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On Auslander-Reiten components for group algebras of finite groups

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Throughout G is a finite group and k denotes an algebraically closed field of characteristic $p > 0$. Let B be a block of the group algebra kG. Let $\Gamma_s(B)$ be the stable Auslander-Reiten quiver of B and Θ a connected component of $\Gamma_s(B)$. Then it is known that if Θ is not a tube and a defect group of B is not a Kleinian four group, Θ is isomorphic to $\mathbf{Z}A_\infty$, $\mathbf{Z}D_\infty$ or $\mathbf{Z}A_\infty^\infty$ (see [Bn], [Bs], [E1], [E-S] and [W]). In Section 1, we give some condition which implies that Θ is isomorphic to $\mathbf{Z}A_\infty$. In Section 2, we consider a connected component of the form $\mathbf{Z}A_\infty$ which contains a simple module.

The notation is almost standard. All kG-modules considered here are finite dimensional over k. For a non-projective indecomposable kG-module W, we write $\mathscr{A}(W)$ to denote the Auslander-Reiten sequence (AR-sequence for short) $0 \to \Omega^2 W \to m(W) \to W \to 0$ terminating at W, where Ω is the Heller operator, and we write $m(W)$ to denote the middle term of $\mathscr{A}(W)$. Concerning some basic facts and terminologies used here, we refer to [Bn] and [E1].

1. $\mathbf{Z}A_\infty$–components

The purpose of this section is to show the following theorem.

Theorem 1.1. Let Θ be a connected component of $\Gamma_s(B)$ and M an indecomposable kG-module in Θ. Let P be a vertex of M, S a P-source of M and Δ the connected component of $\Gamma_s(kP)$ containing S. Suppose that Δ is isomorphic to $\mathbf{Z}A_\infty$. Then Θ is isomorphic to $\mathbf{Z}A_\infty$.

Assume the same hypothesis as in Theorem 1.1. Then since Δ is isomorphic to $\mathbf{Z}A_\infty$, P is not cyclic, dihedral, semidihedral or generalized quaternion (see for example [E1]). Moreover Θ is isomorphic to either $\mathbf{Z}A_\infty$, $\mathbf{Z}D_\infty$ or $\mathbf{Z}A_\infty^\infty$ since k is algebraically closed. By [Bn, Theorem 2.30.6], if we have an unbounded additive function on Θ, we can conclude that Θ is isomorphic to $\mathbf{Z}A_\infty$. Following the argument of [E2, Section 5], we will construct an unbounded additive function.

In order to prove Theorem 1.1, we recall the result of Okuyama and Uno[O-U].

Theorem 1.2([O-U, Theorem]). Let Γ be a connected component of $\Gamma_s(kG)$. Suppose that Γ is not a tube. Then one of the following holds.

(i) All the modules in Γ have the vertices in common.

(ii) We can take $T: X_1 - X_2 - X_3 - \cdots X_n - \cdots$ in Γ with $\Gamma \cong \mathbf{Z}T$ and $\mathrm{vx}(X_1) \leqq \mathrm{vx}(X_2) \leqq \mathrm{vx}(X_3) \leqq \mathrm{vx}(X_4) = \mathrm{vx}(X_5) = \cdots = \mathrm{vx}(X_n) = \cdots$.

(iii) $p = 2$, $\Gamma = \mathbf{Z}A_\infty^\infty$, and only two distinct vertices P and Q occur, with $Q < P$. Moreover, one of the following holds.

(iiia) $|P : Q| = 2$ with $|Q| > 4$, and the modules with vertex Q lie in a subquiver Γ_Q such that both Γ_Q and $\Gamma \setminus \Gamma_Q$ are isomorphic to $\mathbf{Z}A_\infty$ as graphs.

(iiib) Q is a Kleinian four group and P is a dihedral group of order 8, and the modules with vertex Q lie in two or four adjacent τ-orbits.

Let $\mathrm{a}_k(G)$ be the Green ring. For an exact sequence of kG-modules $\mathcal{S}: 0 \to A \to B \to C \to 0$, let $[\mathcal{S}] \in \mathrm{a}_k(G)$ be the element $[\mathcal{S}] = B - A - C$.

Lemma 1.3. Let V and W be non-projective indecomposable kG-modules with the same vertex P, and S a P-source of W. Suppose that there is an irreducible map from V to W. Then for some P-source U of V, there exists an irreducible map from U to S.

Proof. Let $\mathcal{A}(W)$ be the AR-sequence $0 \to \Omega^2 W \to m(W) \to W \to 0$ terminating at W. Then $V \mid m(W)$. By [K2, Lemma 1.6(2)], we have $[\mathcal{A}(W){\downarrow_P}] = t(\Sigma_{g \in N/H}[\mathcal{A}(S^g])$, where $N = \mathbf{N}_G(P)$, $H = \{ g \in N \mid S^g \cong S \}$ and t is the multiplicity of M in $S{\uparrow^G}$. This implies that some

P-source U of V is isomorphic to a direct summand of the middle term $m(S)$ of the AR-sequence $\mathscr{A}(S)$.

