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A LOCAL PARAMETRIZATION OF THE TEICHMÜLLER SPACE OF CLOSED HYPERBOLIC SURFACES, IN TERMS OF TRIANGULATIONS

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1. Introduction

In this note, we will introduce a local parametrization of the Teichmüller space of closed hyperbolic surfaces. It is given by geodesic cellular decompositions on closed hyperbolic surfaces. We will explain it briefly. Let Σ_g be a closed surface of genus g ($g \geq 2$). Let $\mathcal{T}_g$ be the Teichmüller space of Σ_g. Let σ_0 be a hyperbolic structure on Σ_g. Let Δ_0 be a geodesic cellular decomposition of the hyperbolic surface (Σ_g, σ_0). For each hyperbolic structure σ which is very close to σ_0 in $\mathcal{T}_g$, there exists a geodesic cellular decomposition Δ of (Σ_g, σ) isotopic to Δ_0. (It is not unique. See the proof of Proposition 2.2.) Lengths of edges of Δ are the coordinates of the parametrization around σ_0. See § 2 for the precise description of the space of the parameters, which is denoted by V_g. The space V_g has dimension $e - v$, where e (resp. v) is the number of the edges (resp. vertices) of Δ_0. The dimension of the Teichmüller space $\mathcal{T}_g$ is $6g - 6$. Note that $(e - v) - (6g - 6) = 2v > 0$, which is shown by an elementary calculation. See the end of § 2 about the surplus of the dimension.

Bowditch-Epstein [Bow-E] gave a global parametrization of the Teichmüller space of punctured hyperbolic surfaces times an open simplex representing "weightings of the cusps". They also use geodesic cellullar decompositions of punctured hyperbolic surfaces, which are called spinal triangulations, to construct the space of their parameters. In this sense, our formulation is similar to theirs.

2. Formulation of the Results

In this section, we will formulate the parametrization of the Teichmüller space of closed hyperbolic surfaces.

Let Σ_g be an oriented closed surface of genus g ($g \geq 2$). Let σ_0 be a hyperbolic structure on Σ_g. Let Δ_0 be a cellular decomposition of (Σ_g, σ_0), made of geodesic triangles. We will denote the number of its vertices and the number of its edges, by v and e respectively.

Now we will consider cellular decompositions isotopic to Δ_0. Let $\mathbf{r}^0 := (r_1^0, \ldots, r_e^0) \in \mathbf{R}^e$ be the vector whose components are the lengths of the edges of Δ_0. Let D be an open neighborhood of $\mathbf{r}^0$ in $\mathbf{R}^e$. Consider a map

$$h_g : \mathbf{R}^e \supset D \longrightarrow \mathbf{R}^v$$

defined as follows. First, we will make hyperbolic geodesic triangles whose edge-lengths are given by $\mathbf{r} \in D$ and glue them together so that the vertices match up. Which triangle to glue to which, and along which edge, is determined by the combinatorial data of Δ_0. Then, generally we will have a hyperbolic structure on Σ_g with cone singularities at v points. We define h_g so that the components of $h_g(\mathbf{r})$ are equal to the cone angles at vertices obtained by gluing the triangles together. For example, $h_g(\mathbf{r}^0) = (2\pi, \ldots, 2\pi)$. We will show the following proposition in the next section.

Proposition 2.1. $(2\pi, \ldots, 2\pi) \in \mathbf{R}^v$ *is a regular value of* h_g.

Let V_g be the inverse image of $(2\pi, \ldots, 2\pi)$ by h_g, i.e., $h_g^{-1}(2\pi, \ldots, 2\pi) \cap D = V_g$. Then, by Proposition 2.1, V_g is an $(e - v)$-dimensional submanifold of $\mathbf{R}^e$ at $\mathbf{r}^0$. Each point of V_g gives a hyperbolic metric on Σ_g. Then there is a natural map

$$\phi_g : V_g \longrightarrow \mathcal{T}_g.$$

Also, the following proposition will be shown in the next section.

Proposition 2.2. *The map* ϕ_g *is a smooth submersion at* $\mathbf{r}^0$.

By Propositions 2.1 and 2.2, we obtain the local parametrization of the Teichmüller space $\mathcal{T}_g$, by means of geodesic cellular decompositions of closed hyperbolic surfaces.

