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Derivation property of the Lévy Laplacian

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Introduction

In his book [11] P. Lévy introduced an infinite dimensional analogue of a finite dimensional Laplacian and developed an infinite dimensional potential theory, see also [12]. (For subsequent developments see e.g., [6], [7], [8], [9], [13], [15], and references cited therein.) The operator, presently called the *Lévy Laplacian*, is defined as the Cesàro mean of second order differential operators:

$$\Delta_L = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \frac{\partial^2}{\partial x_n^2},$$

where $x_1, x_2, \cdots$ constitute a coordinate system of the infinite dimensional vector space under consideration. Although the Lévy Laplacian inherits some typical properties of a finite dimensional Laplacian such as a natural relation with spherical means, it bears some pathological properties and has been discussed more or less in its own interests.

The situation is, however, changing with a recent series of works [1]–[3], [16]. The rediscovery of somehow unexpected relationship between the Lévy Laplacian and the Yang-Mills equation is openning a new approach to infinite dimensional stochastic analysis based on the Lévy Brownian motion and its quantization. (In fact, the relation was first found by Aref'eva and Volovich [4].)

The purpose of this paper is to clarify the derivation property of the Lévy Laplacian. It has been observed in a common discussion that the Lévy Laplacian behaves like a first order differential operator, i.e., a derivation. Moreover, this property is needed to characterize the Lévy Laplacian in terms of its group invariance [14]. However, as we shall show, this is typical when the Lévy Laplacian acts on functions on a Hilbert space. In this paper, employing some ideas in [10] where the Lévy Laplacian is defined as an operator acting on functions on a nuclear space, we study when the Lévy Laplacian is a derivation. As application we discuss the heat semigroup constructed in [2].

1 Lévy Laplacian on a nuclear space

Here we do not deal with a fully general nuclear space but a standad countably Hilbert nuclear space which is also known for the standard framework of white noise calculus.

Let H be a real separable (infinite dimensional) Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $|\cdot|_0 = |\cdot|$ and let A be a positive selfadjoint operator in H with Hilbert-Schmidt inverse. Then there exist a sequence of positive numbers

$$0 < \lambda_1 \le \lambda_2 \le \cdots$$

and a sequence of vectors $\{e_n\}_{n=1}^{\infty} \subset \mathrm{Dom}\,(A)$ such that

$$Ae_n = \lambda_n e_n, \quad |e_n|_0 = 1, \quad \sum_{n=1}^{\infty} \lambda_n^{-2} = \| A^{-1} \|_{HS}^2 < \infty.$$

Note that $\{e_n\}_{n=1}^{\infty}$ forms a complete orthonormal system of H. For every $p \in \mathbb{R}$ we put

$$|\xi|_p^2 = \sum_{n=1}^{\infty} \langle \xi, e_n \rangle^2 \lambda_n^{2p} = |A^p \xi|_0^2, \qquad \xi \in H.$$

For $p \ge 0$ the space E_p of all $\xi \in H$ with $|\xi|_p < \infty$ becomes a Hilbert space with norm $|\cdot|_p$. Note that H is no longer complete with respect to the norm $|\cdot|_{-p}$, $p \ge 0$. The completion E_{-p} is then Hilbert space with norm $|\cdot|_{-p}$. We have thus constructed a chain of Hilbert spaces $\{E_p\}_{p\in\mathbb{R}}$ with natural inclusion relation. Since A^{-1} is of Hilbert-Schmidt type,

$$E = \operatorname*{proj\,lim}_{p\to\infty} E_p = \bigcap_{p\ge 0} E_p$$

becomes a countably Hilbert nuclear space. Such a nuclear space constructed from an operator A is called *standard.* For the strong dual space E^* we have

$$E^* \cong \operatorname*{ind\,lim}_{p\to\infty} E_{-p} \cong \bigcup_{p\ge 0} E_{-p}.$$

Thus we come to a Gelfand triple:

$$E \subset H \subset E^*.$$

Being compatible to the inner product of H, the canonical bilinear form on $E^* \times E$ is denoted by $\langle \cdot, \cdot \rangle$ again.