Lemma 1.4. Under the same hypothesis as in Theorem 1.1, assume that Θ is isomorphic to either $\mathbf{Z}D_\infty$ or $\mathbf{Z}A_\infty^\infty$. Then;

(1) We have a connected subquiver Ξ of Θ and a tree T_1 : $M \leftarrow M_1 \leftarrow M_2 \leftarrow \cdots \leftarrow M_n \leftarrow \cdots$ in Ξ such that $\Xi \cong \mathbf{Z}T_1$ and $P = \mathrm{vx}(M) = \mathrm{vx}(M_i)$ for all i.

(2) We have a tree $T_2 : U_1 \leftarrow \cdots \leftarrow U_m \leftarrow S \leftarrow S_1 \leftarrow \cdots \leftarrow S_n \leftarrow \cdots$ in Δ such that $\Delta \cong \mathbf{Z}T_2$ and S_i is a P-source of M_i for all i (m may be zero, and in this case S lies at the end of Δ).

Proof. (1) follows immediately from Theorem 1.2.

(2) By Lemma 1.3, we have P-sources S_i of M_i and a subquiver $S \leftarrow S_1 \leftarrow \cdots \leftarrow S_n \leftarrow \cdots$ in Δ. Thus we have only to show that $S_{i+1} \not\cong \Omega^2 S_{i-1}$ for all $i \geqq 1$. Assume contrary that $S_{i+1} \cong \Omega^2 S_{i-1}$ for some i. Let r_i be the multiplicity of S_i in $M_i{\downarrow}P$. By [K2, Lemma 1.6(2)], we have $[\mathscr{A}(M_i){\downarrow}_P] = t_i(\Sigma_{g\in N/H}[\mathscr{A}(S_i^g)])$, where $N = \mathrm{N}_G(P)$, $H = \{g \in N \mid S_i^g \cong S_i\}$ and t_i is the multiplicity of M_i in $S_i{\uparrow}^G$. Since Δ is isomorphic to $\mathbf{Z}A_\infty$, it follows that $r_{i-1} + r_{i+1} \leqq t_i \leqq r_i$ and $r_{i+1} < r_i$. On the other hand, we have $[\mathscr{A}(M_{i+1}){\downarrow}_P] = t_{i+1}(\Sigma_{g\in N/H}[\mathscr{A}(S_{i+1}{}^g)])$, where t_{i+1} is the multiplicity of M_{i+1} in $S_{i+1}{\uparrow}^G$. This implies that $r_i \leqq t_{i+1} \leqq r_{i+1}$, a contradiction.

Proof of Theorem 1.1. We continue to use the same notation in Lemma 1.4. Let Q be a minimal p-subgroup of G such that $M{\downarrow}_Q$ is not projective. Since M is not projective, $M{\downarrow}_Q$ is periodic from [C, Lemma 2.5]. By the Mackey decomposition $M{\downarrow}_Q \mid (S{\uparrow}^G){\downarrow}_Q \cong \oplus_{g\in P\backslash G/Q}(S^g{\downarrow}_{P^g\cap Q}){\uparrow}_Q$. Since $M{\downarrow}_Q$ is not projective, $S^g{\downarrow}_{P^g\cap Q}$ is not projective for some $g \in G$. Then $S^g{\downarrow}_{P^g\cap Q} \mid M{\downarrow}_{P^g\cap Q}$ and thus $M{\downarrow}_{P^g\cap Q}$ is not projective. This implies that $Q = P^g \cap Q$ and $Q < P^g$ by our choice of Q. Therefore we may assume that $Q < P$ and $S{\downarrow}_Q$ is periodic and non-projective (if necessary, replace P, S and Δ by P^g, S^g and Δ^g). We claim that Q satisfies the following two conditions for any indecomposable kG-module W in Θ (and any kP-module V in Δ):

(A1) W and V are not Q-projective; (A2) $W{\downarrow}_Q$ and $V{\downarrow}_Q$ are not projective. Indeed, since both $M{\downarrow}_Q$ and $S{\downarrow}_Q$ are periodic and non-projective, it follows that for any W in Θ and any V in Δ, $W{\downarrow}_Q$ and $V{\downarrow}_Q$ are periodic and non-projective, and thus both W and V are not Q-projective. Let $d_Q(W)$ (resp. $d_Q(V)$) be the number of non-projective indecomposable direct summands of $W{\downarrow}_Q$ (resp. $V{\downarrow}_Q$). Then d_Q is an additive function on Θ and also on Δ (see, e. g., [O], [E-S] and [K3]). Note that d_Q commutes with $\tau = \Omega^2$.

