数理解析研究所講究録
第 877 巻 1994 年 19-33

Basic Sets and Degree Equations for Blocks of Finite Groups

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1. Degree Equations

Let G be a finite group and p a prime. Let G^0 be the set of p-regular elements of G and $\{\chi_1, \ldots, \chi_n\}$ be the irreducible ordinary characters of G. For a subset J of the index set $\{1, \ldots, n\}$, let $\{\chi_J\}$=$\{\chi_j | j \in J\}$.

There are several methods to distibute the irreducible ordinary characters of G into p-blocks. Most available one is to use the central characters. Another one is to use Osima's Theorem.

Theorem 1. (Osima) *For $J \subseteq \{1, \ldots, n\}$, if $\sum_{j \in J} \chi_j(x)\chi_j(y) = 0$ whenever $x \in G^0$ and $y \in G - G^0$, then $\{\chi_J\}$ is a union of p-blocks.*

Put $\rho_J = \sum_{j \in J} \chi_j(1)\chi_j$ for $J \subseteq \{1, \ldots, n\}$. In his paper [8], Harada stated the following;

Conjecture A. *If ρ_J vanishes on $G - G^0$, then $\{\chi_J\}$ is a union of p-blocks of G.*

As in [8], the proof of Conjecture A is reduced to the case where $\{\chi_J\}$ is contained in a single block as follows;

Conjecture A$'$. *Let B be a p-block with the irreducible ordinary characters $\chi_1, \ldots, \chi_k$. For $J \subseteq \{1, \ldots, k\}$, assume that ρ_J vanishes on $G - G^0$. Then $\{\chi_J\} = B$ or $\emptyset$.*

On the other hand, we prove the following;

Theorem 2. *Let B be a p-block of G with defect d which contains the irreducible ordinary characters $\chi_1, \ldots, \chi_k$ and the principal indecomposable characters $\Phi_1, \ldots, \Phi_l$. Let $D= [d_{is}]$ denote the decomposition matrix of B. Then the following assertions hold.*

(i) *There exist $m_i \in Z$ $(i = 1, \ldots, k)$ which satisfy $[m_1 \cdots m_k]\, D = [w_1 \cdots w_l]$, where $\Phi_s(1) = p^a u w_s$ $(s = 1, \ldots, l)$ with* GCD $\{\Phi_s(1)\} = p^a u$.

(ii) *If we set $\chi_i(1) = p^a u m_i + p^{a-d} u \varepsilon_i$ $(i = 1, \ldots, k)$, then all ε_i are integers which satisfy $[\varepsilon_1 \cdots \varepsilon_k]\, D = O$ and $\eta_B = \sum_{i=1}^{k} \varepsilon_i \chi_i$ vanishes on G^0. In particular, we have a degree equation $\eta_B(1) = \sum_{i=1}^{k} \varepsilon_i \chi_i(1) = 0$.*

Proof. (i) Since D has rank l and its invariant factors are all 1, there are integral invertible matrices X and Y such that $D = X \begin{bmatrix} E \\ O \end{bmatrix} Y$, where E is the $l \times l$ identity matrix. If we put here $[w_1 \cdots w_l] Y^{-1} [E \;\; O] X^{-1} = [m_1 \cdots m_k]$, then $m_i \in Z$ and

$$[m_1 \cdots m_k]\, D = [m_1 \cdots m_k]\, X \begin{bmatrix} E \\ O \end{bmatrix} Y = [w_1 \cdots w_l]$$

as required.

(ii) As is well-known, $\mathrm{GCD}\{\chi_i(1)\} = p^{a-d} u$ (see Brauer [1]) and so all ε_i are in Z. By (i) we have

$$p^{a-d} u\, [\varepsilon_1 \cdots \varepsilon_k]\, D = [\chi_1(1) \cdots \chi_k(1)]\, D - p^a u\, [m_1 \cdots m_k]\, D = O.$$

Hence for $x \in G^0$,

$$\eta_B(x) = \sum_{i=1}^{k} \varepsilon_i \chi_i(x) = \sum_{i=1}^{k} \varepsilon_i \sum_{s=1}^{l} d_{is} \varphi_s(x) = \sum_{s=1}^{l} \Big(\sum_{i=1}^{k} \varepsilon_i d_{is}\Big) \varphi_s(x) = 0,$$

where $\{\varphi_s\}$ are the irreducible Brauer characters of B. This completes the proof of Theorem 2.

