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A REMARK ON THEOREMS OF DE FRANCHIS AND SEVERI

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1. INTRODUCTION

The purpose of this paper is to study holomorphic maps between compact Riemann surfaces. There are two famous finiteness theorems related to this problem. Let $\widetilde{X}$ be a compact Riemann surface of genus > 1. Then one is that, for fixed compact Riemann surface X of genus> 1, the number of nonconstant holomorphic maps $\widetilde{X} \to X$ is finite, and another is that there are only finitely many compact Riemann surfaces $\{X_i\}$ of genus > 1 such that, for each X_i, there exists a nonconstant holomorphic map $\widetilde{X} \to X_i$. The first assertion is due to de Franchis, and second one is due to Severi.

Let $S(\widetilde{X}) = \{X_i\}$, where $\{X_i\}$ is as in Severi's theorem. Let

$$n = \sum_{X \in S(\widetilde{X})} \#\{h : \widetilde{X} \to X | \text{ nonconstant holomorphic}\}.$$

Then, by the theorems above, we see $n < \infty$ at once. Howard and Sommese [2] showed that there is a bound on n which depends only on the genus of $\widetilde{X}$, by giving an explicit estimate.

Here we will give some theorems related to rigidity of holomorphic maps between compact Riemann surfaces, and show that we may take an explicit bound on n depending only on the genus of $\widetilde{X}$ smaller than one in [2].

2. PRELIMINARIES

Let $\widetilde{X}, X$ be compact Riemann surfaces of genera $\tilde{g}, g(> 1)$. We denote by $H_1(X)$ the first homology group (with integer coefficients) of X. Any basis of $H_1(X)$ with intersection matrix

$$J = \begin{pmatrix} 0 & E \\ -E & 0 \end{pmatrix}$$

will be called a canonical homology basis, where E is the identity matrix of $g \times g$ sized. Similarly for $\widetilde{X}$. Let $\{\tilde{\chi}_1, \ldots, \tilde{\chi}_{2\tilde{g}}\}$ ($\{\chi_1, \ldots, \chi_{2g}\}$) be a canonical homology basis for $H_1(\widetilde{X})$ ($H_1(X)$ respectively). Let $\{\tilde{w}^1, \ldots, \tilde{w}^{\tilde{g}}\}$ and $\{w^1, \ldots, w^g\}$ be dual bases for holomorphic differentials on $\widetilde{X}, X$ (i.e. $\int_{\chi_j} w^k = \delta_{jk}$ where δ_{jk} is Kronecker's delta), and $\widetilde{\Pi} = (\widetilde{E}, \widetilde{Z}), \Pi = (E, Z)$ be the associated period matrices. Let $h : \widetilde{X} \to X$ be a nonconstant holomorphic map. Then h induces a homomorphism $h_* : H_1(\widetilde{X}) \to H_1(X)$. Let $M = (m_{kj}) \in M(2g, 2\tilde{g}; \mathbb{Z})$, where $h_*(\tilde{\chi}_j) = \sum_{k=1}^{2g} m_{kj}\chi_k$.(We denote by $M(m, n; K)$ the set of $m \times n$ matrices with K-coefficients.) We will call M the matrix representation of h with respect to $\{\tilde{\chi}_1, \ldots, \tilde{\chi}_{2\tilde{g}}\}$ and $\{\chi_1, \ldots, \chi_{2g}\}$. The integral $\int_{h_*(\tilde{\chi}_j)} w^i$ may be evaluated in two ways; by expressing $h_*(\tilde{\chi}_j)$ in $H_1(X)$ or by expressing the pull back of w^i in terms of the holomorphic differentials on $\widetilde{X}$. This leads us to the Hurwitz relation

$$A\Pi = \widetilde{\Pi}M,$$

where $A \in M(g, \tilde{g}; \mathbb{C})$. The set of $M \in M(2g, 2\tilde{g}; \mathbb{Q})$ such that there exists $A \in M(g, \tilde{g}; \mathbb{C})$ with $A\Pi = \widetilde{\Pi}M$ will be called the space of Hurwitz relations. It is easy to see that it is a $\mathbb{Q}$−vector space.

