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On intersections of representing curves of elements of the fundamental group of a surface

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Let M be a closed, orientable surface with genus greater than one.

Problem Decide whether a given set S of elements of $\pi_1(M)$ is representable by curves which intersect only at the basepoint (or, by an embedded bouquet $S^1 \vee \cdots \vee S^1$ where each S^1 represents each element).

Here I'll give a combinatorial algorithm for this problem.

We represent each element by a sequence of 1-cells, which are derived from some tesselation of M. For such a representation, we require "quasi- transversality", which guarantees each representation to be unique up to some special variations. When we have two quasi-transverse curves representing elements of S and intersecting, we trace them for some finite length and decide whether their intersections are removable. Thus we decide whether S is representable by an embedded bouquet.

First we take a train track τ and its barycentric subdivision as the tesselation of M. Here we use a complete shiftless train track. A train track is a 1-dimensional complex on M with smoothness at each vertex. Edges emanating from a vertex are bundled into two classes, which joint "smoothly" at the vertex. A shiftless train track has two or more edges on both sides of each vertex. τ is complete if each component of $M - \tau$ is a triangle in the sense of train track. We first take the barycentric subdivision of τ, then every element of $\pi_1(M)$ is represented by a sequence of 1-cells of it. Hereafter an edge of τ is called a capillary, a component of $M - \tau$ a face, an edge of a face (triangle) is called an edge.

A curve which is a sequence of 1-cells is "quasi-transverse (to τ)" when (1) it consists of following parts : (i) an arc inside a face made of two 1-cells, connecting an edge of the face to another edge, (ii) an arc as such connecting an angle to the opposite edge, (iii) whole of a capillary, (iv) any 1-cell emanating from an endpoint which is a face barycenter, or (v) a half of a capillary emanating from an endpoint which is a capillary barycenter and (2) these parts joint "smoothly". In case of a closed curve, if in addition its basepoint satisfies the condition, it is called "quasi-transverse as a closed curve".

A subarc of the type (i) is a sector around a vertex.

Lemma. *For a given (closed) curve, we can construct a quasi- transverse (as a closed curve) curve (freely) homotopic to it.*

There are three variations of arcs connecting the barycenters of the upper face and the lower face of a vertex. One goes from the upper face, penetrates the vertex and goes to the lower face. Another one goes down the left side of the vertex. It goes from the upper face, goes down the consecutive sectors on the left side of the vertex and goes to the lower face. The other one goes down the right side of the vertex. They make a bigon and the middle line of it (the first arc is the middle line).

If a quasi-transverse curve contains such an arc, we can replace it by another arc above. The resulting curve is homotopic to the original one and is quasi- transverse. Such a deformation is called a bigon deformation. A quasi- transverse curve may allow consecutive bigon deformations, which we call a composed bigon deformation. Then,

Lemma. *For a given 1-cell of a given (finite or closed) quasi- transverse curve, we can decide whether it takes part in some composed bigon deformation.*

Lemma. *If two quasi-transverse curves bound a disk, they are deformed to each other by composed bigon deformations.*

Consequently, if two quasi-transverse curves pass the same barycenter and part from each other, we can decide whether they meet again bounding a disk.

We fix a hyperbolic metric on M. Then its universal covering is the Poincaré disk. For a closed curve on M, a "lift component" of it means an infinite line on the Poincaré dick made of consecutive lifts of it.

Now consider a curve quasi-transverse as a closed curve and its lift components. When a pair of lift components pass a common barycenter, we trace them from it. In tracing, we operate bigon deformations on them so that they go along the same1-cells as long as possible. We trace them for one period of the closed curve, and if they don't part from each each other, then the closed curve represents a proper power of an element of $\pi_1(M)$ and hence is not homotopic to a simple closed curve. If they part, they never meet again. Such a parting is called a proper parting. The situation near the proper parting tells us how their limit points stand on S^1_∞.

Now we see the algorithm.

(1) We take an element from S and a quasi-transverse representation d of it as a closed curve, and see if this element is representable by a simple closed curve. It is known by checking if there is a pair of lift components with separating limit points. A pair of lift components is said to have separating limit points if the limit points stand on S^1_∞ in alternating way. It is known by tracing the lift components from a common barycenter. If we have a proper parting for each pair of halves, then we can decide the order of the limit points on S^1_∞. In case they don't part properly, this element is a proper power of an element of $\pi_1(M)$ and is not representable by a simple closed curve.

If d clears this check, we operate composed bigon deformations on it so that it becomes an actually simple curve D.

(2) Next we take another element and see if it is representable by a curve intersecting D only at the basepoint. We take a quasi-transverse representation a of it, trace a and D from a common barycenter and see if the limit points are separating. Here instead of the limit points of a lift component of a we see the limit points of the lift components of D on which the endpoints of a lift of a lie. If we reach an endpoint of it before a proper parting with D, the order of the limit points follows the situation near the endpoint.

If a clears this check, we operate composed bigon deformations on a so that it becomes actually disjoint from D except its endpoints.

We repeat this step for the remainder of S.

(3) Finally, we see if the intersections of the curves thus obtained are removable. It is known by checking the endpoints of the lifts of the curves. The lift components of D divides the Poincaré disk into connected components, each of which is a topological dick. We see if there is a pair of lifts with separating endpoints. For this,

we trace lifts as before, and see the order how the endpoints stand on the boundary of the topological disk. We trace each pair of lifts from a common barycenter, then reach a proper parting or an endpoint. In the latter case, we follow the situation near the endpoint as before. Thus we can decide whether there are separating endpoints.

If and only if S clears all these steps, it is representable by an embedded bouquet.