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Length parameters for Teichmüller space of punctured surfaces

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1. Introduction

Let F be the oriented closed surface of genus g and P a set of s points of F. The condition $2g-2+s>0$ is assumed throughout this paper. The *Teichmüller space* $\mathbf{T}_{g,s}$ is the set of marked surfaces with complete hyperbolic metric of finite area whose underlying topological surface is $F\setminus P$. If an injective map $f:\mathbf{T}_{g,s}\rightarrow\mathbf{R}^{d}$ for some $d\geq 0$ is given, then f gives a global parametrization for the space $\mathbf{T}_{g,s}$. Among several global parametrizations, the one by the geodesic length functions is well known ([4],[7],[8],[9],[10],[11]).

In case $P=\{x_1,...,x_s\}$ is non-empty, there are other parametrizations originally introduced by R. C. Penner for the decorated Teichmüller space ([6]); If an ideal triangulation of the punctured surface $F\setminus P$ is given, then the $h-$length coordinates and $L-$length coordinates associated with it give global parametrizations for the Teichmüller space $\mathbf{T}_{g,s}$ (for the terminology, see Section 2. We remark that the $L-$length differs from Penner's λ-length by a constant factor.) The advantage of the parametrization by $L-$length coordinates (or the $h-$length coordinates) is that it allows $\mathbf{T}_{g,s}$ a real-algebraic representation determined by comparatively simple equations. The representation by the $L-$lengths is found in [3]. In terms of the $h-$lengths, the representation of $\mathbf{T}_{g,s}$ is described by s equations and $6g-6+3s$ so-called coupling equations whose geometric meanings are almost trivial. In Section 2 we construct $L-$ and $h-$length coordinates associated with a special ideal triangulation and give the representations of $\mathbf{T}_{g,s}$.

In Section 3 we establish a relation between the $L-$lengths of ideal arcs and the lengths of closed geodesics on a punctured hyperbolic surface and obtain an explicit real-algebraic representation of $\mathbf{T}_{g,s}$ by geodesic length functions.

In Section 4 we present a changing rule from the $h-$length coordinates defined in Section 2 to the Fricke coordinates, that is, entries of the marked canonical generators (each is a 2×2 matrix) of the Fuchsian group corresponding to the point of $\mathbf{T}_{g,s}$. This supplies another proof of the fact that the $h-$ and $L-$length coordinates give global parametrizations for the

Teichmüller space $\mathbf{T}_{g,s}$.

2. Coordinates for the Teichmüller space associated with an ideal triangulation of a punctured surface

2.1. In this paper we employ the upper half plane model $\mathbf{H}$ with the metric $|dz|/(\mathrm{Im} z)$ as the hyperbolic plane. Let c be a complete hyperbolic line. Then c has two endpoints v_a, v_b in the boundary $\partial\mathbf{H}$ of $\mathbf{H}$ viewed as a subregion of the Riemann sphere. Choose horocycles C_a and C_b based at v_a and v_b, respectively. Let l denote the signed distance between C_a and C_b along c, taken with positive sign if $C_a \cap C_b = \emptyset$ and with negative sign if $C_a \cap C_b \neq \emptyset$. We call $e^{l/2}$ the *L-length* of c between C_a and C_b and denote it by $L(c; C_a, C_b)$.

Let T be a hyperbolic surface bounded by three complete lines. We say that T is an *ideal triangle* if T has a finite area which necessarily equals π. An ideal triangle has three ends. If an ideal triangle T is embedded in $\mathbf{H}$, then the ends determine three vertices in $\partial\mathbf{H}$. We adopt the notation in Figure 2.1 (a). Suppose that a horocycle C_α based at v_a is given. We call the hyperbolic length of the part of C_α between the edges b and c the *h-length* of the end α with respect to the horocycle C_α and denote it by $h(\alpha, C_\alpha)$.

For the ideal triangle T equipped with horocycles as in Figure 2.1 (a), the $L-$lengths of the edges and $h-$lengths of the ends associated with the horocycles are related as in the following formulae:

(2.2)
$$h(\alpha, C_\alpha) = \frac{L(a; C_\beta, C_\gamma)}{L(b; C_\gamma, C_\alpha)L(c; C_\alpha, C_\beta)}, \quad L(a; C_\beta, C_\gamma) = \frac{1}{\sqrt{h(\beta, C_\beta)h(\gamma, C_\gamma)}}$$

2.3. Coupling equations. Consider a hyperbolic quadrilateral which is cut into two ideal triangles by a diagonal. We adopt the notation of Figure 2.1 (b). Then the $h-$lengths of the ends which abut on the diagonal e satisfy the following *coupling equation*:

$$h(\alpha, C_\alpha)h(\beta, C_\beta) = h(\gamma, C_\beta)h(\delta, C_\alpha). \tag{2.4}$$

This equation follows easily from (2.2) if the expression of the $L-$length of e in terms of the $h-$lengths is considered in each of the two triangles.