Now Θ is isomorphic to either $\mathbf{Z}A_\infty$, $\mathbf{Z}D_\infty$ or $\mathbf{Z}A_\infty^\infty$. Assume by way of contradiction that Θ is isomorphic to either $\mathbf{Z}D_\infty$ or $\mathbf{Z}A_\infty^\infty$. Then by [Bn, Lemma 2.30.5] any additive function on Θ which commutes with Ω^2 is bounded. On the other hand, since Δ is isomorphic to $\mathbf{Z}A_\infty$, an additive function d_Q on Δ is unbounded. Since $S_i{\downarrow}_Q \mid M_i{\downarrow}_Q$ by Lemma 1.4, it follows that $d_Q(S_i) \leqq d_Q(M_i)$ for all i. This implies that an additive function d_Q on Θ is unbounded, a contradiction.

Corollary 1.5. Assume that k is algebraically closed and let Θ be a connected component of $\Gamma_s(kG)$. Let M be an indecomposable kG-module in Θ, P a vertex of M and S a P-source of M. Suppose that P is not cyclic, dihedral, semidihedral or generalized quaternion and that the k-dimension of S is not divisible by p. Then Θ is isomorphic to $\mathbf{Z}A_\infty$.

Proof. By [K2, Theorem 2.1], the connected component of $\Gamma_s(kP)$ containing S is isomorphic to $\mathbf{Z}A_\infty$. Hence the result follows by Theorem 1.1.

In particular we have the following.

Corollary 1.6. Let B be a block of kG whose defect group is not cyclic, dihedral, semidihedral or generalized quaternion and M a simple module in B of height 0. Then M lies in a $\mathbf{Z}A_\infty$-component.

Remark. In [E2], Erdmann proved that if a p-group P is not cyclic, dihedral, semidihedral or generalized quaternion, then there are infinitely many kP-modules of dimension 2 or 3 lying at the ends of $\mathbf{Z}A_\infty$–components ([E2, Propositions 4.2 and 4.4]). Consequently she showed that for a wild block B over an algebraically closed field, the stable Auslander-Reiten quiver $\Gamma_s(B)$ has infinitely many $\mathbf{Z}A_\infty$–components ([E2, Theorem 5.1]).

2. $\mathbf{Z}A_\infty$–components and simple modules

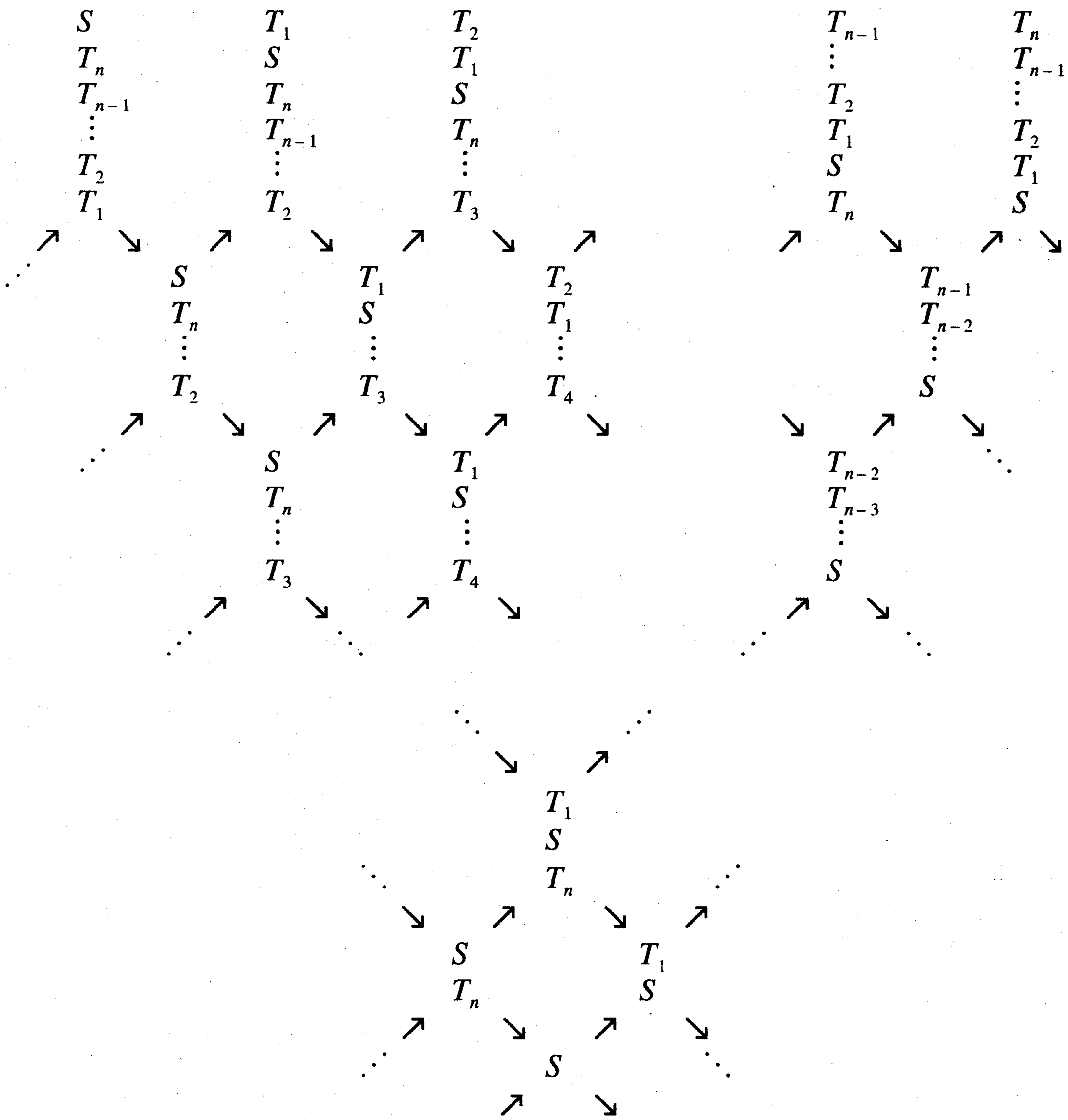
In this section we consider a $\mathbf{Z}A_\infty$–component which contains a simple module. Note that if B is a wild block (i. e., a defect group of B is not cyclic, dihedral, semidihedral or generalized quaternion), then $\Gamma_s(B)$ has a $\mathbf{Z}A_\infty$–component containing a simple module by Corollary 1.6.

