NEW CLASSES OF MEROMORPHICALLY MULTIVALENT FUNCTIONS

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Abstract. In this paper, we introduce new subclasses $C_{n,p}(\alpha)$ of meromorphically multivalent functions defined by the subordination relation. We also obtain the inclusion relations for the classes $C_{n,p}(\alpha)$ and investigate the integral preserving properties of functions in $C_{n,p}(\alpha)$.

1. Introduction

Let $\sum_p$ denote the class of functions of the form

\begin{equation}
    f(z) = \frac{a_{-p}}{z^{p}} + \sum_{k=0}^{\infty} a_k z^k \quad (a_{-p} \neq 0, p \in N = \{1, 2, \ldots\})
\end{equation}

which are regular in the punctured disk $E = \{z : 0 < |z| < 1\}$. Following Uralegaddi and Somanatha [4], we define

\begin{equation}
    D^0 f(z) = f(z),
\end{equation}

\begin{equation}
    D^1 f(z) = \frac{a_{-p}}{z^{p}} + (p + 1)a_0 + (p + 2)a_1 z + (p + 3)a_2 z^2 + ... = (z^{p+1} f(z))'z^{p},
\end{equation}

\begin{equation}
    D^2 f(z) = D(D^1 f(z)),
\end{equation}

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and for \( n = 1, 2, \ldots \),

\[
D^n f(z) = D(D^{n-1} f(z)) = \frac{a_{-p}}{z^p} + \sum_{m=1}^{\infty} (p+m)^n a_{m-1} z^{m-1}
\]

Using the operator \( D^n \), Cho and Lee [2] introduced the subclasses \( B_{n,p}(\alpha) \) of \( \sum_p \) whose members are characterized by the condition

\[
Re \left\{ z^{p+1}(D^n f(z))' \right\} < -p \frac{n + \alpha}{n+1} \quad (z \in U = \{ z : |z| < 1 \})
\]

for some \( \alpha (0 \leq \alpha < 1) \) and \( n \in N_0 = N \cup \{0\} \). They proved that \( B_{n+1,p}(\alpha) \subset B_{n,p}(\alpha) \), and since \( B_{0,p}(\alpha) \) is the class of meromorphically \( p \)-valent functions [3]), all functions in \( B_{n,p}(\alpha) \) are \( p \)-valent. Also they considered some properties in connection with certain integral transform.

In this paper, we introduce the new classes \( C_{n,p}(\alpha) \) of meromorphically \( p \)-valent functions in \( U \).

Let \( C_{n,p}(\alpha) \) denote the class of functions \( f \in \sum_p \) which satisfy the condition

\[
-z^{p+1}(D^n f(z))' \prec p + \frac{p(1-\alpha)}{n+2} z \quad (0 \leq \alpha < 1, z \in U);
\]

where \( \prec \) denotes the subordination relation. From (1.7), we have that \( C_{n,p}(\alpha) \subset B_{n,p}(\alpha) \) for \( n \in N_0 \). Hence the classes \( C_{n,p}(\alpha) \) are subclasses of meromorphically \( p \)-valent functions. Also we shall prove that \( C_{n+1,p}(\alpha) \subset C_{n,p}(\alpha) \). Furthermore we consider certain integral transform of functions in \( C_{n,p}(\alpha) \).
2. Properties of the classes $C_{n,p}(\alpha)$

For the proofs of coming theorems, we need the following lemma due to Jack [1].

**Lemma 1.** Let $w$ be non-constant regular in $U = \{ z : |z| < 1 \}$, $w(0) = 0$. If $|w|$ attains its maximum value on the circle $|z| = r < 1$ at $z_0$, we have $z_0w'(z_0) = kw(z_0)$ where $k$ is a real number, $k \geq 1$.