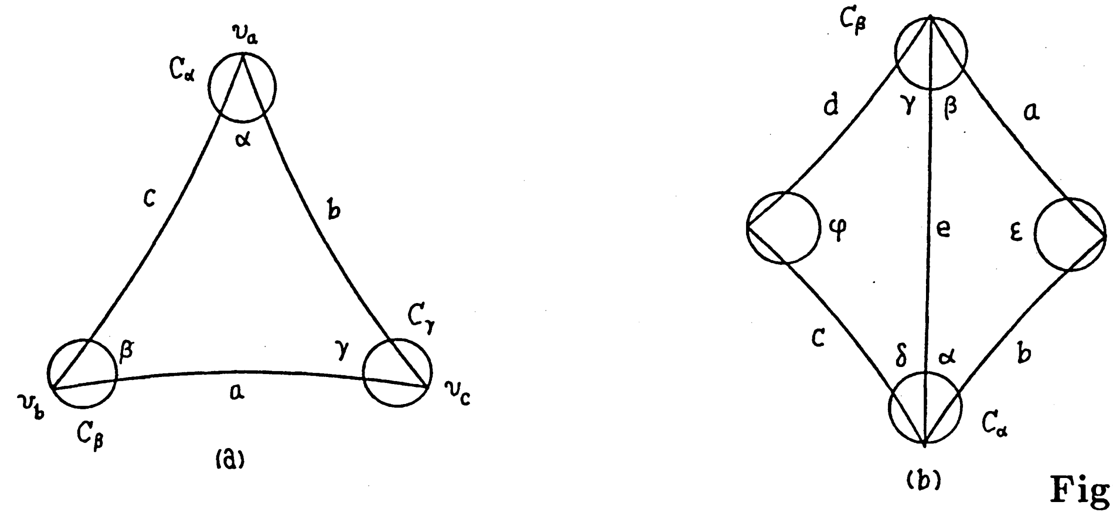

Figure 2.1

2.5. Let R be a hyperbolic surface with finite area whose underlying topological surface is $F \setminus P$. Then there is a Fuchsian group Γ acting on the upper half plane $\mathbf{H}$ such that $\mathbf{H}/\Gamma = R$. Every puncture of R defines a conjugacy class of parabolic cyclic subgroups of Γ. Let H be a parabolic cyclic group in this class and h a generator of H. Let C be a horocycle based at the fixed point of H. We say that C has length α with respect to Γ (or the hyperbolic surface R) if the length of the segment on C between z and $h(z)$ is α, where z is any point of C.

An *ideal geodesic arc* c on R is a geodesic arc connecting punctures. It is possible that c ends in the same puncture. The *$L-$length* $L_\alpha(c)$ of c with respect to horocycles of length α is defined to be $L(\tilde{c}; C_a, C_b)$, where $\tilde{c}$ is a lift of c to $\mathbf{H}$ and C_a, C_b are the horocycles of length α based at the endpoints of $\tilde{c}$.

2.6. This section refers to Figure 2.2. Let $\tilde{F}$ denote the surface $F \setminus \{x_2, ..., x_s\}$. Choose simple closed curves $a_1, b_1, ..., a_g, b_g, c_1, ..., c_{s-1}$ on $\tilde{F}$ which cut $\tilde{F}$ into $(4g+2s-2)$-gon D' and $s-1$ punctured discs $D_1, ..., D_{s-1}$ where D_i is bounded by c_i $(i = 1, ..., s-1)$. We add arcs $d_1, ..., d_{s-1}$ such that d_j connects x_1 and x_{j+1} in D_j. Let $v_0, v_1, ..., v_{p-1}$, where $p = 4g+2s-2$, denote the vertices of D'. We add also $p-3$ disjoint curves $e_1, ..., e_{p-3}$ which connect the vertices of D

as illustrated in Figure 2.2. Then the system of arcs

$$a_1, b_1, ..., a_g, b_g, c_1, ..., c_{s-1}, d_1, ..., d_{s-1}, e_1, ..., e_{p-3}, \quad p = 4g + 2s - 2$$

forms an ideal triangulation of $F \setminus P$ which we denote by Δ. Let D denote the union of D' and $D_1, ..., D_{s-1}$.