Now consider the kernel of the derivative of the mapping ϕ_g. The euler number of Σ_g is $2 - 2g$, and e, v satisfy the equation $3v = 2e$. Then an elementary calculation shows that $(e - v) - (6g - 6) = 2v$. Thus the dimension of

$$\ker(\mathrm{d}\phi_g : \mathrm{T}_{\mathbf{r}^0} V_g \longrightarrow \mathrm{T}_{\phi_g(\mathbf{r}^0)} \mathcal{T}_g)$$

is $2v$. There are elements of ker $d\phi_g$, which will be called infinitesimal flat moves of vertices. Take a vertex ν of Δ_0. Denote the non-ν ends of all edges of Δ_0 emanating from ν by $\nu_1, \ldots, \nu_k$. Move ν on (Σ_g, σ_0) a bit, with fixing $\nu_1, \ldots, \nu_k$, and then connect ν with ν_i by a geodesic segment, for each i $(i = 1, \ldots, k)$. Then we obtain a geodesic cellular decomposition of (Σ_g, σ_0) isotopic to Δ_0. Let us call tangent vectors corresponding to this move of ν *infinitesimal flat moves* of ν. Obviously, the infinitesimal flat moves of ν are contained in ker $d\phi_g$. Each vertex has two dimensional directions of this move. If the infinitesimal flat moves of all vertices are linearly dependent, there exists a perturbation of the triangles of Δ_0 which has the following property : the derivatives of all edge-lengths of Δ_0 with respect to the perturbation are zero. Then one can construct a non-trivial Killing vector field on (Σ_g, σ_0). This contradicts the result of Bochner [Boch]. Therefore the infinitesimal flat moves are linearly independent. Thus we have the following proposition.

Proposition 2.3. *ker $d\phi_g$ is generated by the infinitesimal flat moves of all vertices.*

3. Proofs of Propositions 2.1 and 2.2

In this section, we will give the proofs of Proposition 2.1 and Proposition 2.2. By these propositions, as wroted in § 2, we obtain the local parametrization of the Teichmüller space of closed hyperbolic surfaces.

Proof of Proposition 2.1. Take any vertex p of the geodesic cellular decomposition Δ_0. Let $h_{g,p}$ be the cone angle at p given by $\mathbf{r} \in D$. For indicating that V_g is an $(e - v)$-dimensional manifold, we will show that there are deformations of the edge-lengths each of which induces a cone singularity at p (that is, the cone angle $\neq 2\pi$) with the deformed metric and that the derivative of $h_{g,p}$ with respect to such a deformation is not equal to 0.

First, consider the case where p lies on a simple closed geodesic on (Σ_g, σ_0). Take a pentagon in the hyperbolic 2-space $\mathbf{H}^2$ as indicated in Fig.1.

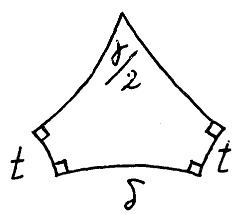

Fig.1

Let δ be the length of the closed geodesic and t be some arbitrary small number. The number t is the parameter of the deformation which we need. Now cut (Σ_g, σ_0) along the geodesic. Along the two boundary components of the surface cut just above, paste the two copies of the pentagon symmetrically as in Fig.2. Glue the broken edges of the pentagons by an isometry. For each $t > 0$, Σ_g has a hyperbolic metric σ_t with singularity of cone angle γ at the vertex of the pentagon.

