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On Asymptotic Solutions of Nonlinear and Linear Abel-Volterra Integral Equations

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Abstract

The paper is devoted to consider nonlinear Abel-Volterra integral equations of the form

$$\varphi^m(x)=\frac{a(x)}{\Gamma(\alpha)}\int_0^x\frac{\varphi(t)dt}{(x-t)^{1-\alpha}}+f(x)\quad(0<x<d\leqq\infty)$$

with $\alpha>0, m\neq 0,-1,-2,\cdots$, which includes the linear case for $m=1$. The asymptotic behavior of the solution $\varphi(x)$, as $x\to 0$, is obtained, provided that $a(x)$ and $f(x)$ have the special asymptotic behavior near zero.

1. Introduction

The nonlinear Volterra convolution integral equation

$$\varphi^m(x)=\int_0^x k(x-t)\varphi(t)dt+f(x)\quad(x>0)\tag{1.1}$$

with $m>1$ and more general non-convolution equation

$$\varphi^m(x)=a(x)\int_0^x k(x-t)\varphi(t)dt+f(x)\quad(x>0)\tag{1.2}$$

with $m>1$ were studied in [1], [4], [6], [14], [17]-[19] and [2], [3], [5], [7], respectively. The interest to these equations is caused by their applications in nonlinear theory of water perlocation [8], [19]. The above papers were devoted to investigate existence, uniqueness and stability of the nontrivial solution $\varphi(x)$ and the method of succesive approximation to the homogeneous and nonhomogeneous equations (1.1) and (1.2). In particular, if $k(u)=u^{\alpha-1}$ $(\alpha>0)$ the equations (1.1) and (1.2) are Abel's type integral equations which have

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applications in nonlinear theory of wave propagation [15], [22] (see [10] and [21] for the theory and other applications of Abel's type integral equations).

The problem to find solutions of the equations (1.1) and (1.2) in closed forms or their asymptotic solutions near zero and infinity, provided that such a solution exists, is also of importance. The solution in closed form of the homogeneous Abel-Volterra integral equation

$$\varphi^m(x) = \frac{a}{\Gamma(\alpha)} \int_0^x \frac{\varphi(t)dt}{(x-t)^{1-\alpha}} \quad (x > 0) \tag{1.3}$$

with $\alpha > 0, a \in \mathbf{R}$ was obtained for $m > 1$ in [22] (see also [1] and [4]), where $\mathbf{R}$ is meant the real number field. The asymptotic behavior, of the solution $\varphi(x)$ as $x \to 0$ and $x \to \infty$ of the Abel-Volterra integral equation of the form

$$\varphi(x) = \frac{1}{\Gamma(\alpha)} \int_0^x \frac{f(t) - [\varphi(t)]^m dt}{(x-t)^{1-\alpha}} + f(x) \quad (x > 0) \tag{1.4}$$

with $\alpha > 0$ and $m > 1$ in the cases when $f(x)$ has the general power asymptotics near zero and infinity was studied in [13] and [20] for $\alpha = 1/2$ and in [9] for any $\alpha > 0$ (see also [11] in this connection), and several first terms of asymptotics of $\varphi(x)$ were obtained. It should be noted that the asymptotic behavior of solutions of nonlinear Volterra equations more general than (1.2) was considered by many authors (see the results and bibliography in the book [5; Chapters 15, 17-20]), but most of the results are given only the first asymptotic term of the solutions.

Our paper deals with the investigation of the asymptotic behavior of a solution $\varphi(x)$, as $x \to 0$, of the Abel-Volterra equations of the form (1.2)

$$\varphi^m(x) = \frac{a(x)}{\Gamma(\alpha)} \int_0^x \frac{\varphi(t)dt}{(x-t)^{1-\alpha}} + f(x) \quad (0 < x < d \leqq \infty) \tag{1.5}$$

with $\alpha > 0, m \neq 0, -1, -2, \cdots$, provided that $a(x)$ and $f(x)$ have the asymptotics,

$$a(x) \sim x^{\alpha pm} \sum_{k=-l}^{\infty} a_k x^{\alpha k} \quad (x \to 0) \tag{1.6}$$

with $a_{-l} \neq 0$ and

$$f(x) \sim x^{\alpha pm} \sum_{k=-n}^{\infty} f_k x^{\alpha k} \quad (x \to 0) \tag{1.7}$$

with $f_{-n} \neq 0$, respectively, where $p = -1, 0, 1, \cdots, l, n \in Z$ being the set of integers. We show that under certain assumptions on parameters m, p, l and n the solution $\varphi(x)$ of the equation (1.5) has the asymptotic expansion

$$\varphi(x) \sim \sum_{k=s}^{\infty} \varphi_k x^{\alpha k} \quad (x \to 0) \tag{1.8}$$

with $s \geqq -1$, where the coefficients φ_k are expressed via a_k and f_k.

It should be noted that our method allows us to find the asymptotic solution, as $x \to 0$, of the linear Abel-Volterra integral equation

$$\varphi(x) = \frac{a(x)}{\Gamma(\alpha)} \int_0^x \frac{\varphi(t)dt}{(x-t)^{1-\alpha}} + f(x) \quad (0 < x < d \leqq \infty) \tag{1.9}$$

with $\alpha > 0$, provided that $a(x)$ and $f(x)$ have the asymptotics (1.6) and(1.7) with $m = 1$. Such an asymptotic solution in the particular cases $p = 0$, $l = 1$ and $n = 0, 1$ was obtained by authors in [16].

In Section 2 we prove two lemmas on an asymptoptic representation of power of a function being given asymptotics. Sections 3 - 5 deal with the asymptotic solutions of the equations (1.5). Section 6 is devoted to the nonlinear equation (1.5) with the integer $m = 2, 3, \cdots$. In section 7 we give asymptotic solutions of the linear equations (1.9).

2. Preliminaries

First we formulate the preliminary assertion.

Lemma 1. *Let $p \in \mathbf{Z}, \alpha \in \mathbf{R}$ and $\{\varphi_k\}_{k=p}^{\infty}$ be a sequence of real numbers. If m is a real number such that $m \neq 1, 0, -1, -2, \cdots$ and*

$$\varphi(x) \sim \sum_{k=p}^{\infty} \varphi_k x^{\alpha k} \quad (x \to 0), \tag{2.1}$$

then

$$\varphi^m(x) \sim x^{\alpha p m} \sum_{k=0}^{\infty} \Phi_{p,k} x^{\alpha k} \quad (x \to 0), \tag{2.2}$$

where the coefficients $\Phi_{p,k}$ are expressed via the coefficients φ_k:

$$\Phi_{p,0} = \binom{m}{0} \varphi_p^m;$$

$$\Phi_{p,1} = \binom{m}{1} \varphi_p^{m-1} \varphi_{p+1};$$

$$\Phi_{p,2} = \binom{m}{1} \varphi_p^{m-1} \varphi_{p+2} + \binom{m}{2} \varphi_p^{m-2} \varphi_{p+1}^2;$$

$$\Phi_{p,3} = \binom{m}{1} \varphi_p^{m-1} \varphi_{p+3} + \binom{m}{2}\binom{2}{1} \varphi_p^{m-2} \varphi_{p+1}\varphi_{p+2} + \binom{m}{3} \varphi_p^{m-3} \varphi_{p+1}^3;$$

$$\Phi_{p,4} = \binom{m}{1} \varphi_p^{m-1} \varphi_{p+4} + \binom{m}{2} \varphi_p^{m-2} \left[\varphi_{p+2}^2 + \binom{2}{1} \varphi_{p+1}\varphi_{p+3}\right];$$

$$+ \binom{m}{3}\binom{3}{1} \varphi_p^{m-3} \varphi_{p+1}^2 \varphi_{p+2} + \binom{m}{4} \varphi_p^{m-4} \varphi_{p+1}^4;$$

$$\Phi_{p,5} = \binom{m}{1}\varphi_p^{m-1}\varphi_{p+5} + \binom{m}{2}\binom{2}{1}\varphi_p^{m-2}\left[\varphi_{p+1}\varphi_{p+4} + \varphi_{p+2}\varphi_{p+3}\right]$$
$$+\binom{m}{3}\varphi_p^{m-3}\left[\binom{3}{1}\varphi_{p+1}^2\varphi_{p+3} + \binom{3}{2}\varphi_{p+1}\varphi_{p+2}^2\right]$$
$$+\binom{m}{4}\binom{4}{1}\varphi_p^{m-4}\varphi_{p+1}^3\varphi_{p+2} + \binom{m}{5}\varphi_p^{m-5}\varphi_{p+1}^5, \qquad etc. \tag{2.3}$$