Lemma[4]. *In the space of Hurwitz relations,* $< M, N >= tr(\tilde{J}\,{}^tMJ^{-1}N)$ *defines an inner product* (${}^tM$ *denotes transposition of* M).

In particular, when M is a matrix representation of a holomorphic map $h : \widetilde{X} \to X$, $< M, M >= 2dg$, where d is the degree of the holomorphic map h.

The Jacobian variety of X is $J(X) = \mathbb{C}^g/\Gamma$, where Γ is the lattice (over $\mathbb{Z}$) generated by 2g-columns of Π. Similarly for $J(\widetilde{X})$. For any holomorphic map $h : \widetilde{X} \to X$, there exists a homomorphism $H : J(\widetilde{X}) \to J(X)$ with $\kappa \circ h = H \circ \tilde{\kappa}$, where $\tilde{\kappa}, \kappa$ are canonical injections.

By an underlying real structure for $J(X)$, we mean the real torus $\mathbb{R}^{2g}/\mathbb{Z}^{2g}$ together with a map $\mathbb{R}^{2g}/\mathbb{Z}^{2g} \to J(X)$ induced by a linear map $\mathbb{R}^{2g} \ni x \mapsto \Pi x \in \mathbb{C}^g$. It is well-known that for any homomorphism $H : J(\widetilde{X}) \to J(X)$, there are $A \in M(g, \tilde{g}; \mathbb{C})$ and $M \in M(2g, 2\tilde{g}; \mathbb{Z})$ such that the following diagram is commutative (precicely, apart from an additive constant).

$$\begin{array}{ccccc}
\mathbb{R}^{2\tilde{g}} & \xrightarrow{\tilde{\Pi}} & \mathbb{C}^{\tilde{g}} & \longrightarrow & J(\widetilde{X}) \\
\downarrow{\scriptstyle M} & & \downarrow{\scriptstyle A} & & \downarrow{\scriptstyle H} \\
\mathbb{R}^{2g} & \xrightarrow{\Pi} & \mathbb{C}^{g} & \longrightarrow & J(X)
\end{array}$$

In particular, if h is induced by a holomorphic map $h : \widetilde{X} \to X$, M is the matrix representation of h.

Giving a nonconstant holomorphic map $h : \widetilde{X} \to X$, we dnote by $h^*(Q) \subset \widetilde{X}$ a divisor of the preimages of $Q \in X$ with multiplicities. Defining $\tilde{\kappa}(h^*(Q))$ by linearity, we get a holomorphic map $X \to J(\widetilde{X})$, which can be extended to a homomorphism $H^* : J(X) \to J(\widetilde{X})$. H^* is called the Rosati adjoint of H. H^* is induced by the matrix $M^* = \tilde{J}\,{}^tMJ^{-1}$ acting on the underlying real tori[4].

3. Statements

Theorem 1. *Let* $\widetilde{X}, X$ *be compact Riemann surfaces of genera* $\tilde{g}, g(> 1)$. *Let* $h_i : \widetilde{X} \to X$ *be a nonconstant holomorphic map, and* $M_i \in M(2g, 2\tilde{g}; \mathbb{Z})$ *be a matrix representation of* $h_i (i = 1, 2)$. *Suppose that there is an integer* $l > \sqrt{8(\tilde{g} - 1)}$ *with* $M_1 \equiv M_2$ *(mod.* l*). Then* $h_1 = h_2$.

Let $\mathfrak{m}_i^j$ denote the j-th row vector of $M_i (i = 1, 2)$.

Theorem 2. *Let* h_1, h_2*,and* M_1, M_2 *be as in Theorem 1. Suppose that there is an integer* $l > 8(\tilde{g} - 1)$ *with* $\mathfrak{m}_1^j \equiv \mathfrak{m}_2^j$ *(mod.* l*) for every* $j \in \{1, \ldots, g\}$. *Then* $h_1 = h_2$.

