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A remark on Alperin's conjecture

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Let p be a prime number and F be an algebraically closed field of characteristic p. Let G be a finite group and B be a (p-)block of G with abelian defect group D and Brauer correspondent B_0, a block of $N_G(D)$. Alperin's weight conjecture states that the number of isomorphism classes of irreducible (FG)B-modules is equal to that of isomorphism classes of irreducible $(FN_G(D))B_0$-modules (see [1]). So if Alperin's conjecture is true, then the number of isomorphism classes of indecomposable (FG)B-modules with source module $F_{\{1_G\}}$ is equal to that of isomorphism classes of indecomposable $(FN_G(D))B_0$-modules with source module $F_{\{1_G\}}$. We generalize this to an arbitrary source module. Let Q be a p-subgroup of G and L be a finitely generated indecomposable FQ-module with vertex Q. We denote by Ind(B|L) the set of isomorphism classes of indecomposable FG-modules which belong to B and have L as a source module. Ind(B|L) is a finite set. We denote by |Ind (B|L)| the cardinality. In this paper we prove the following .

PROPOSITION. *Under the above notation,* if Alperin's conjecture is true, *then we have*

$$|\mathrm{Ind}(B|L)| = \sum_{x \in T(L) \backslash G/N_G(D)}' |\mathrm{Ind}(B_0|L^x)|,$$

where $T(L) = \{t \in N_G(Q) \mid L \otimes_{FQ} t\ (\subset L^G)$ *is isomorphic to* $L\}$, $L^x = L \otimes_{FQ} x$, *and where* x *ranges over a complete set of representatives for* $(T(L), N_G(D))$*-double cosets* $T(L)xN_G(D)$ *of* G *such that* $D \supset Q^x$.

§ 1. Preliminaries.

Let M be a finitely generated right FG-module and put $E = \mathrm{End}_{FG}M$. There is a one to one correspondence between the decompositions of M into direct sums of indecomposable FG-submodules and the decompositions of E into direct sums of principal indecomposable right modules. Moreover, if $\mathrm{id}_M = \varepsilon_1 + \varepsilon_2 + \dots + \varepsilon_n$ is a decomposition of the identity map of M into the sum of primitive idempotents of E, then $\varepsilon_i(M)$ and $\varepsilon_j(M)$ are isomorphic if and only if $\varepsilon_i E$ and $\varepsilon_j E$ are isomorphic as E-modules. Hence the number of isomorphism classes of indecomposable components of M is equal to that of isomorphism classes of irreducible E-modules. Let Z(FG) be the center of FG and for $z \in Z(FG)$, $z_R \in E : m \longrightarrow mz$, $m \in M$. Then the map φ defined by $\varphi(z) = z_R$ is a homomorphism from Z(FG) into Z(E). Let B be a block and assume that $\varphi(B) \neq 0$. It is clear that, by the above, the number of isomorphism classes of indecomposable components of M belonging to B is equal to that of isomorphism classes of irreducible $E\varphi(B)$-modules.

Let Q be a normal abelian subgroup of G and L be a finitely generated G-invariant indecomposable FQ-module. We put $E_G = \mathrm{End}_{FG}\, L^G$, where $L^G = L \otimes_{FQ} FG$. For $x \in G$ let $E_x = \{f \in E_G |\ f(L \otimes_{FQ} 1)$