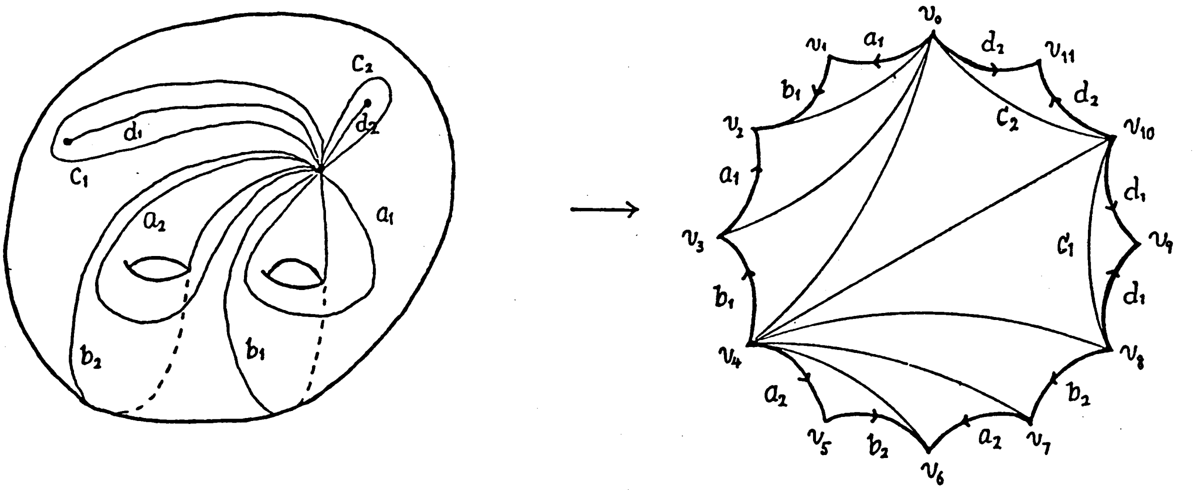

Figure 2.2

2.7. L–length coordinates for the Teichmüller space. Let R_m be a point of the Teichmüller space $\mathbf{T}_{g,s}$. By definition R_m is represented by a hyperbolic surface R together with an orientation-preserving homeomorphism $f : F \setminus P \to R$ ([1, Chap.6]). We send the curves in Δ to R by f and replace the images with geodesic curves homotopic to them relative to the punctures. If $c \in \Delta$ and $\tilde{c}$ is the geodesic curve on R homotopic to $f(c)$ relative to the punctures, then we denote by $L_\alpha(c, R_m)$ the L–length of $\tilde{c}$ relative to horocycles of length α.

2.8. Theorem. *There is a mapping* $f : \mathbf{T}_{g,s} \to \mathbf{R}_+^{6g-6+3s}$ *defined by*

$$f(R_m) = (L_\alpha(c, R_m) |\ c \in \Delta) \tag{2.9}$$

which gives a global parametrization for the Teichmüller space $\mathbf{T}_{g,s}$.

A proof of this theorem is found in [3]. In Section 4, we shall give another proof and for this purpose we need h–length coordinates defined in the next section.

2.10. $h-$length coordinates for the Teichmüller space. We consider triangles in the ideal triangulation Δ of $F \setminus P$ constructed in 2.6. Note that a triangle may be bounded by two curves in Δ. Examples are the triangles bounded by c_j and d_j for $j = 1, ..., s-1$. Such a triangle lifts to an ordinary triangle in the universal covering surface of $F \setminus P$. If we think of $P = \{x_1, ..., x_s\}$ as the ideal boundary of $F \setminus P$, then each triangle in Δ has three ends. Since there are $4g + 2s - 4$ triangles in Δ, there are $12g + 6s - 12$ ends. Let E_i denote the set of ends of triangles in Δ which abut on x_i. Then E_1 contains $12g + 5s - 11$ ends and if $i \neq 1$, E_i contains only one end. Let E denote the set of all ends.

