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On some results of the cohomology of extra special p-groups

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Extra special p-groups are central extrensions of Z/p by elementary abelian p-groups. These groups occupy a distinctive in the cohomology and representation theories of finite groups. Quillen decided mod 2 cohomology of the extra special 2-groups [Q]. However the corresponding calculation for odd p is still unknown. Tezuka-Yagita studied the varieties defined from its mod p cohomology[T-Y]. Extending these reults, Benson-Carlson decided the mod p cohomology modulo Jacobson radical [B-C]. The radical parts seem very difficult. For the group of the order p^3, Lewis decided the integral cohomology and Leary wrote down the mod p cohomology completely [Lw],[L2]. Minh computed the mod 3 cohomology of the group with the order 3^5 and of the exponent 3^2 [M].

One of main results of this paper, is to give the additive structutre of the mod p cohomology of the group with the order p^5 and its exponent p (Theorem 8.25). The another results are existence of groups and their modules which period is exactly $2p^n$ for each n.

§1. Extra special p-group.

An extra special p-group G is a group such that its center is Z/p and there is the central extension

(1.1) $$1 \longrightarrow Z/p \longrightarrow G \xrightarrow{\pi} V \longrightarrow 1 \qquad \text{where } V=\oplus^n Z/p.$$

Such group is isomorphic to the n-th central product $E\ldots E=E_n$ or $E_{n-1}M$ where E (resp.M) is the non abelian group of the order p^3 and exponent p (resp.p^2). Hence we can explicitely write

(1.2) $E_n = \langle a_1, \ldots a_{2n}, c \mid [a_{2i-1}, a_{2i}] = c,\ c \in \text{Center}$

$[a_i, a_j] = 1$ for $i<j$, $(i,j) = (2k-1, 2k)$

$a_k{}^p = c^p = 1 \quad \rangle$.

The group $E_{n-1}M$ is written similarly except for $a_{2n}{}^p = c$.

Let us write by $x_i \in H^1(V) = \mathrm{Hom}(V, Z/p)$ the dual of $\pi(a_i)$ and write $y_i = \beta x_i$

Then the cohomology of V is

$$H^*(V) \cong S_{2n} \otimes \Lambda_{2n} \quad \text{with } S_{2n} = Z/p[y_1, \ldots, y_{2n}] \text{ and } \Lambda_{2n} = \Lambda(x_1, \ldots, x_{2n}).$$

Proposition 1.3. The extension (1.1) represents element in $H^2(V)$

$$f = \sum^n x_{2i-1} x_{2i} \quad (\text{resp. } \sum^n x_{2i-1} x_{2i} + y_{2n}) \quad \text{for } G = E_n \ (\text{resp. } E_{n-1}M).$$

We consider spectral sequence induced from (1.1)

(1.4) $E_2^{*,*} = H^*(V; H^*(Z/p))$

$$\cong S_{2n} \otimes \Lambda_{2n} \otimes Z/p[u] \otimes \Lambda(z) \Longrightarrow H^*(G)$$

with $\beta z = u$. From Proposition 1.3, we know

(1.5) $d_2 z = f$.

By transgression theorem,

(1.6) $d_3 u = \beta d_2 z = \sum y_{2i-1} x_{2i} - y_{2i} x_{2i-1}$

(1.7) $d_{2p^s+1} u^{p^s} = \mathcal{P}^{p^{s-1}} \cdots \mathcal{P} d_3 u = \sum y_{2i-1}{}^{p^s} x_{2i} - y_{2i}{}^{p^s} x_{2i-1}$.