$\subset L \otimes_{FQ} x$ }. We have

$$E_G = \sum_{x \in G/Q} E_x.$$

By the assumption there exists a unit ϕ_x in E_x, and it holds that $E_x = E_1\phi_x = \phi_x E_1$ and $E_x E_y = E_{xy}$. Moreover if $c \in C_G(Q)$, then we can choose (and do so) ϕ_c such that $\phi_c(l \otimes y) = l \otimes cy$, $l \in L$ and $y \in G$. The F-linear map $\phi : FC_G(Q) \longrightarrow E_G$ defined by $\phi(c) = \phi_c$, $c \in C_G(Q)$ is an algebra homomorphism. Moreover, for $z \in FC_G(Q) \cap Z(FG)$, $\phi(z) = z_R$, $z_R : l \otimes x \longmapsto l \otimes xz = l \otimes zx$, $l \in L$ and $x \in G$. Let $J(E_1)$ be the radical of E_1 and let $\bar{E}_G = E_G / (J(E_1)E_G)$. $E_1/J(E_1) \cong F$ and $J(E_1)E_G \subset J(E_G)$. For each $\bar{x} \in \bar{G} = G/Q$, let $\phi_{\bar{x}}$ be one of $\overline{\phi_y}$, $y \in Qx$. Here $\overline{\phi_y} = \phi_y + J(E_1)E_G$. We have

$$(1) \qquad \phi_{\bar{x}}\, \phi_{\bar{y}} = \alpha(\bar{x}, \bar{y})\phi_{\bar{x}\bar{y}}, \qquad \bar{x}, \bar{y} \in \bar{G},$$

where $\alpha(\bar{x}, \bar{y}) \in F^{\times} = F - \{0\}$. It is clear that α is an F-cocycle of $\bar{G}$ and $\bar{E}_G$ is isomorphic to the twisted group algebra $F(\bar{G}, \alpha)$ of $\bar{G}$ with cocycle α. We identify $\bar{E}_G$ with $F(\bar{G}, \alpha)$. Moreover ϕ induces an algebra isomorphism from $F\overline{C_G(Q)}$ into $\bar{E}_G$, because $\overline{\phi_{cz}} = \overline{\phi_c}$ for $c \in C_G(Q)$, $z \in Q$. We embed $F\overline{C_G(Q)}$ into $\bar{E}_G$. Let H be a subgroup of G containing Q. We can embed E_H ($= \mathrm{End}_{FH} L^H$) into E_G, and $\bar{E}_H$ into $\bar{E}_G$. Moreover we have $\bar{E}_H = F(H, \alpha_{\bar{H}})$ and $\bar{E}_{C_G(Q)} = F\overline{C_G(Q)}$.

By [3, Theorem 1.1], there exists a finite group $\tilde{G}$ with the following properties: (i) $\tilde{G}$ has a p'-subgroup A which is contained in the center of $\tilde{G}$. (ii) There is an isomorphism $\sigma : \bar{G} \longrightarrow \tilde{G} / A$. (iii) There is an F-valued linear character λ of A such that the special cocycle of $\bar{G}$ associated with λ is cohomologous to α.

For each $\bar{x} \in \bar{G}$, let $r(\bar{x})$ be an element of $\tilde{G}$ such that $\sigma(\bar{x}) = Ar(\bar{x})$. Then $\tilde{G} = \bigcup_{\bar{x} \in \bar{G}} Ar(\bar{x})$ and there exist $a(\bar{x}, \bar{y}) \in A$ such that

$r(\bar{x})r(\bar{y}) = a(\bar{x}, \bar{y})r(\overline{xy})$, for $\bar{x}, \bar{y} \in \bar{G}$. By the above (iii), there exists a map $\delta : \bar{G} \longrightarrow F^{\times}$ such that

(2) $$\alpha(\bar{x}, \bar{y}) = \delta(\bar{x})\delta(\bar{y})\delta(\overline{xy})^{-1}\lambda(a(\bar{x}, \bar{y})), \quad \bar{x}, \bar{y} \in \bar{G}.$$

Let e_λ be the primitive idempotent of FA corresponding to λ. $(FA)e_\lambda = Fe_\lambda$ and $(F\tilde{G})e_\lambda = \sum_{\bar{x}\in\bar{G}} F(r(\bar{x})e_\lambda)$. Now let θ be an F-linear map such that

$$\theta : F(\bar{G}, \alpha) = \bar{E}_G \longrightarrow (F\tilde{G})e_\lambda$$
$$\bar{x} \longmapsto \delta(\bar{x})r(\bar{x})e_\lambda, \quad \bar{x} \in \bar{G}.$$

From (1) and (2), θ is an algebra isomorphism. For each subgroup H of G containing Q, let $\tilde{H}$ be a subgroup of $\tilde{G}$ such that $\sigma(\bar{H}) = \tilde{H} / A$. θ induces the isomorphism

$$F(\bar{H}, \alpha_{\bar{H}}) = \bar{E}_H \longrightarrow (F\tilde{H})e_\lambda.$$

§ 2. Proof of Proposition.