A function $F : E \to \mathbb{R}$ is called twice differentiable at $\xi \in E$ if there exist $F'(\xi) \in E^*$ and $F''(\xi) \in \mathcal{L}(E, E^*)$ such that

$$F(\xi + \eta) = F(\xi) + \langle F'(\xi), \eta \rangle + \frac{1}{2} \langle F''(\xi)\eta, \eta \rangle + o(\eta), \qquad \eta \in E,$$

where

$$\lim_{t\to 0} \frac{o(t\eta)}{t^2} = 0.$$

Let $C^2(E)$ be the space of everywhere twice differentiable functions $F : E \to \mathbb{R}$ such that both $\xi \mapsto F'(\xi) \in E^*$ and $\xi \mapsto F''(\xi) \in \mathcal{L}(E, E^*)$ are continuous. The topological isomorphisms:

$$(E \otimes E)^* \cong \mathcal{L}(E, E^*) \cong \mathcal{B}(E, E),$$

which follow from the kernel theorem, are often useful. Accordingly, we write

$$\langle F''(\xi)\eta,\, \eta\rangle = \langle F''(\xi),\, \eta\otimes\eta\rangle\,, \qquad \eta\in E.$$

We set

$$\mathcal{D} = \left\{ F\in C^2(E);\, \lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\langle F''(\xi)e_n,\, e_n\rangle \quad \text{exists for all} \quad \xi\in E \right\}$$

and

$$\Delta_L F(\xi) = \lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\langle F''(\xi)e_n,\, e_n\rangle\,, \qquad \xi\in E, \quad F\in\mathcal{D}.$$

The operator Δ_L is called the *Lévy Laplacian* on E (with respect to $\{e_n\}$). Note that the definition depends also on the arrangement of the complete orthonormal sequence $\{e_n\}$.

A *polynomial* on E is by definition a finite linear combination of functions of the form:

$$F(\xi) = \left\langle a,\, \xi^{\otimes\nu}\right\rangle, \qquad a\in(E^{\otimes\nu})^*, \quad \xi\in E.$$

The coefficient a is uniquely determined after symmetrization. Obviously, every polynomial belongs to $C^2(E)$. In fact,

$$\begin{aligned}
\langle F'(\xi),\, \eta\rangle &= \nu\left\langle a,\, \xi^{\otimes(\nu-1)}\otimes\eta\right\rangle = \nu\left\langle a\otimes_{\nu-1}\xi^{\otimes(\nu-1)},\, \eta\right\rangle, \\
\langle F''(\xi),\, \eta\otimes\eta\rangle &= \nu(\nu-1)\left\langle a,\, \xi^{\otimes(\nu-2)}\otimes\eta\otimes\eta\right\rangle = \nu(\nu-1)\left\langle a\otimes_{\nu-2}\xi^{\otimes(\nu-2)},\, \eta\otimes\eta\right\rangle,
\end{aligned}$$

where $\otimes_\nu$ denotes the contraction of the tensor products. Hence,

$$F'(\xi) = \nu a\otimes_{\nu-1}\xi^{\otimes(\nu-1)}, \qquad F''(\xi) = \nu(\nu-1)a\otimes_{\nu-2}\xi^{\otimes(\nu-2)}.$$

Not every polynomial belongs to $\mathcal{D}$. In §5 we shall introduce particular classes of polynomials.

2 Derivation property

We begin with an immediate but important remark.

Lemma 2.1 *Let $F_1, F_2\in\mathcal{D}$. Then $F_1F_2\in\mathcal{D}$ if and only if the limit*

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\langle F_1'(\xi),\, e_n\rangle\,\langle F_2'(\xi),\, e_n\rangle$$

exists for all $\xi\in E$. Moreover,

$$\Delta_L(F_1F_2) = (\Delta_L F_1)F_2 + F_1(\Delta_L F_2)$$

if and only if

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\langle F_1'(\xi),\, e_n\rangle\,\langle F_2'(\xi),\, e_n\rangle = 0, \qquad \xi\in E.$$

PROOF. By definition for any $\xi, \eta \in E$,

$$\langle (F_1F_2)'(\xi), \eta \rangle = \langle F_1'(\xi), \eta \rangle F_2(\xi) + F_1(\xi) \langle F_2'(\xi), \eta \rangle \tag{1}$$

and

$$\langle (F_1F_2)''(\xi), \eta \otimes \eta \rangle =$$

$$= \langle F_1''(\xi), \eta \otimes \eta \rangle F_2(\xi) + 2 \langle F_1'(\xi), \eta \rangle \langle F_2'(\xi), \eta \rangle + F_1(\xi) \langle F_2''(\xi), \eta \otimes \eta \rangle .$$