By Kudo's transgression theorem

(1.8) $d_{2p^s(p-1)+1}(u^{p^s(p-1)} \otimes d_{2p^s+1} u^{p^s}) = \beta \mathcal{P}^{p^s} d_{2p^s+1} u^{p^s}$

$$= \sum y_{2i-1}{}^{p^{s+1}} y_{2i} - y_{2i}{}^{p^{s+1}} y_{2i-1}.$$

Let us write $(1.6) = z_n(1)$, $(1.7) = z_n(s+1)$, $(1.8) = w_n(s+1)$. Hence $E_\infty^{*,0}$ is a quatient of

(1.9) $E = S_{2n} \otimes \Lambda_{2n} / (f, z_n(1), \ldots, z_n(n), w_n(1), \ldots, w_n(n))$

We also know that $u^{p^{n+1}}$ is a permanent cycle because which represents p^{n+1}-th Chern class of induced representation from a maximal elementary abelian p-group. Write by u' a corresponding element in $H^*(G)$. Then $H^*(G)$ is a $E \otimes Z/p[u']$-module. From Benson-Carlson [B-C].

$$H^*(G)/J = E\ Zp[u']/J \quad \text{for the Jacobson radical } J.$$

Note the regularlity of the sequence $w_1(1), \ldots, w_n(n)$ in S_{2n} is shown in Tezuka-Yagita [T-Y].

§2. $\widetilde{G}$; the central product of G and S^1.

The spectral sequence (1.4) is very complicated even for E (see [K-S-T-Y] (5.11)).Hence we consider another arguments which are used by Kropholler, Leary, Huebshmann and Moselle. Embed $\langle c\rangle \cong Z/p \hookrightarrow S^1$ and cosider the central product

(2.1) $\quad G = G\times_{<c>}S^1.$

Note that $\widetilde{E}_n \cong \widetilde{E_{n-1}M}$,indeed, take $a_{2n}c^{-1/p}$ as a_{2n} ,if $a_{2n}{}^p=c$. Then we have the exact sequence

$$(2.2) \qquad 1 \longrightarrow S^1 \longrightarrow \widetilde{G} \longrightarrow V \longrightarrow 1$$

and induced spectral sequence

$$(2.3) \qquad E_2^{*,*} \cong H^*(V;H^*(BS^1))$$
$$\cong S_{2n}\otimes \Lambda_{2n}\otimes Z/p[u] \Longrightarrow H^*(\widetilde{G})$$

Then diffentials (1.6)-(1.8) also hold but $d_2=0$ by the dimensional reason.

Given $H^*(\widetilde{G})$, to see $H^*(G)$ we use the following fibration induced from (2.1)

$$(2.4) \qquad S^1 \cong \widetilde{G}/G \longrightarrow BG \longrightarrow B\widetilde{G}.$$

The induced spectral sequence is

$$(2.5) \qquad E_2^{**} \cong H^*(\widetilde{G};H^*(S^1))$$
$$\cong H^*(\widetilde{G})\otimes\Lambda(z) \Longrightarrow H^*(G)$$

and $d_2z=f$ (1.5). Therefore

Proposition 2.6. $\quad H^*(G) \cong (Ker(f)|H^*(G))\{z\}\oplus H^*(G)/(f).$

§3. 2p-terms for E_n for $n<p$.

In sections 3-5, we consider spectral sequnce (2.3) for E_n for $p<n$. Given graded algebra A and z A^{odd}, we define homology $H(A,z)$ with the differential $d_z(a)=za$. The first non zero differential in (2.3) is $d_3u=z_n(1)$.Hence

$$(3.1) \qquad E_4^{*,2j} \cong \begin{cases} S_{2n}\otimes\Lambda_{2n}/(z_n(1)) & j=0 \mod p \\ H(S_{2n}\otimes\Lambda_{2n},z_n(1)) & 1\leq j\leq p-2 \\ Ker z_n(1) & j=p-1 \end{cases}$$

$$E_4^{*,2j+1} = 0.$$

We can prove

(3.7) $$E_{2p}^{*,2j} \cong \begin{cases} S_{2n}\otimes\Lambda_{2n}/(z_n(1), w_n(1), z_n(2)f^{n-1}) & j=0 \bmod p \\ Z/p\{f^n\} & 1\leqq j< p-1 \\ 0 & j=p-1. \end{cases}$$

§4. E_{2p+2}-term.

The next differential is (1.7)

(4.1) $d_{2p+1}(u^p)=\mathcal{P}z_n(1) = z_n(2)$.