We call this $\{\varepsilon_i\}$ *a residue set* associated to B.

Theorem 3. *Let B be a p-block of G with the irreducible ordinary characters $\chi_1, \ldots, \chi_k$. For $J \subseteq \{1, \ldots, k\}$, assume that $\sum_{j \in J} \varepsilon_j \chi_j$ vanishes on G^0 for every residue set $\{\varepsilon_i\}$ associated to B. Then $\{\chi_J\} = B$*

or $\emptyset$.

Proof. Let D be the decomposition matrix of B. We consider the vector space $V = \langle [x_1 \cdots x_k] | [x_1 \cdots x_k]D = [0 \cdots 0] \rangle$ over the complex field. Since D is of rank l, V has a basis with entries in Z. Let $\boldsymbol{\delta}=[\delta_1 \cdots \delta_k]$ be an element of V with $\delta_i \in$Z and $\boldsymbol{\varepsilon}=[\varepsilon_1 \cdots \varepsilon_k]$ be a residue set with $\{m_i\}$ associated to B. Then $\boldsymbol{\varepsilon}'=\left[\varepsilon_1 - p^d\delta_1 \cdots \varepsilon_k - p^d\delta_k\right]$ is a residue set with $\{m_i + \delta_i\}$ and $\boldsymbol{\delta}= \frac{1}{p^d}(\boldsymbol{\varepsilon}-\boldsymbol{\varepsilon}')$. Hence V is generated by all $[\varepsilon_1 \cdots \varepsilon_k]$ such that $\{\{\varepsilon_i\}\}$ are residue sets associated to B. For every $y \in G - G^0$, $[\chi_1(y) \cdots \chi_k(y)]$ is evidently contained in V by the orthogonality relation and so it is expressed by a linear combination of $\{[\varepsilon_1 \cdots \varepsilon_k]\}$. Hence $\sum_{j\in J} \chi_j(y)\chi_j$ vanishes on G^0 by our assumption. Thus from Osima's Theorem, we obtain $\{\chi_J\} = B$ or $\emptyset$. The proof is now complete.

Replacing the hypothesis of Theorem 3 with weaker one, we state the following;

Conjecture B. *Let B be a p-block with the irreducible ordinary characters $\chi_1, \ldots, \chi_k$. For $J \subseteq \{1, \ldots, k\}$, assume that $\sum_{j\in J} \varepsilon_j\chi_j(1) = 0$ for every residue set $\{\varepsilon_i\}$ associated to B. Then $\{\chi_J\} = B$ or $\emptyset$.*

It is verified that two conjectures A$'$ and B are equivalent.

Theorem 4. *Conjecture* A$'$ *holds if and only if Conjecture* B *holds.*

Proof. First, assume that Conjecture A$'$ holds and $\sum_{j\in J} \varepsilon_j\chi_j(1) = 0$ for every residue set $\{\varepsilon_i\}$. Then the same argument as in the proof of Theorem 3 implies that $\rho_J(y) = \sum_{j\in J} \chi_j(1)\chi_j(y) = 0$ for every $y \in G - G^0$. Hence $\{\chi_J\} = B$ or $\emptyset$.

Conversely, suppose that Conjecture B holds and ρ_J vanishes on $G - G^0$. Since $\eta_B = \sum_{i=1}^k \varepsilon_i\chi_i$ vanishes on G^0 by Theorem 2, we have

$$0 = (\eta_B, \rho_J) = (\sum_{i=1}^k \varepsilon_i\chi_i, \sum_{j\in J} \chi_j(1)\chi_j) = \sum_{j\in J} \varepsilon_j\chi_j(1).$$

Hence $\{\chi_J\} = B$ or $\emptyset$ which satisfies to complete the proof.

By Theorem 4, in order to prove Harada's conjecture, it suffices to show that Conjecture B holds for every block of G. Many examples show that the following hypothesis makes sense.