Then the assertion is immediate. qed

In particular, note that $\mathcal{D}$ is not an algebra, i.e., not closed under pointwise multiplication. Now we put

$$\mathcal{D}_0 = \left\{ F \in \mathcal{D};\ \limsup_{N \to \infty} \frac{1}{N} \sum_{j=1}^{N} | \langle F'(\xi), e_j \rangle |^2 = 0 \right\}.$$

Theorem 2.2 *The space $\mathcal{D}_0$ is closed under pointwise multiplication, i.e., is an algebra, on which the Lévy Laplacian acts as derivation.*

PROOF. Suppose that $F_1, F_2 \in \mathcal{D}_0$. We first prove that $F_1F_2 \in \mathcal{D}$. Obserce that

$$\begin{aligned}
&\frac{1}{N} \left| \sum_{n=1}^{N} \langle F_1'(\xi), e_n \rangle \langle F_2'(\xi), e_n \rangle \right| \\
&\le \frac{1}{N} \left(\sum_{n=1}^{N} | \langle F_1'(\xi), e_n \rangle |^2 \right)^{1/2} \left(\sum_{n=1}^{N} | \langle F_2'(\xi), e_n \rangle |^2 \right)^{1/2} \\
&= \left(\frac{1}{N} \sum_{n=1}^{N} | \langle F_1'(\xi), e_n \rangle |^2 \right)^{1/2} \left(\frac{1}{N} \sum_{n=1}^{N} | \langle F_2'(\xi), e_n \rangle |^2 \right)^{1/2} \\
&\longrightarrow 0 \qquad \text{as} \quad N \to \infty.
\end{aligned}$$

It then follows from Lemma 2.1 that $F_1F_2 \in \mathcal{D}$. We next show that

$$\limsup_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} | \langle (F_1F_2)'(\xi), e_n \rangle |^2 = 0.$$

In fact, since

$$\langle (F_1F_2)'(\xi), e_n \rangle = \langle F_1'(\xi), e_n \rangle F_2(\xi) + F_1(\xi) \langle F_2'(\xi), e_n \rangle ,$$

by Minkowskii's inequality we obtain

$$\begin{aligned}
&\left(\sum_{n=1}^{N} | \langle (F_1F_2)'(\xi), e_n \rangle |^2 \right)^{1/2} \\
&\le \left(\sum_{n=1}^{N} | \langle F_1'(\xi), e_n \rangle F_2(\xi) |^2 \right)^{1/2} + \left(\sum_{n=1}^{N} | F_1(\xi) \langle F_2'(\xi), e_n \rangle |^2 \right)^{1/2}
\end{aligned}$$

and therefore

$$\left(\frac{1}{N}\sum_{n=1}^{N}|\langle (F_1F_2)'(\xi),\, e_n\rangle|^2\right)^{1/2}$$

$$\leq \left(\frac{1}{N}\sum_{n=1}^{N}|\langle F_1'(\xi),\, e_n\rangle|^2\right)^{1/2}|F_2(\xi)| + \left(\frac{1}{N}\sum_{n=1}^{N}|\langle F_2'(\xi),\, e_n\rangle|^2\right)^{1/2}|F_1(\xi)|$$

$$\longrightarrow 0 \qquad \text{as} \quad N\to\infty,$$

as desired. We have thus proved that $F_1F_2 \in \mathcal{D}_0$. Finally it follows immediately from Lemma 2.1 that $\Delta_L(F_1F_2) = \Delta_L F_1 \cdot F_2 + F_1 \cdot \Delta_L F_2$, namely that the Lévy Laplacian acts on $\mathcal{D}_0$ as derivation. qed

Here is an immediate consequence.

Corollary 2.3 *For $p \geq 0$ we put*

$$\mathcal{A}_p = \{F \in \mathcal{D};\ F'(\xi) \in E_p,\ \xi \in E\}.$$

Then $\mathcal{A}_p$ is a subalgebra of $\mathcal{D}_0$. In particular, Δ_L acts on $\mathcal{A}_p$ as derivation.