Let $E=S_{2n}\otimes\Lambda_{2n}/(z_n(1), w_n(1))$. Then we get from (3.7)

(4.2) $$E_{2p+2}^{*,2jp} = \begin{cases} E/(z_n(2)) & \text{for } j=0 \bmod p \\ H(E/z_n(2)f^{n-1}, z_n(2)) & 0<j\leqq p-2 \\ \mathrm{Ker}(z_n(2)\,|E/z_n(2)f^{n-1}) & j=p-1 \end{cases}$$

$$E_{2p+2}^{*,2i+2jp} = Z/p\{f^n\} \qquad 0<i\leqq p-2 .$$

§5. Homology of $H(E_i/E_{i+1})$.

Proposition 5.18. $H(E,z_2(2))^{odd} \simeq H(E,z_2(2))^{even}-Z/p\{f^n\}$ and $H(E,z_2(2))^{odd} \cong S_4\{x_1',..,x_{2n}'\}/(y_{ij}x_j', y_ix_k'=y_kx_i')$ where we express $x_i'=x_if^{n-1}$, $x_i'=y_if^{n-1}$, $y_{ij} = y_i^p-y_j^{p-1}y_i$.

From (4.2)' we get

Corollary 5.19. $H(E/z_n(2)f^{n-1}, z_n(2))$ is generated by f^{n-1} as $S_{2n}\otimes\Lambda_{2n}$-module and

$$H(E/z_n(2)f^{n-1}, z_n(2))^{odd} = H(E,z_n(2))^{odd} ,$$

$$H(E/z_2(2)f^{n-1}, z_n(2))^{even} = S_{2n}/(y_iy_{ji})\{f^{n-1}\}\oplus Z/p\{f^n\}.$$

§6. $E_{2p(p-1)+1}$ -term for $\tilde{G}$.

Let $y_{ij}=y_i^p-y_j^{p-1}y_i$ and $y_{ij}' = (y_i^{p^2}y_j-y_iy_j^{p^2})/y_iy_{ji}$

Therefore we can prove

Proposition 6.14. For n=2 case

$$E_{2(p-1)p+2}^{*,2pj} \cong \begin{cases} S_4\otimes\Lambda_4/(z_2(1), z_2(2), w_2(1), w_2(2), (y_{21}'+y_{43}')\beta(x_1x_2)) & j=0 \bmod p \\ S_4\otimes\Lambda_4/(y_iy_{ji}, y_ix_j-y_jx_i, f, x_hx_k((h,k)=(1,2),(3,4))) & 0<j<p-1 \bmod p \end{cases}$$

$$E_{2(p-1)p+2}^{*,2j} \cong \begin{cases} Z/p\{x_1\dots x_4\} & 0<j<p-1 \bmod p \\ 0 & j=p-1 \bmod p \end{cases}$$

In the next section, we will prove also

Theorem 6.15. $E_{2(p-1)p+2}^{*,*} \cong E_\infty^{*,*}$.

§ 8. Ker f in $H^*(\tilde{E}_2)$.

Theorem 8.25. *There is an additive isomorphism*

$$H^*(E_2) \cong (A/(f) \oplus (\mathrm{Ker}(f)|A)\{z\} \oplus_{1\le s\le p-2} (H'\{f_s\} \oplus H'\{f_s z\}) \oplus_{1\le t\le p^2-3 \text{ and } t\ne -1,\ \ne 0 \bmod p \text{ or } t=p(p-1)} (Z/p\{z_t\} \oplus Z/p\{zz_t\})) \otimes Z/p[u^{p^2}]$$

where

(i) $A \cong S_4\otimes\Lambda_4/(z_2(1), z_2(2), w_2(1)w_2(2), (y_{21}'+y_{43}')\beta(x_1x_2), z_2(3))$ with $S_4\otimes\Lambda_4 = z/p[y_1,\cdots,y_4]\otimes\Lambda(x_1,\cdots,x_4)$, $z_2(1) = \beta f$, $z_2(2) = \mathcal{P}\beta f$, $z_2(3) = \mathcal{P}^p\mathcal{P}\beta f$ and $w_2(1) = \beta z_2(2)$, $w_2(2) = \beta z_2(3)$