Hypothesis 5. *A basic set for a block can be chosen from the set of irreducible ordinary characters.*

Let B be a p-block with the irreducible ordinary characters $\chi_1, \ldots, \chi_k$. Under Hypothesis 5, let $\{\chi_1, \ldots, \chi_l\}$ be a basic set for B and the other characters are expressed as Z-linear combinations of the basic set on G^0 as follows;

$$\chi_\lambda = a_1^\lambda \chi_1 + \cdots + a_l^\lambda \chi_l \ (\lambda = l+1, \ldots, k). \tag{1}$$

Hence the decomposition matrix of B is of the form

$$D = \begin{bmatrix} d_{11} & \cdots & d_{1l} \\ \vdots & & \vdots \\ d_{l1} & \cdots & d_{ll} \\ \sum_{\tau=1}^{l} a_\tau^{l+1} d_{\tau 1} & \cdots & \sum_{\tau=1}^{l} a_\tau^{l+1} d_{\tau l} \\ \vdots & & \vdots \\ \sum_{\tau=1}^{l} a_\tau^{k} d_{\tau 1} & \cdots & \sum_{\tau=1}^{l} a_\tau^{k} d_{\tau l} \end{bmatrix} \begin{matrix} \chi_1 \\ \vdots \\ \chi_l \\ \chi_{l+1} \\ \vdots \\ \chi_k \end{matrix}$$

Then

$$\begin{aligned} \boldsymbol{n}_1 &= [\ -a_1^{l+1} \ \cdots \ -a_l^{l+1} \ \ 1 \ \ 0 \ \ 0 \ \cdots \ 0\ ] \\ \boldsymbol{n}_2 &= [\ -a_1^{l+2} \ \cdots \ -a_l^{l+2} \ \ 0 \ \ 1 \ \ 0 \ \cdots \ 0\ ] \\ & \qquad \vdots \\ \boldsymbol{n}_{k-l} &= [\ -a_1^{k} \ \cdots \ -a_l^{k} \ \ 0 \ \ 0 \ \ 0 \ \cdots \ 1\ ] \end{aligned}$$

are linearly independent solutions of the equation

$$[x_1 \cdots x_k]D = [0 \cdots 0].$$

As in Theorem 2, let $\boldsymbol{m}_0 = [m_1^0 \cdots m_k^0]$ be a Z-solution of the equation

$$[x_1 \cdots x_k]D = [w_1 \cdots w_l]. \tag{2}$$

Then

$$[m_1 \cdots m_k] = \boldsymbol{m}_0 + z_1\boldsymbol{n}_1 + \cdots + z_{k-l}\boldsymbol{n}_{k-l} \quad (z_1, \ldots, z_{k-l} \in \mathsf{Z})$$

are all of Z-solutions of (2). We define, for a residue set $\{\varepsilon_i\}$ with $\{m_i\}$,

$$\begin{aligned} \boldsymbol{\chi}(1) &= [\ \chi_1(1) \ \cdots \ \chi_k(1)\ ] \\ \boldsymbol{\varepsilon} &= [\ \varepsilon_1 \ \cdots \ \varepsilon_k\ ]\ . \end{aligned}$$

Since $\chi_i(1) = p^a u m_i + p^{a-d} u \varepsilon_i$, we have

$$p^{a-d}u\boldsymbol{\varepsilon} = \boldsymbol{\chi}(1) - p^a u(\boldsymbol{m}_0 + z_1\boldsymbol{n}_1 + \cdots + z_{k-l}\boldsymbol{n}_{k-l}). \tag{3}$$

Let $\cdot$ denote the scalar product of vectors. For $J \subseteq \{1, \ldots, k\}$ and a vector $\boldsymbol{v}= [v_1 \cdots v_k]$, let $\boldsymbol{v}^J$ denote the vector of size k whose i-th component is v_i if $i \in J$ and 0 otherwise. Then by (3)

$$\begin{aligned} (\boldsymbol{\chi}(1)^J - p^a u \boldsymbol{m}_0^J - p^a u z_1 \boldsymbol{n}_1^J - \cdots - p^a u z_{k-l} \boldsymbol{n}_{k-l}^J) \cdot \boldsymbol{\chi}(1)^J & \\ = p^{a-d} u \textstyle\sum_{j \in J} \varepsilon_j \chi_j(1). & \qquad (4) \end{aligned}$$