Proof. Using properties of asymptotic expansions [21, Section 16] we have, as $x \to 0$,

$$\varphi^m(x) \sim x^{\alpha pm}\left(\sum_{k=0}^{\infty}\varphi_{k+p}x^{\alpha k}\right)^m \sim x^{\alpha pm}\sum_{j=0}^{\infty}\binom{m}{j}\varphi_p^{m-j}\left(\sum_{k=1}^{\infty}\varphi_{k+p}x^{\alpha k}\right)^j$$
$$\sim x^{\alpha pm}\left\{\binom{m}{0}\varphi_p^m + \binom{m}{1}\varphi_p^{m-1}\left[\varphi_{p+1} + \sum_{k=1}^{\infty}\varphi_{k+p+1}x^{\alpha k}\right]x^{\alpha}\right.$$
$$\left. + \sum_{j=2}^{\infty}\binom{m}{j}\varphi_p^{m-j}x^{\alpha j}\left(\sum_{k=0}^{\infty}\varphi_{k+p+1}x^{\alpha k}\right)^j\right\}$$
$$\sim x^{\alpha pm}\left\{\binom{m}{0}\varphi_p^m + \binom{m}{1}\varphi_p^{m-1}\varphi_{p+1}x^{\alpha}\right.$$
$$+\left[\binom{m}{1}\varphi_p^{m-1}\varphi_{p+2} + \binom{m}{2}\varphi_p^{m-2}\varphi_{p+1}^2\right]x^{2\alpha}$$
$$+\binom{m}{1}\varphi_p^{m-1}x^{2\alpha}\sum_{k=2}^{\infty}\varphi_{k+p+1}x^{\alpha k}$$
$$+\binom{m}{2}\varphi_p^{m-2}x^{2\alpha}\left[\binom{2}{1}\varphi_{p+1}\left(\sum_{k=1}^{\infty}\varphi_{k+p+1}x^{\alpha k}\right) + \left(\sum_{k=1}^{\infty}\varphi_{k+p+1}x^{\alpha k}\right)^2\right]$$
$$\left. + \sum_{j=3}^{\infty}\binom{m}{j}\varphi_p^{m-j}x^{\alpha j}\left(\sum_{k=0}^{\infty}\varphi_{k+p+1}x^{\alpha k}\right)^j\right\}.$$

Continuing this process we obtain (2.5) - (2.6).

If m is an integer, then (2.2) can be written in another form.

Lemma 2. *Let $p \in \mathbf{Z}, \alpha \in \mathbf{R}$ and $\{\varphi_k\}_{k=p}^{\infty}$ be a sequence of real numbers. If $m = 2, 3, \cdots$ and the asymptotic expansion* (2.1) *holds, then, as $x \to 0$,*

$$\varphi^m(x) \sim x^{\alpha pm}\sum_{r=0}^{\infty}\Phi_{p,k}x^{\alpha k}, \tag{2.4}$$

$$\Phi_{p,0} = \varphi_p^m, \quad \Phi_{p,k} = \sum_{i_0=0}^{m-1}\sum_{i_1,i_2,\cdots,i_j}\frac{m!}{i_0!i_1!i_2!\cdots i_j!}\varphi_p^{i_0}\varphi_{p+1}^{i_1}\varphi_{p+2}^{i_2}\cdots\varphi_{p+j}^{i_j}, \tag{2.5}$$

where the summation is taken over all nonnegative integers $i_1, i_2, \cdots, i_j$ such that

$$0 \leqq i_1 \leqq i_2 \leqq \cdots \leqq i_j \leqq k,\ i_0 + i_1 + \cdots + i_j = m,\ i_1 + 2i_2 + \cdots + ji_j = k. \tag{2.6}$$

3. Asymptotic Solutions of Nonlinear Equations in the Space of Locally Integrable Functions

In this section we obtain the asymptotic behavior of the solution $\varphi(x)$ of the equation as $x \to 0$

$$\varphi^m(x) = \frac{a(x)}{\Gamma(\alpha)} \int_0^x \frac{\varphi(t)dt}{(x-t)^{1-\alpha}} + f(x)\ \ (0 < x < d \leqq \infty) \tag{3.1}$$

for $0 < \alpha < 1, m \in \mathbf{R}, m \neq 0, -1, -2, \cdots$, provided that $a(x)$ and $f(x)$ have the asymptotics

$$a(x) \sim x^{-\alpha m} \sum_{k=-l}^{\infty} a_k x^{\alpha k}\ \ (x \to 0) \tag{3.2}$$

with $l \in \mathbf{Z}, a_{-l} \neq 0$, and

$$f(x) \sim x^{-\alpha m} \sum_{k=-n}^{\infty} f_k x^{\alpha k}\ \ (x \to 0) \tag{3.3}$$

with $n \in \mathbf{Z}, f_{-n} \neq 0$.

First we consider the equation (3.1), where $a(x)$ and $f(x)$ have the asymptotics (3.2) and (3.3) in the case $l = n \geqq 0$. We shall seek an asymptotic solution $\varphi(x)$ of (3.1) in the form

$$\varphi(x) \sim \sum_{k=-1}^{\infty} \varphi_k x^{\alpha k}\ \ (x \to 0). \tag{3.4}$$

Then, by Lemma 1

$$\varphi^m(x) \sim x^{-\alpha m} \sum_{k=0}^{\infty} \Phi_{-1,k} x^{\alpha k}\ \ (x \to 0), \tag{3.5}$$

where the coefficients $\Phi_{-1,k}$ are expressed via φ_k by (2.2) - (2.3) (with $p = -1$). Applying (3.4) and Theorem 16.1 of [21], we have, as $x \to 0$,

$$\frac{1}{\Gamma(\alpha)} \int_0^x \frac{\varphi(t)dt}{(x-t)^{1-\alpha}} \sim \sum_{k=-1}^{\infty} \frac{\Gamma(\alpha k + 1)\varphi_k}{\Gamma(\alpha k + \alpha + 1)} x^{\alpha k + \alpha}. \tag{3.6}$$

Using (3.2) and properties of asymptotic expansions, we obtain, as $x \to 0$,

$$\begin{aligned} \frac{a(x)}{\Gamma(\alpha)} \int_0^x \frac{\varphi(t)dt}{(x-t)^{1-\alpha}} &\sim \sum_{k=0}^{\infty} \left(\sum_{i=0}^{k} \frac{a_{k-n-i}\Gamma(\alpha i - \alpha + 1)\varphi_{i-1}}{\Gamma(\alpha i + 1)} \right) x^{\alpha(k-n-m)} \\ &\sim x^{-\alpha m} \sum_{k=-n}^{\infty} \left(\sum_{i=0}^{k+n} \frac{a_{k-i}\Gamma(\alpha i - \alpha + 1)\varphi_{i-1}}{\Gamma(\alpha i + 1)} \right) x^{\alpha k}. \end{aligned} \tag{3.7}$$

From here, taking (3.1), (3.3) and (3.5) into account, we arrive at the following asymptotic relation, as $x \to 0$,

$$x^{-\alpha m} \sum_{k=0}^{\infty} \Phi_{-1,k} x^{\alpha k} \sim x^{-\alpha m} \sum_{k=-n}^{\infty} \left(\sum_{i=0}^{k+n} \frac{a_{k-i}\Gamma(\alpha i - \alpha + 1)\varphi_{i-1}}{\Gamma(\alpha i + 1)} \right) x^{\alpha k} + x^{-\alpha m} \sum_{k=-n}^{\infty} f_k x^{\alpha k}, \quad (3.8)$$

where $\Phi_{-1,k}$ are expressed via the coefficients φ_k by (2.2) and (2.3).