It is already known that $M_1 = M_2$ implies $h_1 = h_2$ (see [3]).

Theorem 3. *Let X_1, X_2 be compact Riemann surfaces of genus $g > 1$. Let $h_i : \widetilde{X} \to X_i$ be a nonconstant holomorphic map, and M_i be a matrix representation of $h_i (i = 1, 2)$. Suppose that there is an integer $l > \sqrt{8}(\tilde{g} - 1)$ with $M_1 \equiv M_2$ (mod. l). Then X_1, X_2 are conformally equivalent and there exists a conformal map $f : X_1 \to X_2$ with $f \circ h_1 = h_2$.*

Only outlines of the proofs are given here. For complete proofs, see [5] which will be published elsewhere.

As we have seen in the lemma before, we have an inner product in the space of Hurwitz relations. Therefore, we may induce a distance in it. Using this distance, we have Theorem 1 and 2. To get Theorem 3, we use the Rosati adjoint. Let $G_i = M_i^* M_i = \widetilde{J}^t M_i J^{-1} M_i (i = 1, 2)$. Then we have endmorphisms of $J(\widetilde{X})$ with the matrices G_1, G_2 acting on the underlying real tori. If $G_1 = G_2$, then the targets X_1, X_2 are conformally equivalent. Using the distance induced by the inner product, we have Theorem 3.

Next we will give an bound on n which was defined in section 1. Let

$$S_g = \{X \in S(\widetilde{X}) | \text{genus}\, g\},$$

and

$$Hol_g(\widetilde{X}) = \bigcup_{X \in S_g} \{h : \widetilde{X} \to X | \text{nonconstant holomorphic}\}.$$

Let $F_l = \mathbb{Z}/(l)$, where l is a prime number $> \sqrt{8}(\tilde{g} - 1)$. By Theorem 1 and 3, we have an injection $Hol_g(\widetilde{X}) \to M(2g, 2\tilde{g}; F_l)$. Thus we consider each matrix representation in $M(2g, 2\tilde{g}; F_l)$, for the convenience of calculation. Let h_i be an element of $Hol_g(\widetilde{X})$ and $M_i \in M(2g, 2\tilde{g}; F_l)$ a matrix representation of $h_i (i = 1, 2)$. If there exists $S \in Sp(2g; F_l)$ with $M_2 = SM_1$, then targets of h_1, h_2, say X_1, X_2 ,are conformally equivalent and there is a conformal map $f : X_1 \to X_2$ with $f \circ h_1 = h_2$ (Sp denotes symplectic groups). M_i satisfies $M_i \tilde{J}\, {}^t M_i = d_i J$, where d_i is the degree of h_i. Therefore, we have

$$\#Hol_g(\widetilde{X}) \leq \sum_d \#\{M \in M(2g, 2\tilde{g}; F_l) | M \tilde{J}\, {}^t M = dJ\} \times 84(g - 1)/\#Sp(2g; F_l),$$

where d runs through all considerable numbers as degrees of holomorphic maps. We have

$$\#Sp(2g; F_l) = l^{g^2}(l^2 - 1)(l^4 - 1) \ldots (l^{2g} - 1)$$

(see [1]), and we may take l with $\sqrt{8}(\tilde{g} - 1) < l < 2\sqrt{8}(\tilde{g} - 1)$. Consequently,

$$n \leq 42(\tilde{g} - 1)(\tilde{g} - 2)2^{2\tilde{g}}(4\sqrt{2}(\tilde{g} - 1))^{\tilde{g}^2 + \tilde{g}/2} + 84(\tilde{g} - 1).$$

Howard and Sommese [2] showed that

$$n \leq (2\sqrt{6}(\tilde{g} - 1) + 1)^{2\tilde{g}^2 + 2}\tilde{g}^2(\tilde{g} - 1)(\sqrt{2})^{\tilde{g}(\tilde{g} - 1)} + 84(\tilde{g} - 1).$$

It is easy to see that our bound is smaller for every $\tilde{g} > 1$.

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