Let R_m be a point of the Teichmüller space $\mathbf{T}_{g,s}$ represented by (R, f). Let $\mathbf{H}$ be the universal covering of R equipped with horocycles of length α with respect to R. Send the curves in Δ to R by f and straighten the images to geodesic arcs by a homotopy relative to the punctures. Then an end $\epsilon \in E$ corresponds to an end $\tilde{\epsilon}$ of a geodesic triangle in R. Lift the triangle to $\mathbf{H}$ and consider the horocycle C of length α based at the vertex v naturally determined by ϵ. Let $h_\alpha(\epsilon, R_m) = h(\tilde{\epsilon}, C)$. We call $h_\alpha(\epsilon, R_m)$ the *h-length* of the end ϵ in R_m with respect to α.

In Section 4 we show that the mapping $g : \mathbf{T}_{g,s} \to \mathbf{R}_+^{12g+6s-12}$ defined by

$$g(R_m) = (h_\alpha(\epsilon, R_m) |\ \epsilon \in E)$$

gives a global parametrization for the Teichmüller space. The set $g(\mathbf{T}_{g,s})$ is determined by $6g - 6 + 3s$ coupling equations, corresponding to the arcs of Δ, and by the following trivial equations:

$$\sum_{\epsilon \in E_i} h_\alpha(\epsilon) = \alpha, \quad i = 1, ..., s. \tag{2.11}$$

Any point of $\mathbf{R}_+^{12g+6s-12}$ satisfying these $6g - 6 + 4s$ equations belongs to $g(\mathbf{T}_{g,s})$. Actually a hyperbolic surface can be constructed from $4g + 3s - 4$ ideal triangles so that the triangulation is combinatorially same as Δ and so that the triangles are equipped with horocycles which assign the same h-length coordinates as the given point.

For $i = 2, ..., s$, E_i contains only one end and the $h-$length of the end is the constant α. Therefore we can eliminate the $h-$lengths of the ends in E_i, $i > 1$ and replace g with the mapping $g' : \mathbf{T}_{g,s} \to \mathbf{R}_+^{12g+5s-11}$ defined by

$$g'(R_m) = (h_\alpha(\epsilon, R_m) |\ \epsilon \in E_1) \tag{2.12}$$

to obtain a global parametrization.

2.13. The defining relation of the Teichmüller space in terms of the L−length coordinates. Let f be the parametrization (2.8) for $\mathbf{T}_{g,s}$ in L−length coordinates. By using (2.2) and (2.11), we can determine the space $f(\mathbf{T}_{g,s})$ explicitly. Let $\epsilon \in E$ be an end and T be the triangle in Δ which contains ϵ. If $c_{1,\epsilon}, c_{2,\epsilon}, c_{3,\epsilon}$ are the edges of T and $c_{3,\epsilon}$ is opposite ϵ, then the equation (2.11) is equivalent to

$$(\mathrm{R}_i) \qquad \sum_{\epsilon \in E_i} \frac{L_\alpha(c_{3,\epsilon})}{L_\alpha(c_{1,\epsilon}) L_\alpha(c_{2,\epsilon})} = \alpha, \quad i = 1, \dots, s. \tag{2.14}$$

For $i = 2, \dots, s$, E_i contains only one end and in this case we have

$$L_\alpha(c_{i-1}, R_m) = \alpha L_\alpha(d_{i-1}, R_m)^2.$$

So we can eliminate the coordinates $L_\alpha(d_{i-1})$, $i = 2, \dots, s$ and replace f by the mapping $f' : \mathbf{T}_{g,s} \to \mathbf{R}_+^{6g-6+2s+1}$ defined by

$$f'(R_m) = (L_\alpha(c, R_m) |\ c \in \Delta \setminus \{d_1, \dots, d_{s-1}\}).$$

The set $f'(\mathbf{T}_{g,s})$ is determined by the single equation (R_1) in (2.14).

3. A real analytic representation of the Teichmüller space by geodesic length functions

Let A denote the annulus $\{1/2 \leq |z| \leq 2\}$ and c' and c'' the boundary curves of A. Let $A^* = A \setminus \{1\}$ and c denote the arc $\{e^{2\pi i\theta} | 0 < \theta < 1\}$. The following lemma is a consequence of elementary hyperbolic geometry.