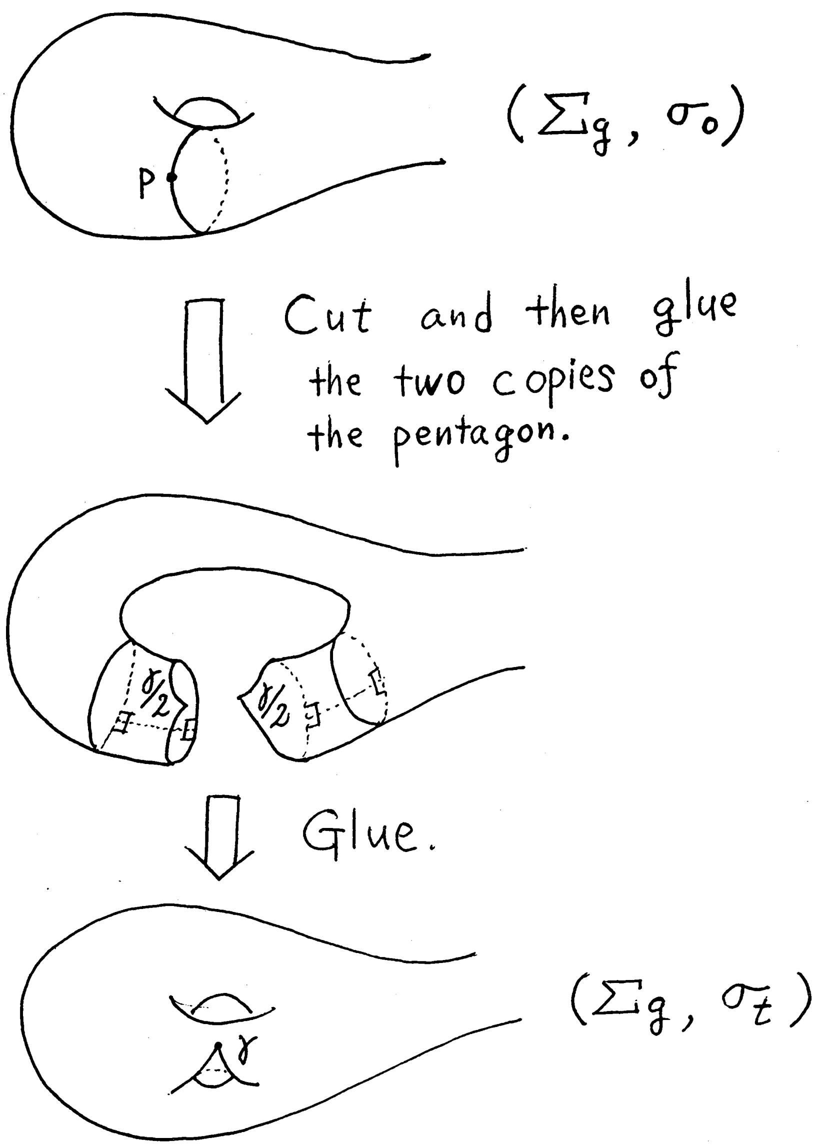

Fig.2

Then connect the cone point with vertices of Δ_0, which are originally connected with p in Δ_0, by geodesic segments on (Σ_g, σ_t). Thus, for each $t > 0$, we have a geodesic cellular decomposition of (Σ_g, σ_t) which is isotopic to Δ_0. In this way, we obtain a deformation of the edge-lengths which makes a cone singularity at p. The edge-lengths smoothly depend on t. By a formula of hyperbolic geometry (cf. [Be]), the cone angle γ given by the parameter t satisfies the following:

$$\cos\frac{\gamma}{2} = (\cosh\delta)(\sinh t)^2 - (\cosh t)^2.$$

Then

$$\left.\frac{d\gamma}{dt}\right|_{t=0} = -2\sqrt{2(\cosh\delta - 1)} < 0.$$

Therefore

$$\left.\frac{dh_{g,p}}{dt}\right|_{t=0} \neq 0.$$

Now consider the case where p does not lie on any simple closed curve on (Σ_g, σ_0). Then, (Σ_g, σ_0) can be cut into pairs of pants so that p lies in the interior of some pants $\mathcal{P}$. Denote the boundary components of $\mathcal{P}$ by $\partial_1\mathcal{P}$, $\partial_2\mathcal{P}$ and $\partial_3\mathcal{P}$, and their lengths by d_1, d_2 and d_3, respectively. As described by Thurston [T § 3.9], $\mathcal{P}$ can be obtained by adequately gluing two ideal hyperbolic triangles and then adjoining limit points. If p lies on the boundary of the ideal triangle, slightly move the edge-lengths of Δ_0 so that p goes to the interior of one of the ideal triangles. Then cut this ideal triangle into three triangles K_1, K_2 and K_3 along three geodesics, each starting from p and going to one of the ideal vertices (Fig.3).