Step 1. We may assume that Q is normal in G.

Proof. Let $\mathcal{S} = \{[P, \beta] \mid P$ is a p-subgroup of G and β is a block of $N_G(P)$ with $\beta^G = B\}$ and let b be a root of B in $C_G(D)$. G acts on $\mathcal{S}$ by conjugation. Since D is abelian, any element of $\mathcal{S}$ is G-conjugate to some $[P, b^{N_G(P)}]$, $P \subset D$. Suppose that $[P, b^{N_G(P)}]$ and $[R, b^{N_G(R)}]$ are G-conjugate : $[P, b^{N_G(P)}] = [R, b^{N_G(R)}]^x$, $x \in G$. b^x is a root of $b^{N_G(P)}$ in $C_G(D^x)$, and hence there exists $y \in N_G(P)$ such that $D = D^{xy}$ and $b = b^{xy}$. So $xy \in T(b)$ and $P = R^{xy}$. Let $\mathcal{P}$ be a complete set of representatives for T(b)-conjugacy classes of subgroups of D, where $T(b) = \{t \in N_G(D) \mid b^t = b\}$. $\{[P, b^{N_G(P)}] \mid P \in \mathcal{P}\}$ forms a complete set of representatives for G-conjugacy classes in $\mathcal{S}$.

By [2, Theorem 2], we have $|\mathrm{Ind}(B|L)| = \sum_{\beta} |\mathrm{Ind}(\beta|L)|$, where β ranges over the set of $\mathrm{Bl}(N_G(Q), B)$ of blocks of $N_G(Q)$ associated with B. On the other hand, for $\beta \in \mathrm{Bl}(N_G(Q), B)$ there exists $P \in \mathcal{P}$ and $u \in G$ such that $[P, b^{N_G(P)}] = [Q, \beta]^u$. Since $|\mathrm{Ind}(\beta|L)| = |\mathrm{Ind}(\beta^u| L^u)| = |\mathrm{Ind}(b^{N_G(P)}|L^u)|$, we can see

$$(3) \qquad |\mathrm{Ind}(B|L)| = \sum_{P\in\hat{\mathcal{P}}_Q} |\mathrm{Ind}(b^{N_G(P)}|L^{u_P})|,$$

where $\hat{\mathcal{P}}_Q = \{P \in \mathcal{P} \mid P \text{ is G-conjugate to } Q\}$ and u_P is an element of G with $P = Q^{u_P}$. We note for P, $|\mathrm{Ind}(b^{N_G(P)} \mid L^{u_P})|$ does not depend on the choice of u_P. For each Q^v, $v \in G$, we set $\mathcal{P}_{Q^v} = \{P \in \mathcal{P} \mid P \text{ is } N_G(D)\text{-conjugate to } Q^v\}$ and for $P \in \mathcal{P}_{Q^v}$, denote by $u_{v,P}$ an element of $N_G(D)$ such that $P = Q^{vu_{v,P}}$. Now, if $Q^z \subset N_G(D)$, $z \in G$, then by applying (3) for $N_G(D)$, B_0 and L^z, we have

$$(4) \qquad |\mathrm{Ind}(B_0|L^z)| = \sum_{P\in\mathcal{P}_{Q^z}} |\mathrm{Ind}((b^{N_G(P)\cap N_G(D)})|L^{zu_{z,P}})|.$$