3.1. Lemma. *Let* f *be an embedding of* A^* *into a surface* R *with complete hyperbolic metric such that* $1 \in A$ *corresponds to a puncture of* R *under* f. *If* $L_\alpha(c)$ *denotes the* L*−length with respect to the horocycle of length* α *of the geodesic arc homotopic to* $f(c)$ *relative to the boundary and* $l(c')$ *(resp.* $l(c'')$*) the infimum of hyperbolic lengths of curves in the free homotopy class of* $f(c')$ *(resp.* $f(c'')$*), then*

$$\alpha L_\alpha(c) = 2\cosh(l(c')/2) + 2\cosh(l(c'')/2). \tag{3.2}$$

3.3. Geodesic length parameters. Let Δ be the ideal triangulation of $F \setminus P$ defined in 2.6. Each arc $c \in \Delta \setminus \{d_1, ..., d_{s-1}\}$ extends to a closed curve in $\tilde{F} = f \setminus \{x_2, ..., x_s\}$. There are at most two simple closed curves c' and c'' in $F \setminus P$ up to free homotopy which are homotopic to the extension of c in $\tilde{F}$. More precisely, choose a small disc D in $\tilde{F}$ around x_1. By deforming c with a homotopy, we can assume that c intersects the boundary circle of D in two points and cuts the boundary circle into two arcs. Then remove from c the part in D and add one of the two arcs. By doing this we obtain the simple closed curves c' and c'' on $F \setminus P$ with the desired property.

Let R_m be a point of the Teichmüller space $\mathbf{T}_{g,s}$. If R_m is represented by the hyperbolic surface R and the orientation-preserving homeomorphism $f : F \setminus P \to R$, let $l(c', R_m)$ and $l(c'', R_m)$ denote the infimum of the hyperbolic lengths of curves freely homotopic to $f(c')$ and to $f(c'')$, respectively. So $l(c', R_m)$ (resp. $l(c'', R_m)$) is either the length of the unique geodesic curve freely homotopic to $f(c')$ (resp. $f(c'')$) or zero. By applying Lemma 3.1 to the punctured annulus bounded by $f(c')$ and $f(c'')$, we obtain

$$\alpha L_\alpha(c) = 2\cosh(l(c')/2) + 2\cosh(l(c'')/2).$$

Note that $2\cosh(l(c')/2)$ and $2\cosh(l(c'')/2)$ are absolute values of the traces of the hyperbolic transformations corresponding to c' and c'' in the Fuchsian group Γ such that $R = \mathbf{H}/\Gamma$. Combining the formula above with the results in 2.13 we obtain a real algebraic representation for $\mathbf{T}_{g,s}$ in terms of the geodesic length functions:

3.4. Theorem. *The mapping* $h : \mathbf{T}_{g,s} \to \mathbf{R}^{6g-6+2s+1}$ *defined by*

$$h(R_m) = (\cosh(l(c')/2) + \cosh(l(c'')/2) \mid c \in \Delta \setminus \{d_1, ..., d_{s-1}\})$$

gives a global parametrization of the Teichmüller space $\mathbf{T}_{g,s}$. *Let* $\lambda(c) = 2\cosh(l(c')/2) + 2\cosh(l(c'')/2)$. *Then the image* $h(\mathbf{T}_{g,s})$ *is determined by the equation*

$$\sum_{\epsilon \in E_1} \frac{\lambda(c_{3,\epsilon})}{\lambda(c_{1,\epsilon})\lambda(c_{2,\epsilon})} = 1. \tag{3.5}$$

3.6. For the once punctured torus, the curves c' and c'' constructed above are identical. Therefore $\lambda(c)/2$ is the absolute value of the trace of the hyperbolic transformations corresponding to c under the universal covering $\mathbf{H} \to R$. The triangulation Δ contains three curves a, b, e. The Teichmüller space $\mathbf{T}_{1,1}$ is therefore parametrized by the trace functions $\lambda(a), \lambda(b), \lambda(e)$ with the relation

$$\lambda(a)^2 + \lambda(b)^2 + \lambda(e)^2 = \lambda(a)\lambda(b)\lambda(e).$$

This is a classical result.