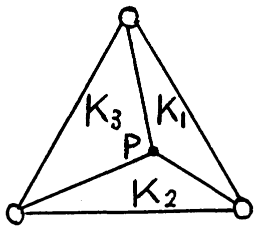

Fig.3

Let γ_1, γ_2 and γ_3 be the angles of K_1, K_2 and K_3 at p, respectively. For each i $(1 \leq i \leq 3)$ and arbitrary small $s > 0$, take a geodesic triangle $K_i(s)$ with angles 0, 0 and $\gamma_i - s$. Paste $K_1(s)$, $K_2(s)$ and $K_3(s)$ with each other to be an ideal triangle with a cone point of angle $2\pi - 3s$. Denote this triangle by $IT(s)$. Now we will make a pair of pants $\mathcal{P}(s)$ with a cone point of angle $2\pi - 3s$, as we obtained $\mathcal{P}$ by gluing the two ideal triangles and adjoining the limit points. We can, and will, glue $IT(s)$ and an ideal triangle IT with sliding the edges of $IT(s)$, exactly same as in the case of $\mathcal{P}$, so that $\mathcal{P}(s)$ has three boundary components $\partial_1\mathcal{P}(s)$, $\partial_2\mathcal{P}(s)$ and $\partial_3\mathcal{P}(s)$, with lengths d_1, d_2 and d_3, respectively. Then glue pairs of pants to be Σ_g with substituting $\mathcal{P}$ for $\mathcal{P}(s)$. In this substitution, we paste the boundaries of $\mathcal{P}(s)$ and $\Sigma_g \setminus \mathcal{P}$ as follows. For each j $(1 \leq j \leq 3)$, fix a point ξ_j in IT near the ideal vertex v_j, which spins around $\partial_j\mathcal{P}(s)$. (See Fig.4.) Let the limit point of the spiral horocycle in $\mathcal{P}(s)$ which passes through ξ_j go to the same place as the case of $\mathcal{P}$ (Fig.5). Then we obtain a hyperbolic 2-orbifold (Σ_g, σ_s) with cone angle $2\pi - 3s$.