From (3) we have

$$(5) \quad |\mathrm{Ind}(B|L)| = \sum_{v\in N_G(Q)\backslash G/N_G(D)} \quad \sum_{P\in\mathcal{P}_{Q^v}} |\mathrm{Ind}(b^{N_G(P)}|L^{vu_{v,P}})|,$$

where v ranges over a complete set of representatives for the

$(N_G(Q),\ N_G(D))$-double cosets of G. If $\mathcal{P}_Q v$ is not empty, then $D \supset Q^v$. Here we assume that Proposition holds for $N_G(Q)$. Then we have for $P \in \mathcal{P}_Q v$,

$$(6)\quad |\mathrm{Ind}(b^{N_G(P)}|L^{vu_{v,P}})| = |\mathrm{Ind}((b^{(vu_{v,P})^{-1}})^{N_G(Q)})|L)|$$

$$= \sum'_{y\in T(L)\backslash N_G(Q)/(N_G(D^{v^{-1}})\cap N_G(Q))} |\mathrm{Ind}((b^{(vu_{v,P})^{-1}})^{N_G(Q)\cap N_G(D^{v^{-1}})})|L^y)|$$

$$= \sum'_{y\in T(L)\backslash N_G(Q)/(N_G(D^{v^{-1}})\cap N_G(Q))} |\mathrm{Ind}((b^{N_G(P)\cap N_G(D)})|L^{yvu_{v,P}})|,$$

where y ranges over a complete set of representatives for $(T(L),\ N_G(D^{v^{-1}})\cap N_G(Q))$-double cosets $T(L)y(N_G(D^{v^{-1}})\cap N_G(Q))$ of $N_G(Q)$. By substituting (6) in (5) and using (4), we can show

$$|\mathrm{Ind}(B|L)|$$

$$= \sum_v \sum_{P\in\mathcal{P}_Q v} \sum'_{y\in T(L)\backslash N_G(Q)/(N_G(D^{v^{-1}})\cap N_G(Q))} |\mathrm{Ind}((b^{N_G(P)\cap N_G(D)})|L^{yvu_{v,P}})|$$

$$= \sum_{v\in N_G(Q)\backslash G/N_G(D)} \sum'_{y\in T(L)\backslash N_G(Q)/(N_G(D^{v^{-1}})\cap N_G(Q))} |\mathrm{Ind}(B_0|L^{yv})|$$

$$= \sum'_{x\in T(L)\backslash G/N_G(D)} |\mathrm{Ind}(B_0|L^x)|,$$

where $D \supset Q^v$ and $D \supset Q$. This completes the proof of step 1.

Step 2. Let b be a block of $C_G(Q)$ covered by B. We may assume that $T(b) = G$, $T(b) = \{x \in G \mid b^x = b\}$

Proof. By step 1, Q is normal in G and $C_G(Q)$ is normal in G. Let B_1 be a block of $T(b)$ which covers b and satisfies $B_1^G = B$. ($B_1 = b$ as elements of FG). Since D is abelian, we may assume that D is a defect group of b and B_1. Let b_0 be a Brauer correspondent of b, a block of $C_G(Q)\cap N_G(D)$, and $(B_1)_0$ be a Brauer correspondent of B_1, a block of $T(b)\cap N_G(D)$. Since there is a one to one correspondence between the indecomposable FG-modules in B and the indecomposable $FT(b)$-modules in B_1 by induction, we can show

$$(7) \qquad |\mathrm{Ind}(B|L)| = \sum_{y\in T(L)\backslash G/T(b)} |\mathrm{Ind}(B_1|L^y)|.$$

On the other hand $T(b_0) = T(b)\cap N_G(D)$, $(B_1)_0$ covers b_0 and $B_0 = ((B_1)_0)^{N_G(D)}$. So we have as in (7)

$$(8) \qquad |\mathrm{Ind}(B_0|L^x)| = \sum_{z\in T(L^x)\cap N_G(D))\backslash N_G(D)/T(b_0)} |\mathrm{Ind}((B_1)_0|L^{xz})|.$$