4. Relations between the Fricke coordinates and the $L-$ and $h-$length coordinates

4.1. The Fricke coordinates. We consider again the triangulation Δ constructed in Section 2.6. Let R_m be a point of the Teichmüller space $\mathbf{T}_{g,s}$. If R_m is represented by the hyperbolic surface R and the orientation-preserving homeomorphism $f : F \setminus P \to R$, send all arcs in Δ into R by f and deform the images to geodesic arcs under a homotopy relative to the boundary. We cut R along the geodesic arcs corresponding to $a_1, b_1, ..., a_g, b_g, d_1, ..., d_{s-1}$. Then we obtain a geodesic $4g+2s-2$-gon D which is triangulated by the images of $c_1, ..., c_{s-1}, e_1, ..., e_{p-3}$ as in Figure 2.2. If we embed D in the hyperbolic plane $\mathbf{H}$, we obtain also the side-pairing transformations which generate a Fuchsian group Γ such that $R = \mathbf{H}/\Gamma$. Let

$$(A_1, B_1, ..., A_g, B_g, D_1, ..., D_{s-1}) \tag{4.2}$$

be the ordered set of the side-pairing transformations which are matrices in $\mathbf{SL}(2, \mathbf{R})$. To determine this ordered set uniquely for each $R_m \in \mathbf{T}_{g,s}$, we assume that

$$(4.3) \qquad trM < 0 \quad \text{for} \quad M \in \{A_1, ..., D_{s-1}\},$$

where trM is the trace of a matrix M, and that $A_1^{-1}B_1A_1(\infty) = 0$ and also that

$$D_s = A_1^{-1}B_1^{-1}A_1B_1 \cdots A_g^{-1}B_g^{-1}A_gB_gD_1^{-1} \cdots D_{s-1}^{-1}$$

is expressed by the matrix

$$\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}.$$

Here we remark that $trD_s < 0$ is due to the choice of matrices of negative traces (4.2) for $A_1, B_1, ..., D_{s-1}$, see [5]. Then entries of matrices determine a point in $\mathbf{R}^{8g+4s-4}$ which gives the Fricke coordinates for R_m. Obviously the Fricke coordinate-system gives a global parametrization for the Teichmüller space $\mathbf{T}_{g,s}$.

4.4. Before establishing the relation between the Fricke coordinates and the h–length coordinates, we introduce the notion of an elementary move. Let D be an ideal geodesic polygon embedded in $\mathbf{H}$ triangulated by ideal geodesic arcs (for our purpose, we need only the polygon D constructed in 4.1, but here we assume that D is an arbitrary polygon). Suppose that for each vertex of D a horocycle is given. Then each end of the triangles in the triangulation has an h–length and each edge has an L–length with respect to these horocycles. Choose an inner edge e of the triangulation. Let S and T be the triangles which share the edge e. Then, by replacing e with another diagonal f of the quadrilateral $S \cup T$, we obtain another triangulation of D, which is said to be the result of an *elementary move* on e. The next lemma refers to Figure 4.1.

4.5. Lemma ([6,p.334]). *L–lengths of edges and h–lengths of ends caused by the elementary move satisfy:* $L_eL_f = L_aL_c + L_bL_d$,

$$\epsilon' = \beta + \gamma, \quad \varphi' = \alpha + \delta,$$

$$\alpha' = \frac{\varphi}{\varphi'}\alpha = \frac{\varphi}{\epsilon'}\gamma, \quad \beta' = \frac{\varphi}{\epsilon'}\beta = \frac{\varphi}{\varphi'}\delta,$$

$$\gamma' = \frac{\epsilon}{\epsilon'}\gamma = \frac{\epsilon}{\varphi'}\alpha, \quad \delta' = \frac{\epsilon}{\varphi'}\delta = \frac{\epsilon}{\epsilon'}\beta.$$

Here and in what follows we make α etc., stand for the h–length of an end α (if relevant horocycles are known) in order to simplify the notation.

For given positive numbers α, β, γ, we define a matrix

$$M(\alpha,\beta,\gamma) = -\sqrt{\frac{\gamma}{\beta}}\begin{pmatrix} (\alpha+\beta)/\gamma & \alpha \\ 1/\gamma & 1 \end{pmatrix}. \tag{4.6}$$