If $\sum_{j \in J} \varepsilon_j \chi_j(1) = 0$ for every residue set $\{\varepsilon_i\}$, then by (4) we have

$$\boldsymbol{n}_1^J \cdot \boldsymbol{\chi}(1)^J = \cdots = \boldsymbol{n}_{k-l}^J \cdot \boldsymbol{\chi}(1)^J = 0. \tag{5}$$

Since $\eta_B(1) = \sum_{i=1}^k \varepsilon_i \chi_i(1) = 0$ by Theorem 2, similarly we have

$$\boldsymbol{n}_1^{J'} \cdot \boldsymbol{\chi}(1)^{J'} = \cdots = \boldsymbol{n}_{k-l}^{J'} \cdot \boldsymbol{\chi}(1)^{J'} = 0, \tag{6}$$

where $J' = \{1, \ldots, k\} - J$. Hence the next is proved.

Lemma 6. *Under Hypothesis* 5, *if there is no non-empty proper subset* J *of* $\{1,\ldots,k\}$ *for which* (5) *or* (6) *holds, then Conjecture* B *holds.*

Lemma 7. *Under Hypothesis* 5, *if a basic set* $\{\chi_1,\ldots,\chi_l\}$ *is contained in* $\{\chi_J\}$ *or* $\{\chi_{J'}\}$ *and* $\sum_{j\in J}\varepsilon_j\chi_j(1)=0$ *for every residue set* $\{\varepsilon_i\}$ *associated to* B, *then* $\{\chi_J\}=B$ *or* $\emptyset$.

Lemma 8. *Under Hypothesis* 5, *if the coefficients* a_s^λ *in the equation* (1) *are all non-negative, then Conjecture* B *holds.*

In particular, we deduce

Corollary 9. *Under Hypothesis* 5, *if* $l=1$ *or* 2, *then Conjecture* B *holds.*

Corollary 10. *If the irreducible Brauer characters of* B *are all liftable, then Conjecture* B *holds.*

Using these Lemmas and Corollaries, we can prove

Theorem 11. *If* $G=PSL(2,q)$ *such that* q *is a power of a prime, then Conjecture* A *holds.*

Theorem 12. *If* G *is the symplectic group* $Sp(4,q)$, *where* q *is a power of an odd prime* e, *then Conjecture* A *holds for every* p *different from* e.

Proof. By Theorem 4, it suffices to show that Conjecture B holds for every block. Basic sets of $Sp(4,q)$ are determined by White in [23-25]. We use the notation of those papers for the characters and the blocks. The order of $G=Sp(4,q)$ is $q^4(q^2+1)(q+1)^2(q-1)^2$, so if p $(\neq e)$ is a prime dividing $|G|$, then $p=2$ or p divides exactly one of q^2+1, $q+1$ or $q-1$. If p is odd and divides q^2+1, then the defect group of each block is cyclic. If p is odd and divides one of $q+1$ or $q-1$, then blocks with non-maximal defect have cyclic defect groups. In these cases, the result is clear by Harada [8].

(i) $p = 2$. For the blocks $b_1(r)$, $b_2(r)$, $b_3(r,s)$, $b_4(r,s)$, $b_5(r,s)$, $b_{67}(r)$ and $b_{89}(r)$, we have $l = 1$ or 2 and for the blocks $b_{\mathrm{I}}(r)$, the irreducible Brauer characters are all liftable. Hence the result follows by Corollaries 9 and 10. For the other blocks, basic sets and the expressions of the other characters as linear combinations of the basic sets are shown in the Tables below. The first row in each Table is a basic set and missing entries are 0.

$b_{\mathrm{III}}(r)$

BS	ξ_3	ξ_3'	ξ_{41}	
ξ_{42}	1	1	-1	
ξ_{41}'		1	-1	
ξ_{42}'	-1		1	
χ_3	1	1		$(q \equiv 1 \pmod 4)$
χ_5	-1	1		$(q \equiv 3 \pmod 4)$

b_0 (the principal block)

BS	1_G	θ_1	θ_7	θ_8	θ_{10}	θ_{12}	θ_{13}
Φ_7	1	1	1	1	1		
Φ_9	1		1	1	1	1	
θ_5		-1					1
.							
.							
.							