(6.5) $y_{ij}' = y_i^{p(p-1)} + y_i^{(p-1)(p-1)}y_j^{p-1} + \cdots + y_j^{p(p-1)}$,

(ii) $f = \{x_1x_2 + x_3x_4\}$,

(iii) *(Proposition 8.2)* Ker $(f)|A$ *is generated as an* S_4*-module by*

$$y_iy_{ji}, y_jx_i - y_ix_j, y_{ji}x_i,$$
$$f, x_kx_h((k,h)\ne(1,2),\ne(3,4)),$$
$$x_ix_jx_k, x_1x_2x_3x_4,$$

where $y_{ji} = y_j^p - y_jy_i^{p-1}$,

(iv) *z coresponds non zero element in* $H^1(S^1) = E_2^{0,1}$ *in (2.4),*

(v) *(Proposition 5.18)* $H'^{odd} \cong H'^{even}/(Z/p\{1\})$ *and* $H'^{odd} \cong S_4\{x_1,\cdots,x_4\}/(y_{ij}x_j, y_ix_k = y_kx_i)$,

(vi) $f_s = \{fu^{p^s}\}$ *in the spectral sequence (2.2),*

(vii) $z_t = \{f^2u^t\} = \{x_1x_2x_3x_4u^t\}$ *in (2.2).*

§2. Hochschild-Serre spectral sequence.

We consider the spectral sequence with E_2-term

(2.1) $E_2^{*,*} = H^*(\oplus^{2n}Z/p; H^*(BS^1))$.

In this paper cohomology $H^*(-)$ always means the Z/p-coefficient $H^*(-;Z/p)$. Let us write

$H(+^{2n}Z/p)=S_{2n}\otimes\Lambda_{2n},\ H^*(BS^1)\cong Z/p[u]$

with $S_{2n}=Z/p[y_1,\dots y_{2n}]$, $\Lambda_{2n}=\Lambda(x_1,\dots x_{2n})$, $\beta x_i=y_i$.

We assume first non zero differential

(2.2) $d_3u = \beta f$ with $f=\sum_{i=1}^{n}x_{2i-1}x_{2i}$.

Then by Cartan-Serre and Kudo transgression theorems, we know

(2.3) $d_{2p^{i-1}+1}(u^{p^{i-1}})=z(i)$, $d_{2p(p^{i-1}-1)+1}(z(i)\otimes u^{(p-1)p^{i-1}})=w(i)$

with $z(i)=\mathcal{P}^{p^{i-2}}\dots\mathcal{P}^1\beta f=\sum y_{2j-1}{}^{p^{i-1}}x_{2j}-y_{2j}{}^{p^{i-1}}x_{2j-1}$,

$w(i)=\beta\mathcal{P}^{p^{i-1}}z(i)=\sum y_{2j-1}{}^{p^i}y_{2j}-y_{2j}{}^{p^i}y_{2j-1}$.

Let us write $S(i)=S_{2n}/(w(1),\dots,w(i))$. Recall $(w(1),\dots,w(n))$ is regular in S_{2n} [7].

Lemma 2.4. For $i\leq n-1$, we get

(i) 1 is $S(i)$-free in $E_{2p^i+1}^{*,0}$,

(ii) $z(i+1)$ is $S(i)$-free in $E_{2p^i+1}^{*,0}$,

(iii) if $x\in E_{2p^i+2}^{*,0}$ is higher $w(i+1)$-torsion, then x is higher $w(j)$-torsion for all $j\leq n$ (i.e., $w(j)^s x=0$ for some s and all $j\leq n$).

(iv) $E_{2p^i+2}^{*,2p}$ is higher $w(j)$-torsion for all $j\leq n$.

For the proof of this lemma, we recall the base wise reduced powers defined by Araki.