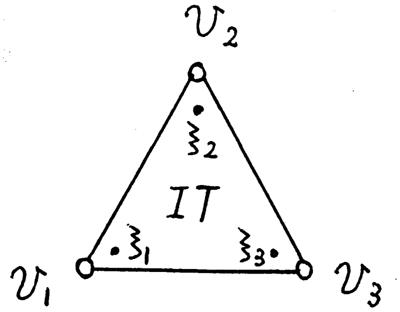

Fig.4

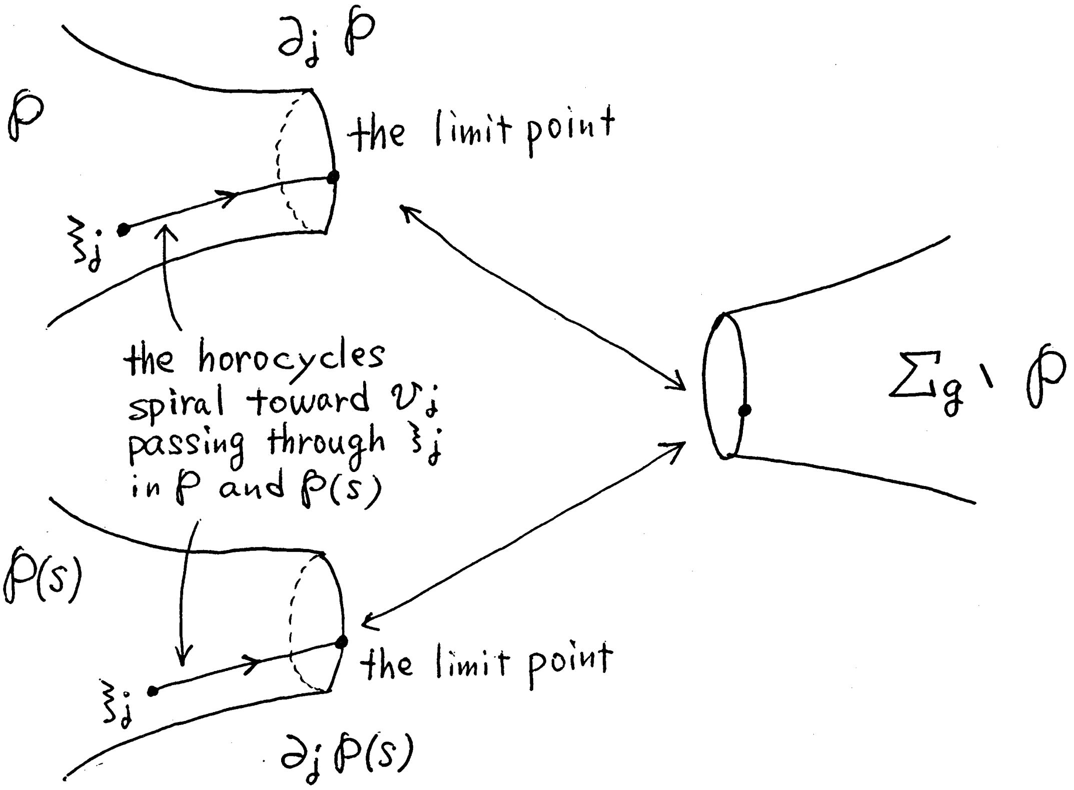

Fig.5

Now consider the vertices of Δ_0 included in $\mathcal{P}$. If some of them, without p, are on the boundaries of K_1 or K_2 or K_3, then slightly move the edge-lengths of Δ_0 so that they lie in the interiors of K_1 or K_2 or K_3, respectively. For each i, take K_i and $K_i(s)$ in $\mathbf{H}^2$ as in Fig.6, and regard the vertices in the interior of K_i as points in $K_i(s)$. Connect the cone point of (Σ_g, σ_s) with these points and the vertices of Δ_0, which lie outside of $\mathcal{P}(s)$, by geodesics, according to the combinatorial data of Δ_0 (Fig.7). Then for each s, we obtain a geodesic cellular decomposition on (Σ_g, σ_s) isotopic to Δ_0. By the construction above, the edge-lengths smoothly depend on s (for verification of this, see Fig.8. Consider a vertex x on $\mathcal{P}(s)$ and a vertex y on some other pairs of pants which are connected by an edge of the decomposition of (Σ_g, σ_s). The edge crosses some component ∂ of $\partial\mathcal{P}(s)$. Draw a geodesic which realizes a minimizing length from x to ∂. Also draw such a geodesic about y. Denote the legs of the geodesics by z and w. The length between z and w along ∂ moves smoothly with respect to s. Also the lengths of the two geodesics moves smoothly with respect to s). The derivative of $h_{g,p}$ is -3 with respect to s.