For the blocks $b_{\mathrm{III}}(r)$ and b_0, assume that a subset J satisfies (5) of Section 2 and some character in the basic set is contained in $\{\chi_J\}$. Then the above each Table shows that the other characters are also contained in $\{\chi_J\}$. Hence the result follows by Lemma 7.

(ii) $p \neq 2$, $p|q-1$. For the blocks $b_3(s,t)$, $b_{\rm III}(s)$, $b_{41}(s)$ and $b_{89}(s)$, we have $l = 1$ or 2 and for the other blocks, the irreducible Brauer characters are all liftable. Thus Conjecture B holds by Corollaries 9 and 10.

(iii) $p \neq 2$, $p|q+1$. For the blocks $b_4(s,t)$, $b_{\rm I}(s)$, $b_{21}(s)$ and $b_{67}(s)$, we have $l = 1$ or 2 and for the blocks b_1 and b_2, the irreducible Brauer characters are all liftable. Hence the result follows by Corollaries 9 and 10. For the principal block b_0, a basic set and the expressions of the other characters as linear combinations of the basic set are as follows;

b_0 (the principal block)

BS	1_G	θ_{10}	θ_{11}	θ_{12}	θ_{13}
χ_4	1	-2	-1	-1	1
χ_6	-1	1		1	
χ_7		-1	-1		1
ξ_1	-1	1	1		
ξ_1'		-1		-1	1

Evidently, the equation (5) or (6) does not occur for any non-empty proper subset J and the result is clear by Lemma 6. This completes the proof of Theorem 12.

Theorem 13. *If G is the finite Chevalley group $G_2(q)$, where q is a power of a prime e, then Conjecture* A *holds for every p different from e.*

2. Basic Sets of Brauer Characters

First, a simple method to distribute irreducible ordinary characters of G into π-blocks is described, where π is a set of primes.

Theorem 14. *Let $\{\chi_1, \dots, \chi_n\}$ be the irreducible ordinary characters of G and $\{x_1, \dots, x_r\}$ the complete set of representatives of π-regular* *conjugate classes of G. Let* $X = [\chi_i(x_j)]$ *be the submatrix of the character table of G. Then by making elementary column operations and interchanging rows, X can be changed of the form*

$$\begin{bmatrix} A_1 & & O \\ & \ddots & \\ O & & A_t \end{bmatrix},$$

where each A_i *is of the form* $A_i = \begin{bmatrix} E \\ A_i' \end{bmatrix}$ *with the identity matrix E and can not be arranged of the form* $\begin{bmatrix} A_i'' & O \\ O & A_i''' \end{bmatrix}$. *Furthermore, each* A_i *forms a single* π*-block of G.*

We can calculate basic sets consisting of the irreducible ordinary characters for the blocks of 21 sporadic groups and their extensions.

Theorem 15. *For every block of the sporadic simple groups* M_{11}, M_{12}, J_1, M_{22}, J_2, M_{23}, HS, J_3, M_{24}, M^cL, He, Ru, Suz, $O'N$, Co_3, Co_2, Fi_{22}, HN, Ly, Th, J_4, *their associated covering and automorphism groups, a basic set of Brauer characters can be chosen from among the irreducible ordinary characters.*

This is proved by displaying basic sets and the expressions of the other characters as Z-linear combinations in the Tables of Appendix. We use the character tables and the notation in the form of ATLAS-style (see Conway *et al* [3] for details).

As in the proof of Theorem 14, let

$$A_i = \begin{bmatrix} 1 & & O \\ & \ddots & \\ O & & 1 \\ a_1^{l+1} & \cdots & a_l^{l+1} \\ \vdots & & \vdots \\ a_1^k & \cdots & a_l^k \end{bmatrix} \begin{matrix} \chi_1 \\ \vdots \\ \chi_l \\ \chi_{l+1} \\ \vdots \\ \chi_k \end{matrix}$$

If we can choose $\{\chi_1, \ldots, \chi_l\}$ such that all a_s^λ are rational integers, then $\{\chi_1, \ldots, \chi_l\}$ is a basic set of Brauer characters for this block and

$$\chi_\lambda = a_1^\lambda \chi_1 + \cdots + a_l^\lambda \chi_l \ \ (\lambda = l+1, \ldots, k)$$

on G^0. Then the Table in Appendix is displayed as

BS	χ_1	$\cdots$	χ_l
χ_{l+1}	a_1^{l+1}	$\cdots$	a_l^{l+1}
$\vdots$	$\vdots$		$\vdots$
χ_k	a_1^k	$\cdots$	a_l^k

Missing entries in the Tables are 0.