If $n \geqq 1$, then we obtain from (3.8) that if the coefficients φ_k satisfy the relations

$$\sum_{i=0}^{k+n} \frac{a_{k-i}\Gamma(\alpha i - \alpha + 1)\varphi_{i-1}}{\Gamma(\alpha i + 1)} + f_k = 0 \quad (k = -n, -n+1, \cdots, -1) \qquad (3.9)$$

$$\Phi_{-1,k} = \sum_{i=0}^{k+n} \frac{a_{k-i}\Gamma(\alpha i - \alpha + 1)\varphi_{i-1}}{\Gamma(\alpha i + 1)} + f_k \quad (k = 0, 1, 2, \cdots), \qquad (3.10)$$

then (3.4) is the asymptotic solution of the equation (3.1). If $n = 0$, then it follows from (3.8) that if the coefficients φ_k satisfy the relations

$$\Phi_{-1,k} = \sum_{i=0}^{k} \frac{a_{k-i}\Gamma(\alpha i - \alpha + 1)\varphi_{i-1}}{\Gamma(\alpha + i + 1)} + f_k \quad (k = 0, 1, 2, \cdots), \qquad (3.11)$$

then (3.4) is also the asymptotic solution of the equation (3.1).

From here we obtain the following result.

Theorem 1. *Let $n = 0, 1, 2, \cdots$ and let functions $a(x)$ and $f(x)$ have asymptotic expansions*

$$a(x) \sim x^{-\alpha n} \sum_{k=-n}^{\infty} a_k x^{\alpha k} \quad (x \to 0) \qquad (3.12)$$

with $0 < \alpha < 1, a_{-n} \neq 0$ and (3.3). Let the coefficients φ_k satisfy the relations (3.9) and (3.10) if $n > 0$ and the relation (3.11) if $n = 0$. Then the integral equation (3.1) is asymptotically solvable in the space of locally integrable functions on $(0, d)$ with $0 < d \leqq \infty$, and its asymptotic solution $\varphi(x)$ has the form (3.4).

Now we consider the case $n < l$ in the asymptotics of (3.2) and (3.3). We shall seek the asymptotic solution $\varphi(x)$ of (3.1) in the form

$$\varphi(x) \sim \sum_{k=l-n-1}^{\infty} \varphi_k x^{\alpha k} \quad (x \to 0) \qquad (3.13)$$

and come to the asymptotic relation

$$x^{\alpha(l-n-1)m} \sum_{k=0}^{\infty} \Phi_{l-n-1,k} x^{\alpha k} \sim x^{-\alpha m} \sum_{k=-n}^{\infty} \left(\sum_{i=l-n}^{k+l} \frac{a_{k-i}\Gamma(\alpha i - \alpha + 1)\varphi_{i-1}}{\Gamma(\alpha i + 1)} \right) x^{\alpha k}$$

$$+ x^{-\alpha m} \sum_{k=-n}^{\infty} f_k x^{\alpha k} \quad (x \to 0), \qquad (3.14)$$

where $\Phi_{l-n-1,k}$ are expressed via the coefficients φ_k by (2.2) and (2.3).

Let now suppose that $q = (l-n)m$ be an integer for $m \neq 0, -1, -2, \cdots$ such that $q \geqq -n$. Then (3.14) is equivalent to the relation

$$x^{-\alpha m} \sum_{k=q}^{\infty} \Phi_{l-n-1,k-q} x^{\alpha k} \sim x^{-\alpha m} \sum_{k=-n}^{\infty} \left(\sum_{i=l-n}^{k+l} \frac{a_{k-i}\Gamma(\alpha i - \alpha + 1)\varphi_{i-1}}{\Gamma(\alpha i + 1)} \right) x^{\alpha k}$$

$$+x^{-\alpha m} \sum_{k=-n}^{\infty} f_k x^{\alpha k} \quad (x \to 0). \tag{3.15}$$

From here we obtain that if $q > -n$ and the coefficients φ_k satisfy the equalities

$$\sum_{i=l-n}^{k+l} \frac{a_{k-i}\Gamma(\alpha i - \alpha + 1)\varphi_{i-1}}{\Gamma(\alpha i + 1)} + f_k = 0 \quad (k = -n, -n+1, \cdots, q-1), \tag{3.16}$$

$$\Phi_{l-n-1,k-q} = \sum_{i=l-n}^{k+l} \frac{a_{k-i}\Gamma(\alpha i - \alpha + 1)\varphi_{i-1}}{\Gamma(\alpha i + 1)} + f_k \quad (k = q, q+1, \cdots), \tag{3.17}$$

then (3.13) is the asymptotic solution of the equation (3.1). If $q = -n$, then it follows from (3.15) that if the coefficients φ_k satisfy the relation (3.17), then (3.13) is also the asymptotic solution of the equation (3.1).

Thus we arrive at the following statement.

Theorem 2. *Let l, n be integers with $l > n$, $q = (l-n)m$ be an integer for $m \neq 0, -1, -2, \cdots$ such that $q \geqq -n$, and let the functions $a(x)$ and $f(x)$ have the asymptotic expansions* (3.2) *and* (3.3). *Let the coefficients φ_k satisfy the relations* (3.16) *and* (3.17) *when $q > -n$ and the relation* (3.17) *when $q = -n$. Then the integral equation* (3.1) *is asymptotically solvable in the space of locally bounded functions on $(0, d)$ with $0 < d \leqq \infty$, and its asymptotic solution $\varphi(x)$ has the form* (3.13).

4. Asymptotic Solutions of Nonlinear Equations in the Space of Locally Bounded Functions

In this section we obtain the asymptotic behavior, as $x \to 0$, of the solution $\varphi(x)$ of the equation

$$\varphi^m(x) = \frac{a(x)}{\Gamma(\alpha)} \int_0^x \frac{\varphi(t)dt}{(x-t)^{1-\alpha}} + f(x) \quad (0 < x < d \leqq \infty) \tag{4.1}$$

with $\alpha > 0, m \neq 0, -1, -2, \cdots$, provided that $a(x)$ and $f(x)$ have the asymptotics

$$a(x) \sim x^{\alpha pm} \sum_{k=-l}^{\infty} a_k x^{\alpha k} \quad (x \to 0) \tag{4.2}$$

with $a_{-l} \neq 0$ and

$$f(x) \sim x^{\alpha pm} \sum_{k=-n}^{\infty} f_k x^{\alpha k} \quad (x \to 0) \tag{4.3}$$

with $f_{-n} \neq 0$, where $p = 0, 1, 2, \cdots, l, n \in \mathbf{Z}$.

First we consider the case $n = l - p - 1 \geqq 0$. We shall seek an asymptotic solution $\varphi(x)$ of (4.1) in the form

$$\varphi(x) \sim \sum_{k=p}^{\infty} \varphi_k x^{\alpha k} \quad (x \to 0). \tag{4.4}$$

Here (4.3) takes the form

$$f(x) \sim x^{\alpha p m} \sum_{k=p+1-l}^{\infty} f_k x^{\alpha k} \quad (x \to 0). \tag{4.5}$$

with $p = 0, 1, 2, \cdots, f_{p+1-l} \neq 0$

Making the same arguments as in Section 2, we arrive at the asymptotic relation, as $x \to 0$, similar to (3.8):

$$x^{\alpha p m} \sum_{k=0}^{\infty} \Phi_{p,k} x^{\alpha k} \sim x^{\alpha p m} \sum_{k=p+1-l}^{\infty} \left(\sum_{i=0}^{k+l-p-1} \frac{a_{k-i-p-1} \Gamma(\alpha[i+p]+1) \varphi_{i+p}}{\Gamma(\alpha[i+p+1]+1)} \right) x^{\alpha k}$$

$$+ x^{\alpha p m} \sum_{k=p+1-l}^{\infty} f_k x^{\alpha k} \quad (x \to 0), \tag{4.6}$$

where the coefficients $\Phi_{p,k}$ are expressed via the coefficients φ_k by (2.2) - (2.3).