Here we assume that Proposition is true for $T(b)$. Since Q is normal in G, the following holds

$$|\mathrm{Ind}(B_1|L^y)| = \sum_{w\in(T(L^y)\cap T(b))\backslash T(b)/(T(b)\cap N_G(D))} |\mathrm{Ind}((B_1)_0|L^{yw})|.$$

Therefore we have from (7) and (8)

$$|\mathrm{Ind}(B|L)|$$

$$= \sum_{y\in T(L)\backslash G/T(\mathcal{B})} \ \sum_{w\in (T(L^y)\cap T(\mathcal{B}))\backslash T(\mathcal{B})/(T(\mathcal{B})\cap N_G(D))} |\mathrm{Ind}((B_1)_0|L^{yw})|$$

$$= \sum_{u\in T(L)\backslash G/(T(\mathcal{B})\cap N_G(D))} |\mathrm{Ind}((B_1)_0|L^u)|$$

$$= \sum_{x\in T(L)\backslash G/N_G(D)} \ \sum_{z\in (T(L^x)\cap N_G(D))\backslash N_G(D)/(T(\mathcal{B})\cap N_G(D))} |\mathrm{Ind}((B_1)_0|L^{xz})|$$

$$= \sum_{x\in T(L)\backslash G/N_G(D)} |\mathrm{Ind}(B_0|L^x)|.$$

This completes the proof of step 2 and we may assume that $B = \mathcal{B}$.

Step 3. We may assume that L is G-invariant.

Proof. By step 2, B is a block of $C_G(Q)$ and hence B is a block of T(L). Similarly B_0 is a block of $T(L)\cap N_G(D)$ and B_0 is a Brauer correspondent of B as a block of T(L). Let N be an indecomposable FG-module with source module L. Then there exists an indecomposable FT(L)-module N_0 with source module L such that $N = N_0{}^G$. N_0 is uniquely (up to isomorphism) determined by N, and N= NB if and only if $N_0 = N_0B$. Conversely if N_0 is an indecomposable FT(L)-module with source module L, then $N_0{}^G$ is an indecomposable FG-module with source module L. Hence the number of isomorphism classes of indecomposable (FG)B-modules with source module L is equal to that of isomorphism classes of indecomposable (FT(L))B-modules with source module L. Similarly, the number of isomorphism classes of indecomposable $(FN_G(D))B_0$-modules with source module L is equal to that of isomorphism classes of indecomposable $(F(T(L)\cap N_G(D))B_0$-modules with source module L. So if Proposition is true for T(L), we have $|Ind(B|L)| = |Ind(B_0|L)|$. By the way, since $T(\mathscr{b}) = G$ and D is a defect group of $\mathscr{b}$ (see step 2), we have $G = C_G(Q)N_G(D) = T(L)N_G(D)$. Hence Proposition holds for B. So we may assume that $T(L) = G$.

Step 4. Conclusion.

Proof. Under the assumption that Q is normal in G, B is a block of $C_G(Q)$ and that L is G-invariant, we will show $|Ind(B|L)| = |Ind(B_0|L)|$, using results and notations in §1. Since any indecomposable FG-module with source module L is a component of L^G and an indecomposable component of L^G has L as a source module by the assumption, we have

$$|Ind(B|L)| = |Irr(F(\bar{G}, \alpha)\bar{B}|,$$

where $Irr(F(\bar{G}, \alpha)\bar{B})$ is the set of isomorphism classes of irreducible $F(\bar{G}, \alpha)\bar{B}$-modules and $\bar{B}$ is a block of $F\overline{C_G(Q)}$ corresponding to B. Similarly we have

$$|Ind(B_0|L)| = |Irr(F(\overline{N_G(D)}, \alpha)\bar{B}_0)|,$$

where $\bar{B}_0$ is a block of $F(\overline{C_G(Q)\cap N_G(D)})$ corresponding to B_0. We note B_0 is a Brauer correspondent of B as a block of $C_G(Q)$.