Note that if $(a,b|c,d) = M(\alpha,\beta,\gamma)$, then

$$\alpha = b/d,\ \beta = 1/cd,\ \gamma = d/c. \tag{4.7}$$

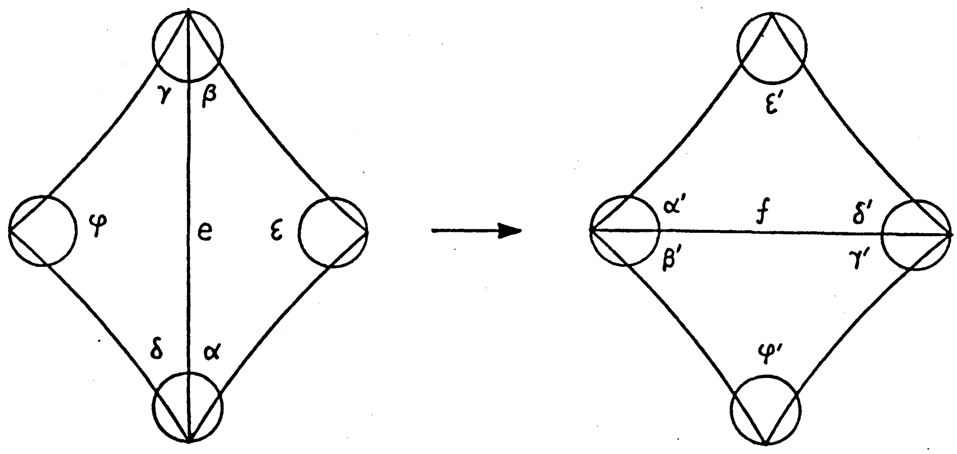

Figure 4.1

The next lemma refers to Figure 4.2 which also indicates two elementary moves starting on Δ.

4.8. Lemma. *Suppose that $L_b = L_c$ and $L_a = L_d$ hold for the L–lengths. Then the linear fractional transformation A which sends the horocycles C_∞, C_0 to $C_{\alpha+\beta}, C_\alpha$, respectively, is $M(\alpha,\beta,\gamma)$ and the linear fractional transformation B which sends the horocycles $C_{\alpha+\beta+\delta}, C_{\alpha+\beta}$ to C_0, C_α, respectively, is*

$$B = R^{-1}M(\beta',\delta',\epsilon'')^{-1}R,$$

where R *is the linear fractional transformation such that* $R(0) = \infty$, $R(\alpha) = 0$, $R(\alpha + \beta) = \beta'$, *and* ϵ'' *is the* h*-length of the end marked by the same letter in Figure 4.2.*

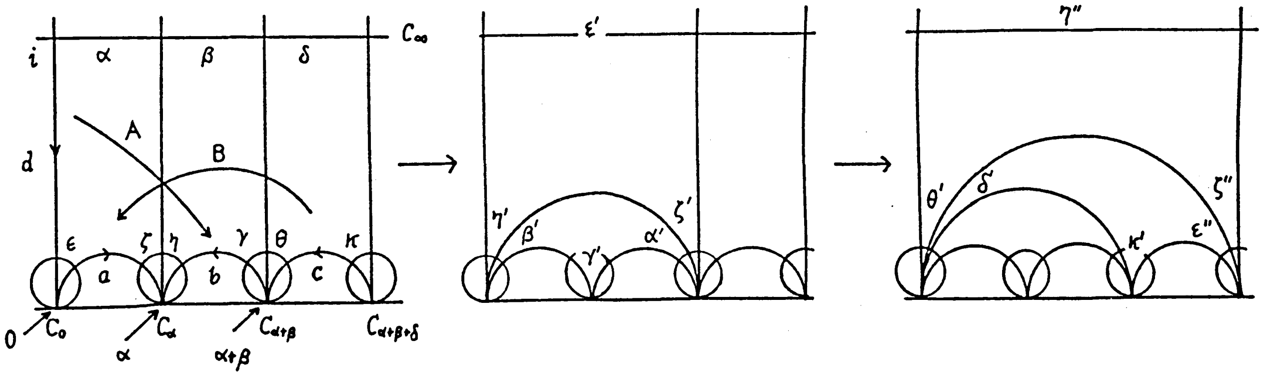

Figure 4.2

4.9. Let $S = (\alpha, \beta, \gamma, \delta, \epsilon, \zeta, \eta, \theta, \kappa)$ be the ordered set of h–lengths of ends as in Figure 4.2. Then we denote by $A(S), B(S)$ the linear fractional transformations A and B in the previous lemma.

4.10. We shall establish relations between the Fricke coordinates and the L–length and h–length coordinates. Since a one-to-one correspondence between the L–length coordinates and the h–length coordinates is easily obtained by using (2.2), we need only to consider the h–length coordinates. In what follows *we consider the case of* $g > 0$ *and* $s > 1$ *and the* h*–lengths of ends are those with respect to horocycles of length* 1. Other cases can be treated in a similar manner.