**Theorem 1.** $C_{n+1,p}(\alpha) \subset C_{n,p}(\alpha)$ for each $n \in N_0$.

**Proof.** Let $f \in C_{n+1,p}(\alpha)$. Then

\[(2.1) \quad -z^{p+1}(D^{n+1}f(z))' \prec p + \frac{p(1-\alpha)}{n+3}z.\]

Define $w(z)$ in $U = \{ z : |z| < 1 \}$ by

\[(2.2) \quad -z^{p+1}(D^n f(z))' = p + \frac{p(1-\alpha)}{n+2}w(z).\]

Clearly $w(0) = 0$. Using the identity

\[(2.3) \quad z(D^n f(z))' = D^{n+1}f(z) - (p+1)D^nf(z),\]

the equation (2.2) may be written as

\[(2.4) \quad -z^p(D^{n+1}f(z) - (p+1)D^n f(z)) = p + \frac{p(1-\alpha)}{n+2}w(z).\]

Differentiating (2.4), we obtain
(2.5) \[-z^{p+1}(D^{n+1}f(z))' = p + \frac{p(1-\alpha)}{n+2}(w(z) + zw'(z)).\]

We claim that $|w(z)| < 1$ in $U$. Suppose that there exists a point $z_0 \in U$ such that $\max_{|z|<|z_0|} |w(z)| = |w(z_0)| = 1(w(z_0) \neq 1)$. Then, by Lemma 1, we have

(2.6) \[z_0 w'(z_0) = kw(z_0),\]

where $k \geq 1$. The equation (2.5) in conjunction with (2.6) yields

(2.7) \[|z_0^{p+1}(D^{n+1}f(z_0))' + p| = \left| \frac{p(1-\alpha)}{n+2}(1+k) \right| > \frac{p(1-\alpha)}{n+3},\]

which is a contradiction to (2.1). Hence $|w(z)| < 1$ in $U$ and from (2.2) it follows that $f \in C_{n,p}(\alpha)$.

**Theorem 2.** Let $f \in C_{n,p}(\alpha)$. Then

(2.8) \[F(z) = \frac{c}{z^{c+p}} \int_0^z t^{c+p-1}f(t)dt \quad (c > 0)\]

belongs to $C_{n,p}(\alpha)$.

**Proof.** Let $f \in C_{n,p}(\alpha)$. Define $w(z)$ in $U$ by

(2.9) \[-z^{p+1}(D^nF(z))' = p + \frac{p(1-\alpha)}{n+2}w(z).\]

Clearly $w(0) = 0$. Using the equation
\[ (2.10) \quad z(D^n F(z))' = cD^n f(z) - (c+p)D^n F(z), \]

the equation \((2.9)\) may be written as

\[ (2.11) \quad -z^p (cD^n f(z) - (c+p)D^n F(z)) = p + \frac{p(1-\alpha)}{n+2} w(z). \]

Differentiating \((2.11)\), we have

\[ (2.12) \quad -z^{p+1} (D^n f(z))' = p + \frac{p(1-\alpha)}{n+2} w(z) + \frac{p(1-\alpha)}{c(n+2)} zw'(z). \]

We claim that \(|w(z)| < 1\) in \(U\). For otherwise, by Lemma 1, there exists \(z_0\), \(|z_0| < 1\) such that \(z_0 w'(z_0) = kw(z_0)\), where \(|w(z_0)| = 1\) and \(k \geq 1\). Applying this result to \((2.12)\), we obtain

\[ (2.13) \quad |z_0^{p+1} (D^n f(z_0))'| + p = \left| \frac{p(1-\alpha)}{n+2} + \frac{p(1-\alpha)k}{c(n+2)} \right| \]

\[ > \frac{p(1-\alpha)}{n+2}, \]

which contradicts our assumption. Hence \(|w(z)| < 1\) in \(U\) and from \((2.12)\) we have that \(F \in C_{n,p}(\alpha)\).

Taking \(n = 0\) and \(c = 1\) in Theorem 2, we have the following

**Corollary 1.** Let \(f \in C_{0,p}(\alpha)\). Then

\[ (2.14) \quad F(z) = \frac{1}{z^{1+p}} \int_0^z t^p f(t) dt \]
belongs to $C_{0,p}(\alpha)$.

References


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