If $l - p - 1 \geqq 1$, then we obtain from (4.6) that when the coefficients φ_k satisfy the relations

$$\sum_{i=0}^{k+l-p-1} \frac{a_{k-i-p-1} \Gamma(\alpha[p+i]+1) \varphi_{i+p}}{\Gamma(\alpha[i+p+1]+1)} + f_k = 0 \quad (k = p+1-l, p+2-l, \cdots, -1), \tag{4.7}$$

$$\Phi_{p,k} = \sum_{i=0}^{k+l-p-1} \frac{a_{k-i-p-1} \Gamma(\alpha[p+i]+1) \varphi_{i+p}}{\Gamma(\alpha[p+i+1]+1)} + f_k \quad (k = 0, 1, 2, \cdots), \tag{4.8}$$

(4.4) is the asymptotic solution of the equation (4.1). If $l - p - 1 = 0$, then it follows from (4.6) that when the coefficients φ_k satisfy the relations

$$\Phi_{p,k} = \sum_{i=0}^{k} \frac{a_{k-i-p-1} \Gamma(\alpha[p+i]+1) \varphi_{i+p}}{\Gamma(\alpha[p+i+1]+1)} + f_k \quad (k = 0, 1, 2, \cdots), \tag{4.9}$$

(4.4) is also the asymptotic solution of the equation (4.1).

Consequently we obtain the following result.

Theorem 3. *Let $p = 0, 1, 2, \cdots$ and l be an integer such that $l \geqq p + 1$ and let the functions $a(x)$ and $f(x)$ have the asymptotic expansions* (4.2) *and* (4.5) *with $n = l - p - 1$. Let the coefficients φ_k satisfy the relations* (4.7) *and* (4.8) *if $l > p+1$ and the relation* (4.9) *if $l = p + 1$. Then the integral* equation (4.1) *is asymptotically solvable in the space of locally bounded functions on $(0, d)$ with $0 < d \leqq \infty$, and its asymptotic solution $\varphi(x)$ has the form* (4.4).

Now we consider the case $n < l - p - 1$. We shall seek an asymptotic solution $\varphi(x)$ of (4.1) in the form

$$\varphi(x) \sim \sum_{k=l-n-1}^{\infty} \varphi_k x^{\alpha k} \quad (x \to 0). \tag{4.10}$$

and come to the asymptotic relation

$$x^{\alpha(l-n-1)m} \sum_{k=0}^{\infty} \Phi_{l-n-1,k} x^{\alpha k} \sim x^{\alpha pm} \sum_{k=-n}^{\infty} \left(\sum_{i=l-n-p-1}^{k+l-p-1} \frac{a_{k-i-p-1}\Gamma(\alpha[i+p]+1)\varphi_{i+p}}{\Gamma(\alpha[i+p+1]+1)} \right) x^{\alpha k}$$
$$+ x^{\alpha pm} \sum_{k=-n}^{\infty} f_k x^{\alpha k} \quad (x \to 0), \tag{4.11}$$

where $\Phi_{l-n-1,k}$ are expressed via φ_k by (2.2) - (2.3).

Let now suppose that $q = (l-n-p-1)m$ be an integer for $m \neq 0, -1, -2, \cdots$ such that $q \geqq -n$. Then (4.11) is equivalent to the relation

$$x^{\alpha pm} \sum_{k=q}^{\infty} \Phi_{l-n-1,k-q} x^{\alpha k} \sim x^{\alpha pm} \sum_{k=-n}^{\infty} \left(\sum_{i=l-n-p-1}^{k+l-p-1} \frac{a_{k-i-p-1}\Gamma(\alpha[i+p]+1)\varphi_{i+p}}{\Gamma(\alpha[i+p+1]+1)} \right) x^{\alpha k}$$
$$+ x^{\alpha pm} \sum_{k=-n}^{\infty} f_k x^{\alpha k} \quad (x \to 0), \tag{4.12}$$

from which we obtain that when $q > -n$ and the coefficients φ_k satisfy the equalities

$$\sum_{i=l-n-p-1}^{k+l-p-1} \frac{a_{k-i-p-1}\Gamma(\alpha[i+p]+1)\varphi_{i+p}}{\Gamma(\alpha[i+p+1]+1)} + f_k = 0 \quad (k = -n, -n+1, \cdots, q-1), \tag{4.13}$$

$$\Phi_{l-n-1,k-q} = \sum_{i=l-n-p-1}^{k+l-p-1} \frac{a_{k-i-p-1}\Gamma(\alpha[i+p]+1)\varphi_{i+p}}{\Gamma(\alpha[i+p+1]+1)} + f_k \quad (k = q, q+1, \cdots), \tag{4.14}$$

(4.10) is the asymptotic solution of the equation (4.1). If $q = -n$, then it follows from (4.12) that when the coefficients φ_k satisfy the relation (4.14), (4.10) is also the asymptotic solution of the equation (4.1).

Hence we arrive at the following statement.

Theorem 4. *Let $p = 0, 1, 2, \cdots$ and l, n be integers with $l - p - 1 > n$ and let $q = (l-n-p-1)m$ be an integer for $m \neq 0, -1, -2, \cdots$ such that $q \geqq -n$. Let the functions $a(x)$ and $f(x)$ have the asymptotic expansions (4.2) and (4.3) and let the coefficients φ_k satisfy the relations (4.13) and (4.14) if $q > -n$ and the relation (4.14) if $q = -n$. Then the integral equation (4.1) is asymptotically solvable in the space of locally bounded functions on $(0, d)$ with $0 < d \leqq \infty$, and its asymptotic solution $\varphi(x)$ has the form (4.10).*

5. Asymptotic Solutions of Nonlinear Equations in More General Case

The results obtained in Sections 2 and 3 allow us to find the asymptotic behavior of the solutions $\varphi(x)$ of the nonlinear integral equations (1.5), as $x \to 0$, provided that $a(x)$ and $f(x)$ have the asymptotic expansions (1.6) and (1.7) under additional assumptions on numbers m, l, n and $p = -1, 0, 1, \cdots$, when $l - p - 1 = n \geqq 0$ and when $l - p - 1 > n$ and $q = (l - n - p - 1)m$, $(m \neq 0, -1, -2, \cdots)$ is an integer such that $q \geqq -n$. The case $p = -1$ was considered in Theorems 1 and 2 and $p = 0, 1, 2, \cdots$ in Theorems 3 and 4. They were caused by our investigations based on the asymptotic relations (3.8), (3.15), (4.6) and (4.12). Such an approach can be applied in some other cases, however the results will be more complicated.

In the present section we illustrate this fact for the nonlinear integral equation (1.5):

$$\varphi^m(x) = \frac{a(x)}{\Gamma(\alpha)} \int_0^x \frac{\varphi(t)dt}{(x-t)^{1-\alpha}} + f(x) \quad (0 < x < d \leqq \infty) \tag{5.1}$$

with $\alpha > 0, m \neq 1, 0, -1, -2, \cdots$ in the case when $a(x)$ and $f(x)$ have the asymptotics (1.6) and (1.7) with $n = l - p - 1 < 0, p = -1, 0, 1, \cdots, l \in \mathbf{Z}$:

$$a(x) \sim x^{\alpha p m} \sum_{k=-l}^{\infty} a_k x^{\alpha k} \quad (x \to 0) \tag{5.2}$$

with $a_{-l} \neq 0$, and

$$f(x) \sim x^{\alpha p m} \sum_{k=p-l+1}^{\infty} f_k x^{\alpha k} \quad (x \to 0) \tag{5.3}$$

with $f_{p-l+1} \neq 0$. The equation (5.1) does not belong to the equations described by Theorems 1 and 3 because $n = l - p - 1 < 0$.