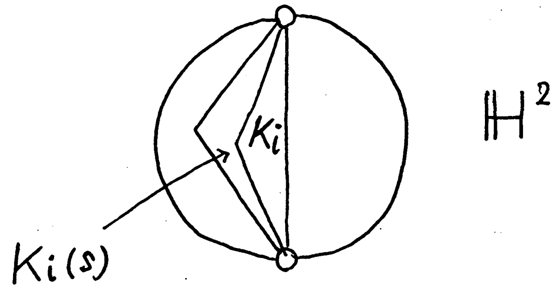

Fig.6

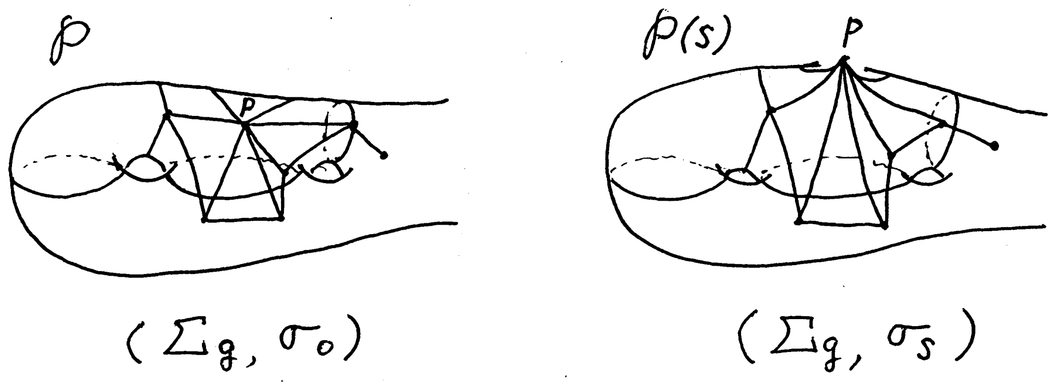

Fig.7

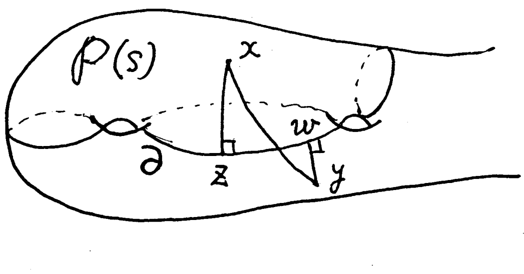

Fig.8

Remark. See Troyanov [Tr] for constructions of hyperbolic structures with cone singularities.

Proof of Proposition 2.2. We will construct a smooth section of ϕ_g around $(\Sigma_g, \phi_g(\mathbf{r}^0)) = (\Sigma_g, \sigma_0)$. Let us take fine meshes of a net made of closed geodesics on $(\Sigma_g, \phi_g(\mathbf{r}^0))$ so that each mesh contains at most one vertex of Δ_0. Let (Σ_g, σ) be a hyperbolic surface which is arbitrarily near $\phi_g(\mathbf{r}^0)$. Then the geodesics of the meshes on $(\Sigma_g, \phi_g(\mathbf{r}^0))$ moves smoothly with respect to changing metrics on Σ_g from $\phi_g(\mathbf{r}^0)$ to σ. For each mesh on (Σ_g, σ), which is obtained by changing some mesh in which a vertex of Δ_0 is contained, take one of its corners (Fig.9). Then we can, and will, connect all such corners by geodesics, according to the combinatorial data of Δ_0. Thus we obtain a geodesic cellular decomposition Δ_σ on (Σ_g, σ) which is combinatorially equivalent to Δ_0 and which moves smoothly with respect to the change of $(\Sigma_g, \phi_g(\mathbf{r}^0))$ to (Σ_g, σ). Denote this correspondence from $\mathcal{T}_g$ to V_g by η_g. This map η_g gives a section of ϕ_g on a neighborhood of $(\Sigma_g, \phi_g(\mathbf{r}^0))$.

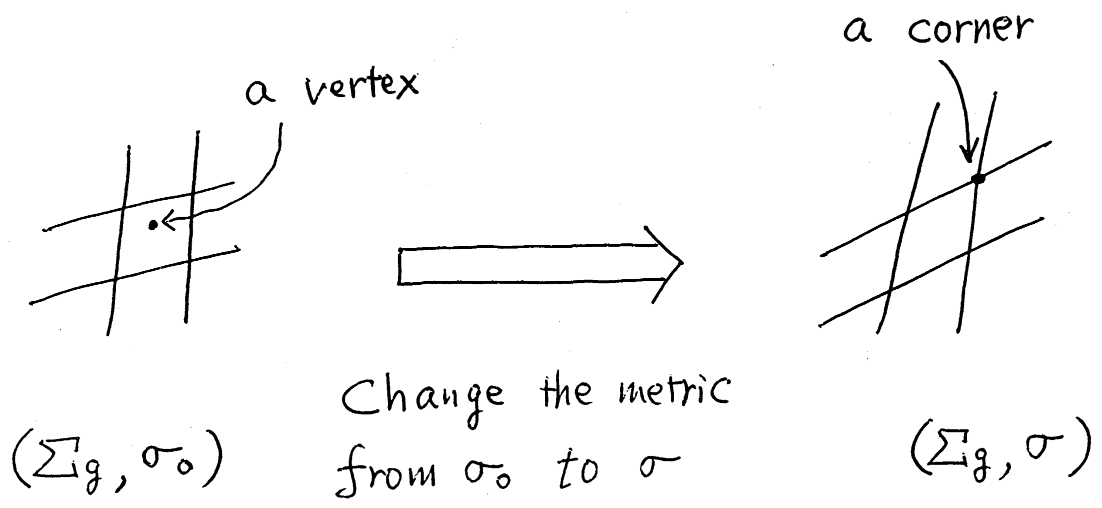

Fig.9

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