Appendix (Examples)

Group: J_2 $\begin{bmatrix} G & G.2 \\ 2.G & 2.G.2 \end{bmatrix}$ Prime: 2 Defect: $\begin{matrix} 7 & 8 \\ 8 & 9 \end{matrix}$

BS	χ_1	χ_2 ·	χ_3 ·	χ_4 ·	χ_5 ·	χ_6	χ_{11}
χ_7 :	−1	−2	−2	2	2	1	
χ_8 ·	−1		−2	2	1	1	
χ_9 ·	−1	−2		1	2	1	
χ_{10} :	−2	−1	−1	2	2	1	
χ_{13} :	−1	−1	−1	1	1	1	1
χ_{14} ·	−1		−1	1	1	1	1
χ_{15} ·	−1	−1		1	1	1	1
χ_{18} :	−1	−2	−2	2	2	2	1
χ_{20} :	−2	−1	−1	1	1	1	2
χ_{21} :	−2	−1	−1	1	1	2	2
χ_{22} ·	−1		−1	1			
χ_{23} ·	−1	−1			1		
χ_{24} :		−1	−1	1	1		
χ_{25} ·		−1	−1	1	1	1	
χ_{26} ·		−1	−1	1	1	1	
χ_{27} ·	−1	−1	−2	2	1	1	
χ_{28} ·	−1	−2	−1	1	2	1	
χ_{31} :				−1	−1		1
χ_{32} ·		−1	1				1
χ_{33} ·		1	−1				1
χ_{34} :	−2	−1	−1	2	2	1	1
χ_{35} :	−2	−1	−1	2	2	2	1
χ_{36} :	−2	−1	−1	1	1	2	2
χ_{37} :	−2	−2	−2	2	2	2	2

Group: Suz $\begin{bmatrix} 2.G & 2.G.2 \\ 6.G & 6.G.2 \end{bmatrix}$ Prime: 3 Defect: $\begin{matrix} 7 & 7 \\ 8 & 8 \end{matrix}$

BS	χ_{44} ..	χ_{49} ..	χ_{64} ..	χ_{65} ..	χ_{66} ..	χ_{67} ..	χ_{68} ..	χ_{69} ..	χ_{72} ..	χ_{74} ..
χ_{45} ·	−1			1	−2	1	−3	1	−1	2
χ_{46} ·	−1			1	−2	1	−3	1	−1	2
χ_{47} ·					−1		−1			1
χ_{48} ·					−1		−1			1
χ_{50} ·	−1	−1	2	1	−1		−4	−1	−2	4
χ_{51} ·	−1	−1	2	1	−1		−4	−1	−2	4
χ_{52} :			2	1^1	1^0		−2	-1^{-1}	−2	2
χ_{53} :			3^2		1^0		−2	-1^{-1}	−2	2
χ_{54} ·	1		−2	−2		−1	4	−1	4	−3
χ_{55} ·	1		−2	−2		−1	4	−1	4	−3
χ_{56} ·	1		−2	−2		−1	4	−1	4	−3
χ_{57} ·	1		−2	−2		−1	4	−1	4	−3
χ_{58} ·			−1	−1	−1		1		1	
χ_{59} ·			−1	−1	−1		1		1	
χ_{62} ·	−1		1	1	−1	1	−3		−2	3
χ_{63} ·	−1		1	1	−1	1	−3		−2	3
χ_{70} ·	−1	−1	2	1	−1		−3	−1	−2	4
χ_{71} ·	−1	−1	2	1	−1		−3	−1	−2	4
χ_{73} :	-1^0		1^1	1^1	-1^{-1}		−2	-1^{-1}	−2	4
χ_{75} :	-1^0		-1^0		-1^{-1}					2
χ_{76} :	2	1^1	−2	−2	2		6		3^2	−4
χ_{115} *				1	−1	1	−2	1	−1	1
χ_{116} *	1			−1		−1	1	−1	1	
χ_{117} *	−1				−2		−2			2
χ_{118} *	−2	−1	3	2	−2	1	−7		−4	6
χ_{119} *		1		1		1	−1	1	−1	
χ_{120} *			3	1	1		−3	−1	−3	3
χ_{121} ·	1			−1	1		2		1	−2
χ_{122} *	1			−1	1		2		1	−2
χ_{123} *			−1		−1				1	
χ_{124} *			−1		−1				1	
χ_{125} *				−1	−1				1	
χ_{126} *	1		−1	−1			2		2	−2
χ_{127} *	−1	−1	1	1	−1		−3	−1	−1	3
χ_{128} *	1	1	−1	−1	1		3		2	−3
χ_{129} ·	1		−2	−2		−1	4	−1	4	−3
χ_{130} *	1		−2	−2		−1	4	−1	4	−3
χ_{131} *			1		1			−1		
χ_{134} *			1	1	1		−1	1	−1	
χ_{135} *	−1		1	1	−1		−2		−2	3
χ_{136} *	1		−1	−1	1	−1	3		3	−3
χ_{137} *	1		−1	−1	1	−1	4	−1	2	−2
χ_{138} *	−1	−1	2	1	−1		−3	−1	−2	4
χ_{139} *	1	1	−3	−2	1		5	1	3	−4
χ_{140} *			1					−1		1
χ_{141} *			1		1		−1	−1	−1	2
χ_{142} *			−2	−1	−1		2		2	
χ_{143} *	−1		1	1	−1	1	−2		−2	4