We shall seek an asymptotic solution $\varphi(x)$ of the equation (5.1) in the form

$$\varphi(x) \sim \sum_{k=q}^{\infty} \varphi_k x^{\alpha k} \quad (x \to 0), \tag{5.4}$$

where q is an unknown integer. If we suppose that $m(q-p)$ is an integer, then applying the same arguments as in Theorems 1 and 3, we come to the asymptotic relation

$$\begin{aligned} x^{\alpha p m} & \sum_{k=m(q-p)}^{\infty} \Phi_{q,k-m(q-p)} x^{\alpha k} \\ & \sim x^{\alpha p m} \sum_{k=q-l+1}^{\infty} \left(\sum_{i=0}^{k+l-q-1} \frac{a_{k-i-q-1}\Gamma(\alpha[i+q]+1)\varphi_{i+q}}{\Gamma(\alpha[i+q+1]+1)} \right) x^{\alpha k} \\ & + x^{\alpha p m} \sum_{k=p-l+1}^{\infty} x^{\alpha k} f_k \quad (x \to 0). \end{aligned} \tag{5.5}$$

If we suppose that $m > 0$ and $(p - l + 1)/m$ is an integer and take q as

$$q = p + \frac{p - l + 1}{m}, \tag{5.6}$$

then $m(q-p) = p-l+1 > 0$ is a positive integer, and $q > p$ and $q \geqq 0$. Hence from (5.5) we obtain that if the coefficients φ_k satisfy the equalities

$$\Phi_{q,k+l-p-1} = f_k \quad (k = p-l+1, p-l, \cdots, q-l), \tag{5.7}$$

$$\Phi_{q,k+l-p-1} = \sum_{i=0}^{k+l-q-1} \frac{a_{k-i-q-1}\Gamma(\alpha[i+q]+1)\varphi_{i+q}}{\Gamma(\alpha[i+q+1]+1)} + f_k \quad (k = q-l+1, q-l+2, \cdots), \tag{5.8}$$

then the equation (5.1) is asymptotically solvable in the space of locally bounded functions on $(0,\infty)$ and its asymptotic solution has the form (5.4).

Thus we obtain the following result.

Theorem 5. *Let $m > 0, p = -1, 0, 1, \cdots, l$ be an integer such that $p-l+1 > 0$ and $(p-l+1)/m$ is integer and let $q = p+(p-l+1)/m$. Let $a(x)$ and $f(x)$ have the asymptotic expansions* (5.2) *and* (5.3)*. If the coefficients φ_k satisfy the relations* (5.7) *and* (5.8)*, then the equation* (5.1) *is asymptotically solvable in the space of locally bounded functions on $(0,d)$ with $0 < d \leqq \infty$, and its asymptotic solution $\varphi(x)$ has the form* (5.4).

Letting $p = -1$, and l be $-l$ in Theorem 5, we have

Corollary 5.1. *Let $m > 0$ and l be a positive integer such that l/m is an integer and set $q = -1 + l/m$. Let $l = 1, 2, \cdots$*

$$a(x) \sim x^{-\alpha m} \sum_{k=l}^{\infty} a_k x^{\alpha k} \quad (x \to 0) \tag{5.9}$$

with $a_l \neq 0$, and

$$f(x) \sim x^{-\alpha m} \sum_{k=l}^{\infty} f_k x^{\alpha k} \quad (x \to 0) \tag{5.10}$$

with $f_l \neq 0$, and let the coefficients φ_k satisfy the relations

$$\Phi_{q,k-l} = f_k \quad (k = l, l+1, \cdots, l+q), \tag{5.11}$$

$$\Phi_{q,k-l} = \sum_{i=0}^{k-l-q-1} \frac{a_{k-i-q-1}\Gamma(\alpha[i+q]+1)\varphi_{i+q}}{\Gamma(\alpha[i+q+1]+1)} + f_k \quad (k = q+l+1, q+l+2, \cdots). \tag{5.12}$$

Then the equation (5.1) *is asymptotically solvable in the space of locally bounded functions on $(0,d)$ with $0 < d \leqq \infty$, and its asymptotic solution $\varphi(x)$ has the form* (5.4).

If we suppose that $m < 1$ and $(p-l+1)/(1-m)$ is an integer and take q as

$$q = p - \frac{p-l+1}{1-m}, \tag{5.13}$$

then $m(q-p)=q-l+1$ is a negative integer and $q<p$. Hence from (5.5) we obtain that if the coefficients φ_k satisfy the equalities

$$\Phi_{q,k+l-q-1}=\sum_{i=0}^{k+l-q-1}\frac{a_{k-i-q-1}\Gamma(\alpha[i+q]+1)\varphi_{i+q}}{\Gamma(\alpha[i+q+1]+1)} \tag{5.14}$$

$$(k=q-l+1,q-l+2,\cdots,p-l),$$

$$\Phi_{q,k+l-q-1}=\sum_{i=0}^{k+l-q-1}\frac{a_{k-i-q-1}\Gamma(\alpha[i+q]+1)\varphi_{i+q}}{\Gamma(\alpha[i+q+1]+1)}+f_k \tag{5.15}$$

$$(k=p-l+1,p-l+2,\cdots),$$

then (5.4) is also the asymptotic solution of the equation (5.1). When $p=0,1,2,\cdots$ and $m=l/(p+1)$, then $q=-1$ and (5.4) is the locally integrable solution on $(0,\infty)$. When $p=1,2,\cdots$ and $pm\leqq l-1$ then $q\geqq 0$ and (5.4) is the locally bounded solution on $(0,\infty)$. Therefore we arrive at the following result.

Theorem 6. *Let $m<1,p=0,1,2,\cdots,l$ be a integer such that $(p+1-l)/(1-m)$ is integer and let $q=p-(p-l+1)/(1-m)$. If $m=l/(p+1)$ and the coefficients φ_k satisfy the relations* (5.14) *and* (5.15), *then the equation* (5.1) *is asymptotically solvable in the space of locally integrable functions on $(0,d)$ with $0<d\leqq\infty$, and its asymptotic solution has the form* (5.4) *with $q=-1$. If $p=1,2,\cdots,pm\leqq l-1$ and φ_k satisfy the relations* (5.14) *and* (5.15), *then the equation* (5.1) *is asymptotically solvable in the space of locally bounded functions on $(0,d)$ with $0<d\leqq\infty$, and its asymptotic solution $\varphi(x)$ has the form* (5.4).

Remark 1. In Sections 3 - 5 we found the power asymptotic solutions (1.8) of the equation (1.5) in the spaces of locally integrable and locally bounded functions on $(0,d)$ with $0<d\leqq\infty$. However, the existence of the solution $\varphi(x)$ itself of the integral equation (1.5) does not follow from the existence of its asymptotic solution.

6. Asymptotic Solution of Nonlinear Equations with Integral Exponent Nonlinearity

In this section we pick out the asymptotic solutions $\varphi(x)$ of the equations (1.5) with the integer $m=2,3,\cdots$ which are met in applications [8], [17], [19]. First we consider this equation in the case $0<\alpha<1$:

$$\varphi^m(x)=\frac{a(x)}{\Gamma(\alpha)}\int_0^x\frac{\varphi(t)dt}{(x-t)^{1-\alpha}}+f(x)\quad(0<x<d\leqq\infty), \tag{6.1}$$

provided that $a(x)$ and $f(x)$ have the asymptotics (3.2) and (3.3). We note that when $l>n$, $q=(l-n)m$ is an integer and, if additionally $n\geqq -m$, the condition $q\geqq -n$ is satisfied. Therefore from Theorems 1 and 2 and Lemma 2 we obtain the following assertion.

