\mathrm{Re}\, z|, 0 < |\mathrm{Im}\, z| < 1 \right\}. \tag{5.6}$$

Choose $\rho \in C_0^\infty(\mathbf{R})$ so that

$$\rho(t) = \begin{cases} 1, & \text{if } |t| < 1/2\,; \\ 0, & \text{if } |t| > 1 \end{cases} \tag{5.7}$$

For each $z \in J$, we define a cutoff function $\gamma_z(\xi)$ on $\mathbf{R}^3$ by

$$\gamma_z(\xi) = \begin{cases} \rho(\langle \xi \rangle - \mathrm{Re}\, z), & \text{if } \mathrm{Re}\, z \geq 2; \\ \rho(\langle \xi \rangle + \mathrm{Re}\, z), & \text{if } \mathrm{Re}\, z \leq -2. \end{cases} \tag{5.8}$$

Using (5.4) and $\gamma_z(\xi)$, we decompose the resolvent of H_0 into three parts:

$$R_0(z) = (-\Delta + 1 - z^2)^{-1} A_z + B_z + z(-\Delta + 1 - z^2)^{-1} \tag{5.9}$$

where

$$
(5.10) \qquad \begin{aligned} A_z &= \mathcal{F}^{-1}\Big[\gamma_z(\xi)\hat{L}_0(\xi)\Big]\mathcal{F}, \\ B_z &= \mathcal{F}^{-1}\Big[\frac{1-\gamma_z(\xi)}{\langle\xi\rangle^2 - z^2}\hat{L}_0(\xi)\Big]\mathcal{F}. \end{aligned}
$$

Applying Lemma 4.2 to A_z, we get

$$
(5.11) \qquad \|A_z f\|_s \le C_1 |z| \|f\|_s,
$$

where C_1 is independent of $z \in J$. Combining (5.11) with Theorem 3.2, we see that

$$
(5.12) \qquad \|(-\Delta + 1 - z^2)^{-1} A_z f\|_{-s} \le C_2 \|f\|_s,
$$

where C_2 is independent of $z \in J$. It is easy to see that there exists a constant $C_3 > 0$ such that

$$
(5.13) \qquad \left|\frac{1-\gamma_z(\xi)}{\langle\xi\rangle^2 - z^2}\hat{L}_0(\xi)\right| \le C_3 \qquad (\xi \in \mathbf{R}^3)
$$

for all $z \in J$. Using (5.13), we have

$$
(5.14) \qquad \|B_z f\|_{-s} \le C_3 \|f\|_s
$$

for all $z \in J$. It follows from Theorem 3.2 that

$$
(5.15) \qquad \|z(-\Delta + 1 - z^2)^{-1} f\|_{-s} \le C_4 \|f\|_s,
$$

where C_4 is independent of $z \in J$. Combining (5.15), (5.14) and (5.12), we get the desired conclusion.

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