Theorem 2.5. (Araki [2]) There are cohomology operations

$_B\mathcal{P}^s : E_r^{a,b}\longrightarrow E_{p(r-2)+2}^{a+(2s-b)(p-1),pb}$

$_B\beta\mathcal{P}^s : E_r^{a,b}\longrightarrow E_{p(r-2)+2}^{a+(2s-b)(p-1)+1,pb}$

which satisfy the naturality and Cartan formula.

Proof of Lemma 2.4. We use induction on i. Suppose (i)-(iv) for i-1. First we will prove (iv) i.e.,

(1) $H^*(E_{2p^i+1}^{*,0}, z(i+1))$ is higher $w(j)$ torsion.

Here $H(A,z)$ means the homology with the differential $da=za$ for $a\in A$. Let us write by $T\subset E_{2p^{i-1}+2}^{*,0}$ the higher $w(j)$-torsion parts and $F=E_{2p^{i-1}+2}^{*,0}/T$. By the inductive assumption, $H(E_{2p^{i-1}+1}^{*,0}, z(i))\cong E_{2p^{i-1}+2}^{*,2p^{i-1}}$ is higher $w(j)$-torsion. Hence for $2p^{i-1}+2\leq r\leq 2(p-1)p^{i-1}$, we see $\mathrm{im}\, d_r\subset T$. Therefore

(2) $E_{r+1}^{*,0}/(\text{ higher } w(j)\text{-torsion})\cong F$.

Next we consider the Kudo transgression $d_{2p^{i-1}(p-1)+1}$. Let us write simply $q=2(p-1)p^{i-1}$. Recall that $E_{2p^{i-1}+2}^{*,q}$ contains $z(i)$ and is a submodule of

$\mathrm{Ker}(z(i))\cong H(E_{2p^{i-1}+1}^{*,0}, z(i))\oplus \mathrm{Im}\, z(i)$.

Since $\mathrm{Im}\, z(i)$ in $E_{q+1}^{*,q}$ is $S(i-1)$-free from (ii), if $\mathrm{Ker}(d_{q+1})\cap \mathrm{Im}\, z(i)\neq 0$, then it is a contradiction because $E_{q+2}^{*,*}$ is $w(i)$-torsion since so is 1. Therefore $\mathrm{Ker}(d_{q+1})\cap \mathrm{Im}\, z(i)=0$. Since $H(E_{2p^{i-1}+1}, z(i))$ is higher $w(j)$-torsion, given $a\in E_{q+1}^{*,q}$ we get $w(i+1)^s a\in \mathrm{Im}\, z(i)$ for some large s. Hence $E_{q+2}^{0,q}$ is higher $w(j)$-torsion. Then we also show, for $2(p-1)p^{i-1}+1\leq r\leq 2p^i$,

(3) $E_{r+1}^{*,0}/(\text{ higher } w(i)\text{-torsion})\cong F/(w(i))/(\text{higher } w(j)\text{-torsion })$.

Let $x\in E_{2p+1}^{*,0}$ and $x\in \mathrm{Ker}\, z(i+1)$. From (3) we can write in $E_{2p^{i-1}+2}^{*,0}$

(4) $z(i+1)x=w(i)a+t$ with t ; higher $w(j)$-torsion mod $(w(i))$.

Therefore for large s, we have

(5) $z(i+1)w(i+1)^s x = w(i)a'$

We consider Araki's reduced powers

$${}_s\beta P^s,\ {}_sP^s :\ E_{2p^{i-1}+2}^{*,0}\longrightarrow E_{2p^i+2}^{*,0}.$$

Act ${}_s\beta P^{p^i}$ to (5). Since $w(i)=z(i+1)=0$ in $E_{2p^i+2}^{*,0}$ and $\beta P^{p^i}z(i+1)=w(i+1)$, we get in $E_{2p^i+2}^{*,0}$

(6) $w(i+1)^{s+1}x = w(i+1)\beta a'$.