Group: $O'N$ $\left[\begin{matrix} G & G.2 \\ 3.G & 3.G.2 \end{matrix}\right]$ Prime: 3 Defect: $\begin{matrix} 4 & 4 \\ 5 & 5 \end{matrix}$

BS	χ_1 ..	χ_3 .	χ_4 .	χ_5 .	χ_6 .	χ_8 .	χ_9 .	χ_{10} ..	χ_{16} .	χ_{17} .	χ_{18} ..	χ_{20} ..	χ_{29} ..	χ_{30} ..
χ_7 :		1	1	1	1	−2	−2		2	2		2	-1^{0}	-1^{0}
χ_{19} :	1^{1}	1	1	2	2	−1	−1	-1^{0}	2	2	−2			
χ_{23} :		−2	−2	−1	−1	3	3	−2	−2	−2	-1^{0}	-3^{-2}	2	2
χ_{24} :		−2	−2	−1	−1	3	3	−2	−2	−2	-1^{0}	-3^{-2}	2	2
χ_{31} ·		−1	−1	−1	−1	2	2	−1	−1	−2		−2	1	1
χ_{32} *		−1	−1	−1	−1	2	2	−1	−2	−1		−2	1	1
χ_{33} ·		1	1			−2	−1	1	1	1	1	2	−1	−1
χ_{34} *		1	1			−1	−2	1	1	1	1	2	−1	−1
χ_{38} *		1	1	1	1	−1	−1		1	1		1		−1
χ_{39} *		1	1	1	1	−1	−1		1	1		1	−1	
χ_{40} *	1	1	1	1	1	−1	−1		1	1	−1			
χ_{41} *	1			1	1			−1	1	1	−1			
χ_{44} *		−1	−1	−1	−1	2	2		−2	−2		−1	1	1
χ_{45} *								1			1			
χ_{48} ·		−1	−1		−1	1	1	−1	−1	−1		−1	1	1
χ_{49} *		−1	−1	−1		1	1	−1	−1	−1		−1	1	1
χ_{54} ·			−1			1	1	−1			−1	−1	1	1
χ_{55} *		−1				1	1	−1			−1	−1	1	1

Group: $O'N$ $\left[\begin{matrix} G & G.2 \\ 3.G & 3.G.2 \end{matrix}\right]$ Prime: 3 Defect: $\begin{matrix} 2 & 2 \\ 3 & 3 \end{matrix}$

BS	χ_2 ..	χ_{11} ..	χ_{12} ..	χ_{13} .	χ_{14} .
χ_{15} :	-1^{-1}	−2	1^{1}	1	1
χ_{35} *		−1	1		
χ_{36} ·		−1			1
χ_{37} *		−1		1	
χ_{42} *	1	1			
χ_{43} *	−1	−1	1	1	1