PROOF. We first prove that $\mathcal{A}_p \subset \mathcal{D}_0$. Suppose $F \in \mathcal{A}_p$. Then, since $0 < \lambda_1 \leq \lambda_2 \leq \cdots$,

$$\begin{aligned}\frac{1}{N}\sum_{n=1}^{N}|\langle F'(\xi),\, e_n\rangle|^2 &= \frac{1}{N}\sum_{n=1}^{N}|\langle F'(\xi),\, e_n\rangle|^2\lambda_n^{2p}\lambda_n^{-2p}\\ &\leq \frac{1}{N}\sum_{n=1}^{N}|\langle F'(\xi),\, e_n\rangle|^2\lambda_n^{2p}\lambda_1^{-2p}\\ &\leq \frac{\lambda_1^{-2p}}{N}|F'(\xi)|_p^2 \longrightarrow 0 \quad \text{as } N\to\infty.\end{aligned}$$

Therefore $F \in \mathcal{D}_0$. It is then straightforward to verify that $\mathcal{A}_p$ is a subalgebra of $\mathcal{D}_0$. qed

In particular, $\mathcal{A}_0$ is an algebra of functions on E on which the Lévy Laplacian acts as derivation. This is the reason why the Lévy Laplacian acting on functions on a Hilbert space is a derivation (note that $E_0 = H$), see e.g., [10], [13], [14], [15].

The derivation property is also observed in a slightly different manner.

Proposition 2.4 *Let $F_1, F_2 \in \mathcal{D}$ and fix $\xi \in E$. If there exists $p \geq 0$ such that*

$$|F_1'(\xi)|_p < \infty, \qquad |F_2'(\xi)|_{-p} < \infty,$$

then

$$\Delta_L(F_1F_2)(\xi) = \Delta_L F_1(\xi)\cdot F_2(\xi) + F_1(\xi)\cdot \Delta_L F_2(\xi).$$

PROOF. We see that

$$\left|\frac{1}{N}\sum_{n=1}^{N}\langle F_1'(\xi),\, e_n\rangle\langle F_2'(\xi),\, e_n\rangle\right|$$

$$\leq \frac{1}{N}\left(\sum_{n=1}^{N}|\langle F_1'(\xi),\, e_n\rangle\lambda_n^{2p}\right)^{1/2}\left(\sum_{n=1}^{N}|\langle F_2'(\xi),\, e_n\rangle\lambda_n^{-2p}\right)^{1/2}$$

$$\leq \frac{1}{N}|F_1'(\xi)|_p|F_2'(\xi)|_{-p} \to 0, \text{ as } N\to\infty.$$

Then we need only to apply Lemma 2.1. qed

3 Lévy Laplacian on positive definite functions

There is an interesting class of functions on E which are related to finite measures on E^*. Let $\mathfrak{B}$ be the σ-field on E^* generated by linear functions:

$$x \mapsto \langle x, \xi \rangle, \qquad x \in E^*,$$

where ξ runs over E. It is easily seen that $\mathfrak{B}$ coincides with the topological σ-field induced from the strong dual topology of E^*.

Let $M_+(E^*)$ be the space of finite measures on E^* and let $M(E^*)$ be the space of all signed measures on $(E^*, \mathfrak{B})$ with finite variation. Every element in $M(E^*)$ is written as $\mu_1 - \mu_2$, $\mu_1, \mu_2 \in M_+(E^*)$. If $\mu \in M(E^*)$, then its Fourier transform $\widehat{\mu}$ is a function on E defined by

$$\widehat{\mu}(\xi) = \int_{E^*} e^{i\langle x, \xi \rangle} \mu(dx), \qquad \xi \in E. \tag{2}$$

We here recall a fundamental result.

Theorem 3.1 (BOCHNER-MINLOS) *There is a one-to-one correspondence between $M_+(E^*)$ and the space $\mathcal{B}_+(E)$ of all continuous positive definite functions on E through the Fourier transform (2).*

Let $\mathcal{B}(E)$ be the space of the Fourier transform of $\mu \in M(E^*)$. Note that $M(E^*)$ is an algebra with convolution product:

$$\int_{E^*} \phi(x) \mu * \nu(dx) = \int_{E^* \times E^*} \phi(x+y) \mu(dx) \nu(dy).$$

Through the Fourier transform $\mathcal{B}(E)$ becomes an algebra with pointwise multiplication. Thus, $\mathcal{B}(E)$ becomes a closed subalgebra of $L^\infty(E)$ and therefore it is an abelian C^*-algebra for itself.

The support of μ is related to the continuity of $\widehat{\mu}$.

Theorem 3.2 *If a positive definite function $C : E \to \mathbb{C}$ admits a continuous extension to E_p, $p \geq 0$, the corresponding measure μ is concentrated on $E_{-(p+q)}$ for any $q \geq 0$ such that the canonical injection $E_{p+q} \to E_p$ is of Hilbert-Schmidt type.*

Lemma 3.3 *Let F be the Fourier transform of $\mu \in M_+(E^*)$. If*

$$\int_{E^*} |x|_p \, \mu(dx) < \infty \tag{3}$$

for some $p \in \mathbb{R}$, then $F'(\xi) \in E_p$ for any $\xi \in E$.