Let D be the geodesic polygon constructed in 4.1. This D is triangulated as illustrated in Figure 4.3. Suppose that the h–length coordinates are given. We shall produce the Fricke coordinates from the h–lengths. For $i = 1, \ldots, g$, let

$$S_i = (\alpha_i, \beta_i, \gamma_i, \delta_i, \epsilon_i, \zeta_i, \eta_i, \theta_i, \kappa_i).$$

Then by Lemma 4.8 we have $A_1 = A(S_1), B_1 = B(S_1)$. For $i = 2, \ldots, g$, consider the polygon with vertices $v_0 (= \infty)$, v_{4i-4}, v_{4i-3}, v_{4i-2}, v_{4i-1}, v_{4i}. By operating elementary moves three times, we can obtain a new triangulation by vertical edges which connect v_0 and other vertices $v_{4i-4}, v_{4i-3}, v_{4i-2}, v_{4i-1}, v_{4i}$. Lemma 4.5 implies that $v_{4i-4}, v_{4i-3}, v_{4i-2}$ can be expressed in terms of the $h-$lengths in S_i and λ_i, μ_i, ν_i. Let R_i be the linear fractional transformation such that $R_i(v_{4i-4}) = \infty, R_i(v_{4i-3}) = 0, R_i(v_{4i-2}) = \alpha_i$. Then we have

$$A_i = R_i^{-1} A(S_i) R_i, \qquad B_i = R_i^{-1} B(S_i) R_i.$$

Next consider the polygon with vertices $v_0, v_{4g+2i-2}, v_{4g+2i-1}, v_{4g+2i}$, for $i = 1, .., s-2$. Operating an elementary move we obtain a triangulation of this polygon by the vertical edges connecting v_0 and vertices $v_{4g+2i-2}, v_{4g+2i-1}, v_{4g+2i} (= v_{4g+2i-2} + \psi_i)$. Then by Lemma 4.5, we can express $v_{4g+2i-1}$ by $\sigma_i, \tau_i, \varphi_i, \psi_i$. Now the transformation D_i is determined, because D_i fixes $v_{4g+2i-1}$ and sends $v_{4g+2i-2}$ to v_{4g+2i}. Finally D_{s-1} is determined by the fact that D_{s-1} fixes $v_{4g+2s-3}$ and sends $v_{4g+2s-4}$ to ∞. Thus the Fricke coordinates are determined by the $h-$length coordinates and hence we conclude:

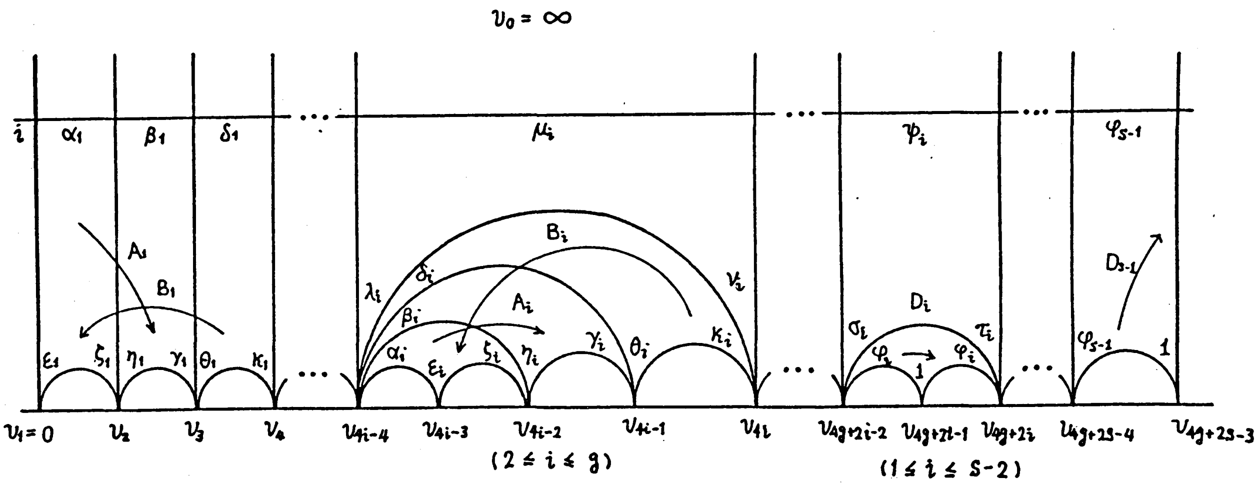

Figure 4.3

4.11. Theorem. *The $h-$length coordinates defined in* 2.10 *give a global parametrization for the Teichmüller space* $\mathbf{T}_{g,s}$.

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