Group: $O'N$ $[G\ G.2]$ Prime: 7 Defect: 3 3

BS	χ_1 ..	χ_2 ..	χ_3 .	χ_4 .	χ_5 .	χ_6 .	χ_7 ..	χ_8 .	χ_9 .	χ_{10} ..
χ_{11} :			1	1	−1	−1		1	1	
χ_{12} :	-1^{0}	-3^{-2}	1	1	−2	−2	2	1	1	-1^{-1}
χ_{25} :	(±1)	−2	1	1	−2	−2	2			-2^{-2}
χ_{29} :					1	1	-1^{-1}			
χ_{30} :					1	1	-1^{-1}			

	χ_{13} .	χ_{14} .	χ_{16} .	χ_{17} .	χ_{21} ..	χ_{22} ..	χ_{26} ..	χ_{27} ..	χ_{28} ..
χ_{11}	−1	−1	1	1					
χ_{12}	−1	−1	2	2	−2	−2	1^{1}	1^{1}	1^{1}
χ_{25}	−1	−1	1	1	-1^{-1}	-1^{-1}	1^{1}	1^{1}	1^{1}
χ_{29}			−1	−1	1^{0}	1^{1}			
χ_{30}			−1	−1	1^{0}	1^{1}			

Group: Fi_{22} [G $G.2$] Prime: 5 Defect: 2 2

BS	$\chi_1^{\mp}$	$\chi_2^{\pm}$	$\chi_3^{\mp}$	$\chi_4^{\pm}$	$\chi_6^{\mp}$	$\chi_{11}^{\pm}$	$\chi_{14}^{\mp}$	$\chi_{20}^{\mp}$
$\chi_{60}^{\pm}$				1	−1	−1	1	−1
$\chi_{63}^{\pm}$					−1	−1		−1
$\chi_{64}^{\pm}$		−1	−1			−1	1	−1
$\chi_{65}^{\mp}$	−1	1		1	−1	−1		

	$\chi_{30}^{\pm}$	$\chi_{37}^{\mp}$	$\chi_{38}^{\pm}$	$\chi_{49}^{\pm}$	$\chi_{50}^{\pm}$	$\chi_{56}^{\mp}$	$\chi_{57}^{\mp}$	$\chi_{58}^{\mp}$
χ_{60}	−1	1		1	1			
χ_{63}	−1		1	1		1	−1	1
χ_{64}	−1		1		1	1		
χ_{65}		1	−1		1		1	

Group: Fi_{22} [$2.G$ $2.G.2$] Prime: 5 Defect: 2 2

BS	χ_{66} ..	χ_{69} .	χ_{70} .	χ_{73} .	χ_{74} .	χ_{75} ..	χ_{76} .	χ_{77} .
χ_{71} ·	1		−1		−1	−1	1	1
χ_{72} ·	1	−1		−1		−1	1	1
χ_{113} ·					−1	−1		1
χ_{114} ·				−1		−1	1	

	χ_{81} ..	χ_{82} .	χ_{83} .	χ_{85} .	χ_{86} .	χ_{102} .	χ_{103} .	χ_{104} ..
χ_{71}	−1	−1		1				
χ_{72}	−1		−1		1			
χ_{113}	−1	−1	−1		1	1		1
χ_{114}	−1	−1	−1	1			1	1

Group: Fi_{22} [$3.G$ $3.G.2$] Prime: 5 Defect: 2 2

BS	χ_{115} *	χ_{116} *	χ_{117} .	χ_{118} *	χ_{120} *	χ_{121} *	χ_{124} *	χ_{127} *
χ_{145}*		−1	1	1	1	−1		1
χ_{157}*					−1			−1
χ_{160}*			−1	−1		1	1	−1
χ_{163}*	−1	1	−1	−1	−1			−1

	χ_{128} *	χ_{137} *	χ_{138} *	χ_{139} *	χ_{140} *	χ_{148} .	χ_{149} *	χ_{158} *
χ_{145}	−1	1			1			
χ_{157}	1	−1	−1	1	−1	1	1	
χ_{160}	1		1	−1				1
χ_{163}	2	−2	−1		−1	1	1	1

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