PROOF. Since

$$\left| i \langle x, \eta \rangle e^{i\langle x, \xi \rangle} \right| \leq |x|_p \, |\eta|_{-p},$$

it follows from Lebesgue's convergence theorem that

$$\langle F'(\xi), \eta \rangle = \int_{E^*} i \langle x, \eta \rangle e^{i\langle x, \xi \rangle} \mu(dx), \qquad \eta \in E.$$

Moreover,

$$|\langle F'(\xi), \eta\rangle| \le \int_{E^*} |x|_p |\eta|_{-p} \mu(dx) = |\eta|_{-p} \int_{E^*} |x|_p \mu(dx),$$

which implies that $F'(\xi) \in E_p$. qed

Remark. It follows from (3) that $\mu(E_p) = 1$. In fact, there exists a null set N such that $|x|_p < \infty$ for any $x \in E^* - N$. Hence $E^* - N \subset E_p$ and therefore $1 = \mu(E^* - N) \le \mu(E_p)$. Note also that p in (3) can be replaced with an arbitrary smaller one.

Example. Let μ_α be the Gaussian measure with variance α^2. Then

$$F(\xi) = \widehat{\mu}(\xi) = \exp\left(-\frac{\alpha^2}{2} |\xi|_0^2\right), \qquad \xi \in E.$$

By a direct calculation we obtain

$$F'(\xi) = -\alpha^2 e^{-\alpha^2 |\xi|_0^2/2} \xi = -\alpha^2 F(\xi)\xi,.$$

and therefore $F'(\xi) \in E = \bigcap_{p \ge 0} E_p$. Consequently, $F = \widehat{\mu_\alpha} \in \mathcal{A}_p$ for any $p \ge 0$.

4 Cauchy problem and semigroup

We recapitulate some results obtained in [2]. For the fixed complete orthonormal basis $\{e_n\}_{n=1}^\infty$ of H, which are in fact contained in E, let S denote the shift with respect to the basis $\{e_n\}$, i.e., the unique linear continuous (in fact isometric) map from H to H such that

$$Se_n = e_{n+1}, \qquad n = 1, 2, \cdots.$$

We note the following

Lemma 4.1 *$S \in \mathcal{L}(E, E)$ if and only if*

$$\sup_{n \ge 0} \frac{\lambda_{n+1}}{\lambda_n^{1+r}} < \infty$$

for some $r \ge 0$.

PROOF. Suppose first that $S \in \mathcal{L}(E, E)$. Take an arbitrary $p > 0$. Then there exist $q \ge 0$ and $C \ge 0$ such that

$$|S\xi|_p \le C |\xi|_{p+q}, \qquad \xi \in E.$$

In particular, putting $\xi = e_n$ we have

$$|e_{n+1}|_p = |Se_n|_p \le C |e_n|_{p+q}.$$

Hence

$$\lambda_{n+1}^p \le C \lambda_n^{p+q}, \qquad n = 1, 2, \cdots,$$

and

$$\sup_{n \ge 1} \frac{\lambda_{n+1}}{\lambda_n^{1+q/p}} \le C^{1/p} < \infty,$$

as desired. Conversely, we assume that there exists $r \geq 0$ with

$$M = \sup_{n \geq 0} \frac{\lambda_{n+1}}{\lambda_n^{1+r}} < \infty.$$

Consider an element $\xi \in E$ which admits an expansion:

$$\xi = \sum_{n=1}^{\infty} c_n e_n,$$

where $c_n = 0$ except finitely many n. Then by definition,

$$S\xi = \sum_{n=1}^{\infty} c_n S e_n = \sum_{n=1}^{\infty} c_n e_{n+1}.$$

For any $p \geq 0$ we have

$$|S\xi|_p^2 = \sum_{n=1}^{\infty} |c_n|^2 |e_{n+1}|_p^2 = \sum_{n=1}^{\infty} |c_n|^2 \lambda_{n+1}^{2p} \leq M^2 \sum_{n=1}^{\infty} |c_n|^2 \lambda_n^{2p(1+r)} = M^2 |\xi|_{p(1+r)}^2 .$$