Let $\tilde{B} = \theta(\bar{B})$. If $\bar{B} = \sum_{\bar{x}\in\overline{C_G(Q)}} a_{\bar{x}}\bar{x}$, $a_{\bar{x}} \in F$, then $\tilde{B} = \sum_{\bar{x}\in\overline{C_G(Q)}} a_{\bar{x}}\delta(\bar{x})r(\bar{x})e_\lambda$. Since $\bar{D}$ is a defect group of $\bar{B}$ as a block of $\overline{C_G(Q)}$ and A is a central p'-subgroup of $\tilde{G}$, a defect group $\tilde{D}$ of $\tilde{B}$ as a block of $C_{\tilde{G}}(Q)$ is a p-subgroup of $C_{\tilde{G}}(Q)$ such that $\sigma(\bar{D}) = \tilde{D}A/A$. We see $N_{\tilde{G}}(\tilde{D})/A = N_{\tilde{G}}(D)$ and $((C_{\tilde{G}}(Q)\cap N_{\tilde{G}}(\tilde{D}))/A = \widetilde{C_G(Q)\cap N_G(D)}$. Let $\tilde{B}_0 =$

$\theta(\bar{B}_0)$. $\tilde{B}_0$ is a block of $\widetilde{C_G(Q)\cap N_G(D)}$ and $\tilde{D}$ is a defect group of $\tilde{B}_0$. Since $\bar{B}_0$ is a Brauer correspondent of $\bar{B}$ as a block of $\overline{C_G(Q)}$, we can show that $\tilde{B}_0$ is a Brauer correspondent of $\tilde{B}$ as a block of $\tilde{C_G}(Q)$.

Let $\tilde{B}_1$, $\tilde{B}_2$,...., $\tilde{B}_n$ be the blocks of $\tilde{G}$ which cover $\tilde{B}$. Since $\bar{B}$ belongs to $Z(F(\bar{G}, \alpha))$, and $\tilde{B}$ belongs to $Z(F\tilde{G})$, the following holds.

(9) $$\tilde{B} = \tilde{B}_1 + \tilde{B}_2 + \dots + \tilde{B}_n.$$

Therefore we have

(10) $$|\mathrm{Ind}(B|L)| = |\mathrm{Irr}(F(\bar{G}, \alpha)\bar{B})| = |\mathrm{Irr}((F\tilde{G})\tilde{B}))| = \sum_{i=1}^{n} |\mathrm{Irr}((F\tilde{G})\tilde{B}_i)|,$$

because $F(\bar{G}, \alpha)\bar{B}$ and $(F\tilde{G})\tilde{B}$ are isomorphic by θ. By the assumption that B is a block of $C_G(Q)$ and since D is abelian, $p \nmid |G : C_G(Q)|$ and hence $p \nmid |\tilde{G} : \tilde{C_G}(Q)|$. So $\tilde{D}$ is a defect group of $\tilde{B}_i$. Let $(\tilde{B}_i)_0$ be a Brauer correspondent of $\tilde{B}_i$ which is a block of $N_{\tilde{G}}(\tilde{D})$. From (9),

$$\tilde{B}_0 = (\tilde{B}_1)_0 + (\tilde{B}_2)_0 + \dots + (\tilde{B}_n)_0.$$

Applying (10) for $N_G(D)$ and B_0, we have

(11) $$|\mathrm{Ind}(B_0|L)| = \sum_{i=1}^{n} |\mathrm{Irr}((FN_{\tilde{G}}(\tilde{D}))(\tilde{B}_i)_0)|.$$

Now if Alperin's conjecture is true, then $|\mathrm{Irr}((F\tilde{G})\tilde{B}_i)| = |\mathrm{Irr}(FN_{\tilde{G}}(\tilde{D}))(\tilde{B}_i)_0)|$, hence from (10) and (11), $|\mathrm{Ind}(B|L)| = |\mathrm{Ind}(B_0|L)|$. This completes the proof of Proposition.

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