Multiply $z(i+1)$ to (5), we know $w(i)z(i+1)a'=0$. Act ${}_s\beta P^{2p^i}$ to this, and we have $w(i+1)^2 a'=0$ in $E_{2p^i+2}^{*,0}$. From (6)

(7) $w(i+1)^{s+2}x=0$ in $E_{2p^i+2}^{*,0}$.

From (3), this means

$$w(i+1)^{s+2}x = w(i)a''+t' \quad \text{in} \quad E_{2p^{i-1}+2}{}^{*,0}.$$

as (4). Multiply $w(i+2)^{s'}$ to this for large s', we get $w(i+1)^{s''}x=w(i)a'''$. Operate ${}_BP^{p^{i+1}s''}$ on this. Thus we prove

(8) $\quad w(i+2)^{s''}x = 0 \quad \text{in} \quad E_{2p^{i}+2}{}^{*,0}$.

Continue this argument and we show (1), i.e., (iv). The arguments (7) to (8) implies (iii).

We already know, for $q=2(p-1)p^{i-1}$, $d_{q+1} : S(i-1)\{z(i)\} \cong \mathrm{Ideal}(w(i))$ in S(i-1), by the arguments before (3). Suppose $d_{q+1}(au^{q/2})=w \neq 0 \bmod (w(i))$ in S(i-1) (or $\neq 0 \bmod (w(i)z(i))$ in $S(i-1)\{z(i)\}$). Then $w(i+1)^s w \neq 0$ for all s in $S(i-1)/(w(i))=S(i)$ since w(i+1) is non zero divizer in S(i). On the other hand $H(E_{2p^{i-1}+1}, z(i))$ is higher w(j)-torsion, we get $w(i+1)^s a \in \mathrm{Im}\, z(i)$ for large s. This means $w(i+1)^s w=0 \bmod (w(i))$ and this is a contradiction. Hence 1 and z(i+1) are S(i)-free in $E_{q+2}{}^{*,0}$. From (3), so are in $E_{2p^{i}+1}{}^{*,0}$. Therefore we show(i) and (ii). q.e.d.

From (3) in the above proof, we also get;

Corollary 2.6. With modulo higher w(j)-torsion , there is the isomorphism

$$E_{2p^{i}+1}{}^{*,0} \cong S_{2n} \otimes \Lambda_{2n}/(z(1),\dots,z(i),w(1),\dots,w(i)).$$

§3. Extra special p-groups

let E_n be the extra special p-group of the order $2p^{n+1}$ and the exponent p

(3.1) $\quad E_n = \langle a_1,\dots,a_{2n},c \mid a_i{}^p=c^p=1,\ c \in \text{Center}$

$$[a_i,a_j] = \begin{cases} c & i=2k-1, j=2k \\ 1 & \text{other } i<j \end{cases} \quad \rangle$$

Consider central products $\tilde{E}_n=E_n \times_{<c>} S^1$ and $\tilde{E}(s)_n=E_n \times_{<c>} Z/p^s$. Then there are central extensions

(3.2) $\quad 1 \longrightarrow S^1 \longrightarrow \tilde{E}_n \longrightarrow \oplus^{2n} Z/p \longrightarrow 1$

(3.3) $\quad 1 \longrightarrow Z/p^s \longrightarrow \tilde{E}(s)_n \longrightarrow \oplus^{2n} Z/p \longrightarrow 1$

and induced spectral sequence $E_r^{*,*}$ and $E(s)_r^{*,*}$ from (3.2) and (3.3) respectively. The spectral sequence $E_r^{*,*}$ satisfies (2.2) and hence Lemma 2.4.

Let $H^*(Z/p^s) \simeq Z/p[u] \otimes \Lambda(z)$. If $s \geq 2$, then $d_2 z=0$ by $\beta z=0$ and the symmetry of $\tilde{E}(s)_n$. Thus

(3.4) $\qquad E(s)_r^{*,*} \cong E_r^{*,*} \otimes \Lambda(z) \quad$ for $s \geq 2$.

Therefore (i),(ii) in Lemma 2.4 satisfies for (3.3).