This implies that S is a continuous operator on E. qed

From now on we assume that $S \in \mathcal{L}(E, E)$. Then the adjoint $S^* \in \mathcal{L}(E^*, E^*)$ becomes a measurable map from E^* into E^*. Let $M_S(E^*) \subset M(E^*)$ be the space of measures on E^* which are invariant under S^*. We put

$$M_S^2(E^*) = \left\{ \mu \in M_S(E^*) \; ; \; \int_{E^*} |\langle x, \eta \rangle|^2 \mu(dx) < \infty \text{ for all } \eta \in E \right\}.$$

Let $\mathcal{H}$ be the subspace of all $x \in E^*$ such that the limit

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} |\langle x, e_n \rangle|^2 < \infty$$

exists. Then,

$$\| x \| = \lim_{N \to \infty} \left(\frac{1}{N} \sum_{n=1}^{N} |\langle x, e_n \rangle|^2 \right)^{1/2}, \qquad x \in \mathcal{H}.$$

becomes a seminorm of $\mathcal{H}$.

Lemma 4.2 *Let $\mu \in M_S^2$. Then $x \in \mathcal{H}$ for μ-a.e. $x \in E^*$. In other words, the limit*

$$\| x \|^2 = \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} |\langle x, e_n \rangle|^2 < \infty$$

exists for μ-a.e. $x \in E^*$. Moreover, the limit converges in $L^1(E^*, \mu)$.

PROOF. For simplicity we put

$$F(x) = |\langle x, e_1 \rangle|^2.$$

Then, clearly $F \in L^1(E^*, \mu)$. Since S^* is a μ-preserving measurable map from E^* into itself, it follows from the ergodic theorem (e.g., [5, Chap.VIII]) that

$$F^*(x) = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} F(S^{*(n-1)}x)$$

converges μ-a.e. $x \in E^*$ as well as in the L^1-sense. In that case $F^* \in L^1(E^*, \mu)$. On the other hand, since

$$\sum_{n=1}^{N} F(S^{*(n-1)}x) = \sum_{n=1}^{N} \langle S^{*(n-1)}x,\, e_1\rangle^2 = \sum_{n=1}^{N} \langle x,\, S^{(n-1)}e_1\rangle^2 = \sum_{n=1}^{N} \langle x,\, e_n\rangle^2,$$

we see that $F^*(x) = \| x \|^2$. The assertion then follows immediately. qed

In a similar manner,

Lemma 4.3 *Let $\mu, \nu \in M_S^2(E^*)$. Then the limit*

$$\langle\langle x,\, y\rangle\rangle = \frac{1}{N} \sum_{n=1}^{N} \langle x,\, e_n\rangle \langle y,\, e_n\rangle$$

exists for $\mu \times \nu$-a.e. $(x, y) \in E^* \times E^*$.

Proposition 4.4 *If $\mu \in M_S^2(E^*)$, then $F = \widehat{\mu} \in \mathcal{D}$ and*

$$\Delta_L F(\xi) = -\int_{E^*} \| x \|^2 e^{i\langle x, \xi\rangle} \mu(dx).$$

PROOF. It is easily verified from definition that

$$\langle F''(\xi),\, e_n \otimes e_n\rangle = -\int_{E^*} \langle x,\, e_n\rangle^2 e^{i\langle x, \xi\rangle} \mu(dx).$$

Then we need only to apply Lemma 4.2. qed

Consider the Cauchy problem for the Laplace equation:

$$\frac{\partial}{\partial t} F(\xi, t) = \Delta_L F(\xi, t), \qquad F(\xi, 0) = F_0(\xi), \tag{4}$$

where F_0 is a certain function on E. For some particular initial condition the Cauchy problem is solved satisfactorily in Accardi-Roselli-Smolyanov [2].

Theorem 4.5 *Let $\mu \in M_S^2(E^*)$ and put $F_0 = \widehat{\mu}$. Then the solution of the Cauchy problem (4) is given as*

$$F(\xi, t) = \widehat{\mu_t}(\xi), \qquad \mu_t(dx) = e^{-t\| x \|^2} \mu(dx), \qquad t \geq 0.$$

PROOF. By Lemma 4.2 μ_t is well defined and belongs to $M_+(E^*)$. Moreover, obviously μ_t is S^*-invariant and

$$\int_{E^*} \langle x,\, \eta\rangle^2 \mu_t(dx) \leq \int_{E^*} \langle x,\, \eta\rangle^2 \mu(dx) < \infty,$$

namely, $\mu_t \in M_S^2(E^*)$. It then follows from Proposition 4.4 that $\widehat{\mu_t} \in \mathcal{D}$ and

$$\Delta_L F(\xi, t) = -\int_{E^*} \| x \|^2 e^{-t\|x\|^2} e^{i\langle x, \xi\rangle} \mu(dx).$$