Theorem 7. *Let $m = 2, 3, \cdots, l$ and n be integers with $n \leqq l$ such that $n \geqq 0$ if $l = n$ and $n \geqq -m$ if $l > n$. Let $a(x)$ and $f(x)$ have the asymptotic expansions (3.2) and (3.3). Let the coefficients φ_k satisfy the relations*

$$\sum_{i=l-n}^{k+l} \frac{a_{k-i}\Gamma(\alpha i - \alpha + 1)\varphi_{i-1}}{\Gamma(\alpha i + 1)} + f_k = 0 \quad (k = -n, -n+1, \cdots, (l-n)m - 1), \tag{6.2}$$

$$\Phi_{l-n-1,k-l(n-m)} = \sum_{i=l-n}^{k+l} \frac{a_{k-i}\Gamma(\alpha i - \alpha + 1)\varphi_{i-1}}{\Gamma(\alpha i + 1)} + f_k \quad (k = (l-n)m, (l-n)m+1, \cdots), \tag{6.3}$$

if $(l-n)m + n > 0$ and the relation (6.3) if $(l-n)m + n = 0$. Then the nonlinear integral equation (6.1) is asymptotically solvable in the space of locally integrable functions on $(0, d)$ with $0 < d \leqq \infty$ when $l = n$ and in the space of locally bounded functions on $(0, d)$ with $0 < d \leqq \infty$, when $l > n$. Its the asymptotic solution $\varphi(x)$ has the form

$$\varphi(x) \sim \sum_{k=l-n-1}^{\infty} \varphi_k x^{\alpha k} \quad (x \to 0). \tag{6.4}$$

Now we consider the equation in the case $\alpha > 0$:

$$\varphi^m(x) = \frac{a(x)}{\Gamma(\alpha)} \int_0^x \frac{\varphi(t)dt}{(x-t)^{1-\alpha}} + f(x) \quad (0 < x < d \leqq \infty) \tag{6.5}$$

with $m = 2, 3, \cdots$, provided that $a(x)$ and $f(x)$ have the asymptotics (4.2) and (4.3). As earlier we see that if $l-p-1 > n$, then $q = (l-n-p-1)m$ is an integer and if additionally $n \geqq -m$, then the condition $q \geqq -n$ is satisfied. Therefore from Theorems 3 and 4 and Lemma 2 we obtain the statement similar to Theorem 7.

Theorem 8. *Let $m = 2, 3, \cdots, p = 0, 1, 2 \cdots, l$ and n be integers with $n \leqq l-p-1$ such that $n \geqq 0$ when $l-p-1 = n$ and $n \geqq -m$ when $l-p-1 > n$. Let $a(x)$ and $f(x)$ have the asymptotic expansions (4.2) and (4.3). Let the coefficients φ_k satisfy the relations*

$$\sum_{i=l-n-p-1}^{k+l-p-1} \frac{a_{k-i-p-1}\Gamma(\alpha[i+p]+1)\varphi_{i+p}}{\Gamma(\alpha[i+p+1]+1)} + f_k = 0 \tag{6.6}$$

$$(k = -n, -n+1, \cdots, (l-n-p-1)m - 1),$$

$$\Phi_{l-n-1,k-(l-n-p-1)m} = \sum_{i=l-n-p-1}^{k+l-p-1} \frac{a_{k-i-p-1}\Gamma(\alpha[i+p]+1)\varphi_{i+p}}{\Gamma(\alpha[i+p+1]+1)} + f_k \tag{6.7}$$

$$(k = (l-n-p-1)m, (l-n-p-1)m+1, \cdots),$$

if $(l-n-p-1)m + n > 0$ and the relation (6.7) if $(l-n-p-1)m + n = 0$. Then the nonlinear integral equation (6.5) is asymptotically solvable in the space of locally bounded functions on $(0, d)$ with $0 < d \leqq \infty$, and its asymptotic solution $\varphi(x)$ has the form (6.4).

The statements of Theorem 5 and Corollary 5.1 are also valid for the integral equation (6.5).

Theorem 9. *Let $m = 2, 3, \cdots, p = -1, 0, 1, \cdots$ and l be an integer such that $(p+1-l)/m$ is a positive integer and let $q = p + (p+1-l)/m$. Let $a(x)$ and $f(x)$ have the asymptotic expansions (5.2) and (5.3) and let the coefficients φ_k satisfy the relations (5.7) and (5.8). Then the equation (6.5) is asymptotically solvable in the space of locally bounded functions on $(0, d)$ with $0 < d \leqq \infty$, and its asymptotic solution $\varphi(x)$ has the form*

$$\varphi(x) \sim \sum_{k=q}^{\infty} \varphi_k x^{\alpha k} \quad (x \to 0). \tag{6.8}$$

Corollary 9.1. *Let $m = 2, 3, \cdots$ and l be a positive integer such that l/m is an integer and $q = -1 + l/m$. Let $a(x)$ and $f(x)$ have the asymptotic expansions (5.9) and (5.10) and let the coefficients φ_k satisfy the relations (5.11) - (5.12). Then the equation (6.5) is asymptotically solvable in the space of locally bounded functions on $(0, d)$ with $0 < d \leqq \infty$, and its asymptotic solution $\varphi(x)$ has the form (6.8).*

We also note a useful result which follows from Corollary 9.1 if we take $l = m = 2, 3, \cdots$.

Theorem 10. *Let $m = 2, 3, \cdots$ and*

$$a(x) \sim \sum_{k=0}^{\infty} a_k x^{\alpha k} \quad (x \to 0) \tag{6.9}$$

with $a_0 \neq 0$ and

$$f(x) \sim \sum_{k=0}^{\infty} f_k x^{\alpha k} \quad (x \to 0) \tag{6.10}$$

with $f_0 \neq 0$, and let the coefficients φ_k satisfy the relations

$$\Phi_{0,0} = f_0, \quad \Phi_{0,k} = \sum_{i=0}^{k-1} \frac{a_{k-i-1}\Gamma(\alpha i + 1)\varphi_i}{\Gamma(\alpha i + \alpha + 1)} + f_k \quad (k = 1, 2, \cdots). \tag{6.11}$$

Then the equation (6.5) is asymptotically solvable in the space of locally bounded functions on $(0, d)$ with $0 < d \leqq \infty$, and its asymptotic solution $\varphi(x)$ has the form

$$\varphi(x) \sim \sum_{k=0}^{\infty} \varphi_k x^{\alpha k} \quad (x \to 0). \tag{6.12}$$

7. Asymptotic Solution of Linear Equation

Let us now discuss the asymptotic behavior of the solution $\varphi(x)$ of the linear equation (1.9), as $x \to 0$. First we consider this equation in the case $0 < \alpha < 1$:

$$\varphi(x) = \frac{a(x)}{\Gamma(\alpha)} \int_0^x \frac{\varphi(t)dt}{(x-t)^{1-\alpha}} + f(x) \quad (0 < x < d \leqq \infty), \tag{7.1}$$

provided that $a(x)$ and $f(x)$ have the asymptotic expansions

$$a(x) \sim x^{-\alpha} \sum_{k=-l}^{\infty} a_k x^{\alpha k} \quad (x \to 0) \tag{7.2}$$

with $l \in \mathbf{Z}, a_{-l} \neq 0$, and

$$f(x) \sim x^{-\alpha} \sum_{k=-n}^{\infty} f_k x^{\alpha k} \quad (x \to 0) \tag{7.3}$$

with $n \in \mathbf{Z}, f_{-n} \neq 0$, respectively. In this case the asymptotic relation (3.14) is simplified:

$$x^{-\alpha} \sum_{k=l-n}^{\infty} \varphi_{k-1} x^{\alpha k} \sim x^{-\alpha} \sum_{k=-n}^{\infty} \left(\sum_{i=l-n}^{k+l} \frac{a_{k-i}\Gamma(\alpha i - \alpha + 1)\varphi_{i-1}}{\Gamma(\alpha i + 1)} \right) x^{\alpha k}$$

$$+ x^{-\alpha} \sum_{k=-n}^{\infty} f_k x^{\alpha k} \quad (x \to 0). \tag{7.4}$$