Corollary 3.5. ([7]) In $H^*(\tilde{E}_n)$ or $H^*(\check{E}(s)_n)$, $s \geq 2$, the S_{2n}-submodule generated by 1 is $S_{2n}/(w(1),\ldots,w(n))$.

Moreover for n=2, the spectral sequemce $E_r^{*,*}$ is given completely in [8].

§4. Periodic modules with large period.

Let k be an algebraic closure of F_p. Let $\Omega_G^r(M)$ be the r-th kernel in the minimal resolution of k(G)-module M, i.e., if

(4.1) $\qquad 0 \rightarrow M_r \longrightarrow Q_{r-1} \longrightarrow \ldots \rightarrow Q_0 \rightarrow M \rightarrow 0$

is exact and if each Q_i is projective, then $M_r \cong \Omega_G^r(M) \oplus Q$ for some projecive module Q. A G-module M is said to be periodic if $\Omega_G^m(M) \simeq M$ for some $m \geq 0$. The smallest of such m is called the period of M.

We denote by $V_G(k)$, the variety defined by commutative ring $H^*(G;k)/\sqrt{0}$. For a G-module M, let $I_G(M)$ be the annihilator in $H^*(G;k)$ of $\mathrm{Ext}_{k*(G)}{}^*(M,M) \cong H^*(G,\mathrm{Hom}_k(M,M))$. Let $V_G(M)$ be the subvariety of $V_G(k)$ associated to $I_G(M)$.

Remark that if V is a closed homogeneous subvariety of $V_G(k)$, then there is a K(G)-module M with $V_G(M)=V$ (Proposition 2.1 (vii) in [3]).

We recall arguments of Andrews and Benson-Carlson [3]. Consider a central extension of a finite group

(4.2) $\qquad 1 \longrightarrow Z/p \longrightarrow G \longrightarrow E \longrightarrow 1.$

Let $\overline{Z/p}$ denote the sum $\sum_{g \in Z/p} g$ as an element of the group ring k(Z/p). Then for $r>0$, $\overline{Z/p}\,\Omega_G^{2r}(k)$ is a k(G)-module with Z/p-acting trivially, so we may regard it as a k(E)-module. We set

(4.3) $\quad V_r = V_E(\bar{Z}/p\,\Omega_G^{2r}(k)) \subset V_E(k).$

Theorem 4.4. (Andrews) Let M be an indecomposable k(E)-module regard as a k(G)-module by inflation. Then M is a periodic k(G)=module of periodic dividing 2r if and only if $V_E(M) \cap V_r = \{0\}$.

Theorem 4.4 (Benson-Carlson [3]) Let $E_r^{*,*}$ be the spectral sequence induced from (4.2). Let $I_{p^a} \subset H^*(E)$ be the Kernel of the induced map $E_2^{*,0} \longrightarrow E_{2p^a+1}^{*,0}$. Then $V_{p^a} = V_E(I_{p^a})$.

Lemma 4.5. ([3]Proposition 2.2.) If M is a periodic k(G)-module, then the peiod of M devides 2[G;E] where E is a maximal elementary abelian p-groups of G.

Theorem 4.6. Let G be the p-group $\tilde{E}(s)_n$, $s \geq 2$. Then there are periodic K(G)-modules of period 2^a for $a \leq n$, and no higher period.

Proof. (See the proof of Corollary 6.2 in [3].) From above lemma, the only possible periods are $2p^a$ for $a \leq n$. By Lemma 2.4 in section 2 and Theorem 4.4, for $a \leq n$ we may find a closed homogeneous subvariet V of $V_E(k)$ with $V \cap V_{p^{a-1}} \neq \{0\}$ and $V \cap V_{p^a} = \{0\}$. By the remark after the definition of $V_G(M)$, we may find a k(E)-module M with $V_E(M) = V$. Then by the Andrews theorem $\Omega_E^{2p^{a-1}}(M) \not\cong M$ but $\Omega_G^{2p^a}(M) \cong M$, so M has period exactly $2p^a$. q.e.d.

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