On the other hand, since $\| x \|^2$ belongs to $L^1(E^*, \mu)$ by Lemma 4.2, we see by Lebesgue's theorem that

$$\frac{\partial F}{\partial t} = -\int_{E^*} \| x \|^2 e^{-t\|x\|^2} e^{i\langle x, \xi\rangle} \mu(dx).$$

Therefore $F(\xi, t) = \widehat{\mu_t}(\xi)$ is a solution of the Cauchy problem under consideration. qed

We put

$$(\widehat{P}^t \mu)(dx) = e^{-t\|x\|^2} \mu(dx), \qquad \mu \in M_S^2(E^*), \quad t \geq 0.$$

Then $\widehat{P}^t$ constitutes a one-parameter semigroup of transformations on $M_S^2(E^*)$.

Let $\mathcal{B}_S^2(E)$ be the image space of $M_S^2(E^*)$ under the Fourier transform. The induced one-parameter semigroup of transformations on $\mathcal{B}_S^2(E)$ is denoted by P^t. This is called the *heat semigroup* of the Lévy Laplacian Δ_L.

We note the following

Proposition 4.6 *The subspace $M_S^2(E^*)$ is closed under convolution. Therefore $\mathcal{B}_S^2(E)$ is closed pointwise multiplication.*

However, the Lévy Laplacian is not a derivation on $\mathcal{B}_S^2(E)$ and $\widehat{P}^t$ is not multiplicative; namely,

$$\widehat{P}^t(\mu * \nu) = \widehat{P}^t\mu * \widehat{P}^t\nu$$

does not holds in general. In fact, $\widehat{\mu}$ belongs to $\mathcal{D}$ but not to $\mathcal{D}_0$ on which the Lévy Laplacian acts as derivation, see Theorem 2.2.

5 Normal polynomials

In this section we introduce particular classes of polynomials under an additional structure of E, namely, multiplication. We assume that E is equipped with a multiplication which makes E a commutative algebra. Furthermore we assume that the multiplication is continuous (since E is a Fréchet space, there is no difference between joint and separate continuity) and that

$$\langle \xi\eta, \zeta \rangle = \langle \xi, \eta\zeta \rangle, \qquad \xi, \eta, \zeta \in E.$$

This situation often occurs when E is a function space (the multiplication above is the usual pointwise multiplication of functions). By duality multiplication of $f \in E^*$ and $\xi \in E$, denoted by $f\xi = \xi f$, is defined as a unique element in E^* such that

$$\langle f\xi, \eta \rangle = \langle f, \xi\eta \rangle, \qquad \eta \in E.$$

Obviously, the multiplicatication $E^* \times E \to E^*$ is an extension of $E \times E \to E$.

Consider a quadratic function $\xi \mapsto \langle f, \xi^2 \rangle$, where $f \in E^*$ is fixed. Since $(\xi, \eta) \mapsto \langle f, \xi\eta \rangle$ is a continuous bilinear form on $E \times E$, there exists $g \in (E \otimes E)^*$ such that $\langle f, \xi\eta \rangle = \langle g, \xi \otimes \eta \rangle$. Thus, $\langle f, \xi^2 \rangle = \langle g, \xi^{\otimes 2} \rangle$ and there occurs no new quadratic function in this manner. On the contrary, using the new product in E we may introduce a subclass of polynomials. Namely, if $f \in E^*$ is "regular," the corresponding quadratic functions constitute a certain class of quadratic functions. This is immediately generalized to polynomials of any degree. Thus, a *normal polynomial* on E is a finite linear combination of functions of the form:

$$\langle f, \xi^{\nu_1} \otimes \cdots \otimes \xi^{\nu_n} \rangle , \qquad \nu_1, \cdots, \nu_n = 0, 1, 2, \cdots ,$$

where $f \in (E^{\otimes n})^*$ is a regular element. Here the tensor product and the multiplication of E should be carefully distinguished.