We thus obtain the following assertions:

a) If $l \geqq 0$ and $l \geqq n$ and the coefficients φ_k satisfy the relations

$$\sum_{i=l-n}^{k+l} \frac{a_{k-i}\Gamma(\alpha i - \alpha + 1)\varphi_{i-1}}{\Gamma(\alpha i + 1)} + f_k = 0 \quad (k = -n, -n+1, \cdots, l-n-1), \tag{7.5}$$

$$\varphi_{k-1} = \sum_{i=l-n}^{k+l} \frac{a_{k-i}\Gamma(\alpha i - \alpha + 1)\varphi_{i-1}}{\Gamma(\alpha i + 1)} + f_k \quad (k = l-n, l-n+1, \cdots), \tag{7.6}$$

when $l > 0$ and (7.6) when $l = 0$, then the asymptotic solution $\varphi(x)$ of the equation (7.1) has the form

$$\varphi(x) \sim \sum_{k=l-n-1}^{\infty} \varphi_k x^{\alpha k} \quad (x \to 0). \tag{7.7}$$

b) If $0 > l \geqq n$ and the coefficients φ_k satisfy the relations

$$\varphi_{k-1} = f_k \quad (k = -n, -n+1, \cdots, -n-l-1), \tag{7.8}$$

$$\varphi_{k-1} = \sum_{i=-n}^{l+k} \frac{a_{k-i}\Gamma(\alpha i - \alpha + 1)\varphi_{i-1}}{\Gamma(\alpha i + 1)} + f_k \quad (k = -n-l, -n-l+1, -n-l+2, \cdots), \tag{7.9}$$

then the asymptotic solution $\varphi(x)$ of the equation (7.1) has the form

$$\varphi(x) \sim \sum_{k=-n-1}^{\infty} \varphi_k x^{\alpha k} \quad (x \to 0). \tag{7.10}$$

Therefore we arrive at the following statement.

Theorem 11. *Let and n and l be integers such that $l \geqq n$ and let $a(x)$ and $f(x)$ have the asymptotic expansions (7.2) and (7.3).*

a) *Let $l \geq 0$ and $l \geqq n$ and the coefficients φ_k satisfy the relations (7.5) and (7.6) when $l > 0$ and the relation (7.6) when $l = 0$.* Then the linear integral equation (7.1) *is asymptotically solvable in the space of locally integrable functions on $(0, d)$ with $0 < d \leqq \infty$ when $l = n$ and in the space of locally bounded functions on $(0, d)$ with $0 < d \leqq \infty$ when $l > n$, and its asymptotic solution $\varphi(x)$ has the form (7.7).*

b) *Let $0 > l \geqq n$ and the coefficients φ_k satisfy the relations (7.8) and (7.9). Then the linear integral equation (7.1) is asymptotically solvable in the space of functions locally bounded on $(0, d)$ with $0 < d \leqq \infty$, and its asymptotic solution $\varphi(x)$ has the form (7.10).*

Corollary 11.1. *Let $0 < \alpha < 1$, $a(x)$ and $f(x)$ have the asymptotic expansions*

$$a(x) \sim x^{-\alpha} \sum_{k=0}^{\infty} a_k x^{\alpha k} \quad (x \to 0) \tag{7.11}$$

with $a_0 \neq 0$, and

$$f(x) \sim x^{-\alpha} \sum_{k=0}^{\infty} f_k x^{\alpha k} \quad (x \to 0) \tag{7.12}$$

with $f_0 \neq 0$ and let

$$\Gamma(\alpha k - \alpha + 1) a_0 \neq \Gamma(\alpha k + 1) \quad (k = 0, 1, 2, \cdots). \tag{7.13}$$

Then the linear integral equation (7.1) is asymptotically solvable in the space of locally integrable functions on $(0, d)$ with $0 < d \leqq \infty$, and its asymptotic solution $\varphi(x)$ has the form

$$\varphi(x) \sim \sum_{k=-1}^{\infty} \varphi_k x^{\alpha k} \quad (x \to 0), \tag{7.14}$$

where the coefficients φ_k are given by the recurrent equalities

$$\varphi_{-1} = [1 - a_0 \Gamma(1-\alpha)]^{-1} f_0,$$

$$\varphi_k = \left(1 - \frac{\Gamma(\alpha k + 1) a_0}{\Gamma(\alpha k + \alpha + 1)}\right)^{-1} \left(\sum_{i=0}^{k} \frac{a_{k+1-i} \Gamma(\alpha i - \alpha + 1) \phi_{i-1}}{\Gamma(\alpha i + 1)} + f_{k+1}\right) \tag{7.15}$$

$$(k = 0, 1, 2, \cdots).$$

Corollary 11.2. *Let $a(x)$ and $f(x)$ have the asymptotic expansions (7.11) and (7.12), respectively. Let there exists an integer $j \in \{0, 1, 2, \cdots\}$ such that*

$$\Gamma(\alpha j - \alpha + 1)a_0 = \Gamma(\alpha j + 1). \tag{7.16}$$

Let the coefficients f_k $(k = 0, 1, \cdots, j)$ in (7.12) satisfy the relations

$$f_0 = 0 \text{ when } j = 0 \quad \sum_{i=0}^{j-1} \frac{a_{j-i}\Gamma(\alpha i - \alpha + 1)\varphi_{i-1}}{\Gamma(\alpha i + 1)} + f_j = 0 \text{ when } j = 1, 2, \cdots, \tag{7.17}$$

where φ_{i-1} $(i = 0, 1, \cdots, j-1)$ are expressed via f_i $(i = 0, 1, \cdots, j-1)$ by means of (7.15). Then the equation (7.1) is asymptotically solvable in the space of locally integrable function on $(0, d)$ with $0 < d \leqq \infty$ and its solution $\varphi(x)$ is given by the formula

$$\varphi(x) \sim cx^{\alpha j} + \sum_{k=-1, k\neq j}^{\infty} \varphi_k x^{\alpha k} \quad (x \to 0). \tag{7.18}$$

Here c is an arbitrary constant and φ_k for $k \neq j$ are found from the recurrent relations (7.15). If the conditions (7.17) are not satisfied, the equation (7.1) does not have any asymptotic solution of the form (7.14).

Remark 2. Corollaries 11.1 and 11.2 coincide with Theorems 2.1 and 2.2 in [16].