Proposition 2.1. Let M be a simple kG-module and Θ a connected component containing M. Suppose that $\Theta \cong \mathbf{Z}A_\infty$ and M does not lie at the end. Then ;

(1) For some simple modules $T_1, T_2, \cdots, T_n$, the projective covers P_i of T_i are uniserial of length $n+2$ and the Loewy series for P_i's are as follows for some simple module S :

$$P_1 : \begin{pmatrix} T_1 \\ S \\ T_n \\ T_{n-1} \\ \vdots \\ \\ \\ \vdots \\ T_2 \\ T_1 \end{pmatrix}, \quad P_2 : \begin{pmatrix} T_2 \\ T_1 \\ S \\ T_n \\ T_{n-1} \\ \vdots \\ \\ \vdots \\ T_3 \\ T_2 \end{pmatrix}, \quad \cdots, \quad P_i : \begin{pmatrix} T_i \\ T_{i-1} \\ \vdots \\ T_2 \\ T_1 \\ S \\ T_n \\ T_{n-1} \\ \vdots \\ T_{i+1} \\ T_i \end{pmatrix}, \quad \cdots, \quad P_n : \begin{pmatrix} T_n \\ T_{n-1} \\ \vdots \\ \\ \vdots \\ T_2 \\ T_1 \\ S \\ T_n \end{pmatrix}.$$

(2) A part of Θ or $\Omega\Theta$ is as follows for $(n+1)(n+2)/2$ uniserial modules:



In particular the Cartan matrix of the block containing M is as follows:

$$\begin{pmatrix} 2 & 1 & 1 & \cdots & 1 & 0 & \cdots & 0 \\ 1 & 2 & 1 & \ddots & \vdots & \vdots & & \vdots \\ 1 & 1 & \ddots & \ddots & 1 & 0 & & \vdots \\ \vdots & & & 2 & 1 & 0 & \cdots & 0 \\ 1 & 1 & \cdots & 1 & * & & & \\ 0 & 0 & \cdots & 0 & & & & \\ \vdots & & & \vdots & & & & \\ 0 & \cdots & \cdots & 0 & & & & \end{pmatrix}.$$

In [T], Thushima studied blocks B of p-solvable groups in which the Cartan integer $c_{\varphi\varphi} = 2$ for some $\varphi \in \mathrm{IBr}(B)$. From [T, Theorem], we have

Corollary 2.5 . Assume that G is p-solvable and B is a wild block of kG. Let M be a simple module in B. Suppose that M lies in a $\mathbf{Z}A_\infty$–component. Then M lies at the end of its component. In particular simple modules in B of height 0 lie at the end of $\mathbf{Z}A_\infty$–components.

Also using the result of Tsushima[T, Lemma 3], we have

Corollary 2. 6. Assume that G has a non-trivial normal p-subgroup and B is a wild block of kG. Let M be a simple module in B. Suppose that M lies in a $\mathbf{Z}A_\infty$–component. Then M lies at the end of its component. In particular simple modules in B of height 0 lie at the end of $\mathbf{Z}A_\infty$–components.

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