We now go into a typical situation. Consider a one dimensional torus $T = \mathbb{R}/\mathbb{Z}$. Put $H = L^2(T)$ and consider d/dt. Then $E = C^\infty(T)$ and $\{e_n\}$ consists of trigonometric functions. In that case $\{e_n\}$ possesses additional properties: first $\{e_n\}$ is *uniformly bounded*:

$$\sup_n \sup_{t \in T} |e_n(t)| < \infty;$$

Second, it is *equally dense*, i.e.,

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \int_0^1 f(t) e_n(t)^2 dt = \int_0^1 f(t) dt, \qquad f \in L^\infty(T).$$

Moreover, the pointwise multiplication gives a continuous bilinear map from $E \times E$ into E. We say that $f \in (E^{\otimes n})^*$ is regular if $f \in L^1(T^n)$. This is the usual definition of a regular distribution. Then we have the space of normal polynomials. In other words, a normal polynomial on E is by definition a linear combination of functions of the form:

$$F(\xi) = \int_{T^n} k(t_1, \cdots, t_n) \xi(t_1)^{\nu_1} \cdots \xi(t_n)^{\nu_n} dt_1 \cdots dt_n, \qquad \xi \in E,$$

where k is an integrable function on T^n. If $\nu_i = 1$ for all i, the polynomial is called *regular* after Lévy's original definition.

Lemma 5.1 *Consider a normal polynomial of the form:*

$$F(\xi) = \langle f, \xi^\nu \rangle , \qquad f \in E^*.$$

Then $F \in \mathcal{D}_0$ if and only if

$$\limsup_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} |\langle f \xi^{\nu - 1}, e_n \rangle|^2 = 0$$

for any $\xi \in E$.

The proof is immediate. Then we come to the following

Proposition 5.2 *Every normal polynomial belongs to $\mathcal{D}_0$.*

The above result generalizes the known fact that the Lévy Laplacian is a derivation on normal polynomials, see [10, Proposition 3.2].

References

[1] L. Accardi, P. Gibilisco and I. V. Volovich, *The Lévy Laplacian and Yang-Mills equations*, Centro Vito Volterra preprint series **129**, 1992.

[2] L. Accardi, P. Roselli and O. G. Smolyanov, *The Brownian motion generated by the Lévy Laplacian*, Centro Vito Volterra preprint series **160**, 1993.

[3] L. Accardi and O. G. Smolyanov, *On Laplacians and traces*, Centro V. Volterra preprint series **141**, 1993.

[4] I. Ya. Aref'eva and I. Volovich, *Higher order functional conservation laws in gauge theories*, in "Generalized Functions and their Applications in Mathematical Physics," Proc. Internat. Conf., Moscow, 1981. (Russian)

[5] N. Dunford and J. T. Schwartz, "Linear Operators, Part I: General Theory," Wiley Classical Library Edition, 1988.

[6] M. N. Feller, *Infinite dimensional elliptic equations and operators of Lévy type*, Russian Math. Surveys **41-4** (1986), 119–170.

[7] Y. Hasegawa, *Lévy's Functional Analysis in terms of an infinite dimensional Brownian motion, I–III*, Osaka J. Math. **19** (1982), 405–428; Osaka J. Math. **19** (1982), 549–570; Nagoya Math. J. **90** (1983), 155–173.

[8] T. Hida and K. Saitô, *White noise analysis and the Lévy Laplacian*, in "Stochastic Processes in Physics and Engineering," (S. Albeverio et al. eds.) pp.177–184, D. Reidel Pub., Dordrecht, 1988.

[9] H.-H. Kuo, *On Laplacian operators of generalized Brownian functionals*, in "Stochastic Processes and their Applications," (K. Itô and T. Hida, eds.) pp.119–128, Lecture Notes in Math. Vol. 1203, Springer-Verlag, 1986.

[10] H.-H. Kuo, N. Obata and K. Saitô, *Lévy Laplacian of generalized functions on a nuclear space*, J. Funct. Anal. **94** (1990), 74–92.

[11] P. Lévy, "Leçons d'Analyse Fonctionnelle," Gauthier-Villars, Paris, 1922.

[12] P. Lévy, "Problèmes Concrets d'Analyse Fonctionelle," Gauthier-Villars, Paris, 1951.

[13] N. Obata, *The Lévy Laplacian and mean value theorem*, in "Probability Measures on Groups IX," (H. Heyer ed.), pp. 242–253, Lecture Notes in Math. Vol. 1379, Springer-Verlag, 1989.

[14] N. Obata, *A characterization of the Lévy Laplacian in terms of infinite dimensional rotation group*, Nagoya Math. J. **118** (1990), 111–132.

[15] E. M. Polishchuk "Contiual Means and Boundary Value Problems in Function Spaces," Birkhäuser, 1988.

[16] P. Roselli, *Laplaciani in spazi a infinite dimensioni e applicazioni*, Tesi, Università di Roma La Sapienza, Maggio 1993.