Now we consider the equation (1.9) in the case $\alpha > 0$:

$$\varphi(x) = \frac{a(x)}{\Gamma(\alpha)} \int_0^x \frac{\varphi(t)dt}{(x-t)^{1-\alpha}} + f(x) \quad (0 < x < d \leqq \infty), \tag{7.19}$$

provided that $a(x)$ and $f(x)$ have the asymptotic expansions

$$a(x) \sim x^{\alpha p} \sum_{k=-l}^{\infty} a_k x^{\alpha k} \quad (x \to 0) \tag{7.20}$$

with $a_{-l} \neq 0$, and

$$f(x) \sim x^{\alpha p} \sum_{k=-n}^{\infty} f_k x^{\alpha k} \quad (x \to 0) \tag{7.21}$$

with $f_{-n} \neq 0$, where $p = 0, 1, 2, \cdots, l, n \in \mathbf{Z}$. The asymptotic relation (4.11) is simplified:

$$\begin{aligned} x^{\alpha p} &\sum_{k=l-n-p-1}^{\infty} \varphi_{k+p} x^{\alpha k} \\ &\sim x^{\alpha p} \sum_{k=-n}^{\infty} \left(\sum_{i=l-n-p-1}^{k+l-p-1} \frac{a_{k-i-p-1}\Gamma(\alpha[i+p]+1)\varphi_{i+p}}{\Gamma(\alpha[i+p+1]+1)} \right) x^{\alpha k} \\ &+ x^{\alpha p} \sum_{k=-n}^{\infty} f_k x^{\alpha k}. \end{aligned} \tag{7.22}$$

Thus we obtain the following assertions:

c) If $l-p-1 \geqq 0$ and $l-p-1 \geqq n$ and the coefficients φ_k satisfy the relations

$$\sum_{i=l-n-p-1}^{k+l-p-1} \frac{a_{k-i-p-1}\Gamma(\alpha[i+p]+1)\varphi_{i+p}}{\Gamma(\alpha[i+p+1]+1)} + f_k = 0 \tag{7.23}$$

$$(k=-n,-n+1,\cdots,l-n-p-2),$$

$$\varphi_{k+p} = \sum_{i=l-n-p-1}^{k+l-p-1} \frac{a_{k-i-p-1}\Gamma(\alpha[i+p]+1)\varphi_{i+p}}{\Gamma(\alpha[i+p+1]+1)} + f_k \tag{7.24}$$

$$(k=l-n-p-1,l-n-p,\cdots),$$

when $l-p-1>0$, and (7.24) when $l-p-1=0$, then the asymptotic solution $\varphi(x)$ of the equation (7.19) has the form (7.7).

d) If $0>l-p-1 \geqq n$ and the coefficients φ_k satisfy the relations

$$\varphi_{k+p} = f_k \quad (k=-n,-n+1,\cdots,p-n-l), \tag{7.25}$$

$$\varphi_{k+p} = \sum_{i=-n}^{k+l-p-1} \frac{a_{k-i-p-1}\Gamma(\alpha[i+p]+1)\varphi_{i+p}}{\Gamma(\alpha[i+p+1]+1)} + f_k \tag{7.26}$$

$$(k=p-n-l+1,p-n-l+2,\cdots),$$

then the asymptotic solution $\varphi(x)$ of the equation (7.19) has the form

$$\varphi(x) \sim \sum_{k=p-n}^{\infty} \varphi_k x^{\alpha k} \quad (x \to 0). \tag{7.27}$$

Therefore we arrive at the following statement.

Theorem 12. *Let $p=0,1,2,\cdots$ and l and n be integers such that $l-p-1 \geqq n$ and let $a(x)$ and $f(x)$ have the asymptotic expansions* (7.20) *and* (7.21).

c) *Let $l-p-1 \geqq 0$ and $l-p-1 \geqq n$ and the coefficients φ_k satisfy the relations* (7.23) *and* (7.24) *when $l-p-1>0$ and the relations* (7.24) *when $l-p-1=0$. Then the linear integral equation* (7.19) *is asymptotically solvable in the space of locally bounded functions on $(0,d)$ with $0<d \leqq \infty$, and its asymptotic solution $\varphi(x)$ has the form* (7.7).

d) *Let $0>l-p-1 \geqq n$ and the coefficients φ_k satisfy the relations* (7.25) *and* (7.26). *Then the linear integral equation* (7.19) *is asymptotically solvable in the space of locally bounded functions on $(0,d)$ with $0<d \leqq \infty$, and its asymptotic solution $\varphi(x)$ has the form* (7.27).

Corollary 12.1. *Let $l=1,2,\cdots,\alpha>0$, and $a(x)$ and $f(x)$ have the asymptotic expansions*

$$a(x) \sim x^{\alpha(l-1)} \sum_{k=-l}^{\infty} a_k x^{\alpha k} \quad (x \to 0) \tag{7.28}$$

with $a_{-l} \neq 0$, *and*

$$f(x) \sim x^{\alpha(l-1)} \sum_{k=0}^{\infty} f_k x^{\alpha k} \quad (x \to 0) \tag{7.29}$$

with $f_0 \neq 0$ *and let*

$$\Gamma(\alpha[k+l-1]+1)a_{-l} \neq \Gamma(\alpha[k+l]+1) \quad (k=0,1,2,\cdots). \tag{7.30}$$

Then the linear integral equation (7.19) *is asymptotically solvable in the space of locally bounded functions on* $(0,d)$ *with* $0 < d \leqq \infty$, *and its asymptotic solution* $\varphi(x)$ *has the form*

$$\varphi(x) \sim \sum_{k=l-1}^{\infty} \varphi_k x^{\alpha k} \quad (x \to 0), \tag{7.31}$$

where the coefficients φ_k *are given by the recurrent equalities*

$$\varphi_{l-1} = \left(1 - \frac{\Gamma(\alpha[l-1]+1)a_{-l}}{\Gamma(\alpha l+1)}\right)^{-1} f_0,$$

$$\varphi_{k+l} = \left(1 - \frac{\Gamma(\alpha[k+l]+1)a_{-l}}{\Gamma(\alpha[k+l+1]+1)}\right)^{-1}$$

$$\times \left(\sum_{i=0}^{k} \frac{a_{k-l-i+1}\Gamma(\alpha[i+l-1]+1)\varphi_{i+l-1}}{\Gamma(\alpha[i+l]+1)} + f_{k+1}\right) \quad (k=0,1,2,\cdots). \tag{7.32}$$

Corollary 12.2. *Let* $l = 1,2,\cdots$, *and* $a(x)$ *and* $f(x)$ *have the asymptotic expansions* (7.28) *and* (7.29). *Let there exists an integer* $j \in \{0,1,2,\cdots\}$ *such that*

$$\Gamma(\alpha[j+l-1]+1)a_{-l} = \Gamma(\alpha[j+l]+1). \tag{7.33}$$

Let the coefficients f_k $(k=0,1,\cdots,j)$ *in* (7.29) *satisfy the relations*

$$f_0 = 0 \text{ when } j = 0 \quad \sum_{i=0}^{j-1} \frac{a_{j-l-i}\Gamma(\alpha[i+l-1]+1)\varphi_{i+l-1}}{\Gamma(\alpha[i+l]+1)} + f_j = 0 \text{ when } j = 1,2,\cdots, \tag{7.34}$$

where ϕ_{i+l-1} $(i=0,1,\cdots,j-1)$ *are expressed via* f_i $(i=0,1,\cdots,j-1)$ *by means of* (7.32). *Then the equation* (7.19) *is asymptotically solvable in the space of locally bounded functions on* $(0,d)$ *with* $0 < d \leqq \infty$, *and its solution* $\varphi(x)$ *is given by the formula*

$$\varphi(x) \sim c x^{\alpha j} + \sum_{k=l-1, k\neq j}^{\infty} \varphi_k x^{\alpha k} \quad (x \to 0). \tag{7.35}$$

Here c *is an arbitrary constant and* φ_k *for* $k \neq j$ *are found from the recurrent relations* (7.32). *If the conditions* (7.34) *are not satisfied, then the equation* (7.19) *does not have any asymptotic solution of the form* (7.31).

Remark 3. If $l = 1$ and $0 < \alpha < 1$, Corollaries 12.1 and 12.2 coincide with Theorems 3.1 and 3.2 in [16].

Remark 4. Theorems 11 and 12 allow us to find the asymptotic solutions $\varphi(x)$ of the linear integral equations (7.1) and (7.19) provided that $a(x)$ and $f(x)$ have the asymptotic expansions (7.2), (7.3) and (7.20), (7.21) when $l \geqq n$ and $l - p - 1 \geqq n$, respectively. Moreover, unlike the nonlinear integral equations (see Sections 3 - 5), we consider all connections between the parameters l, n and p.

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