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Ramification of the Galois representation on the pro-l fundamental group of an algebraic curve*

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§0. Introduction.

Let S be a locally noetherian integral normal scheme of dimension 1, η the generic point of S, and $K=\kappa(\eta)$ the function field of S. Let s be a closed point of S, and put $p_s=\mathrm{char}(\kappa(s))$, the characteristic of $\kappa(s)$. For a proper smooth K-scheme X, we say that X has good reduction on S (resp. at s), if there exists a proper smooth S-scheme (resp. $\mathcal{O}_{S,s}$-scheme) $\mathfrak{X}$ whose generic fiber $\mathfrak{X}_\eta$ is isomorphic to X over K. Our main problem is: Are there any criteria for X to have good reduction?

Such a problem is known to be closely related to local monodromy. In fact, a necessary condition of good reduction comes from the proper smooth base change theorem for l-adic étale cohomology groups ([SGA4], Exp. XVI), which asserts that, if $\mathfrak{X}$ is a proper smooth scheme over $\mathcal{O}_{S,s}$, the cospecialization map

$$H^i_{\text{ét}}(\mathfrak{X}_{\overline{s}},\mathbb{Z}_l)\to H^i_{\text{ét}}(\mathfrak{X}_{\overline{\eta}},\mathbb{Z}_l)$$

is an isomorphism for each prime number $l\neq p_s$ and for each $i\geq 0$. In particular, if X has good reduction at s, then the inertia group at s in $\mathrm{Gal}(K^{\mathrm{sep}}/K)$ (determined up to conjugacy) acts trivially on $H^i_{\text{ét}}(X_{\overline{K}},\mathbb{Z}_l)$.

When X is an abelian variety, the converse also holds:

Theorem (Néron-Ogg-Shafarevich-Serre-Tate). *Let X be an abelian variety over K. Then X has good reduction at s if and only if the inertia group at s acts trivially on the l-adic Tate module $T_l(X_{\overline{K}})$ for some $l\neq p_s$.* □

Note

$$H^i_{\text{ét}}(X_{\overline{K}},\mathbb{Z}_l)\simeq \overset{i}{\wedge} H^1_{\text{ét}}(X_{\overline{K}},\mathbb{Z}_l)$$

for each $i\geq 0$, and

$$H^1_{\text{ét}}(X_{\overline{K}},\mathbb{Z}_l)\simeq \mathrm{Hom}(T_l(X_{\overline{K}}),\mathbb{Z}_l).$$

*This lecture was given in Japanese.

On the other hand, when X is a (proper smooth geometrically connected) curve, the converse does not hold in general. In fact, let J be the Jacobian variety of X, then we have

$$H^i_{\text{ét}}(X_{\overline{K}}, \mathbb{Z}_l) \simeq \begin{cases} \mathbb{Z}_l, & i = 0 \\ H^1_{\text{ét}}(J_{\overline{K}}, \mathbb{Z}_l), & i = 1 \\ \mathbb{Z}_l(-1), & i = 2 \\ 0, & i > 2 \end{cases}$$

for $l \neq \operatorname{char}(K)$. Now, it is known that there exists a curve which does not have good reduction at s but whose Jacobian variety has good reduction at s. For such a curve, the inertia group acts trivially on the étale cohomology groups for $l \neq p_s$.

Thus we need another criterion. Here, another necessary condition comes from the proper smooth base change theorem for étale fundamental groups ([SGA1], Exp. X), which assets that, if $\mathfrak{X}$ is a proper smooth geometrically connected scheme over $\mathcal{O}_{S,s}$, the specialization map (determined up to conjugacy)

$$\pi_1^{p'_s}(\mathfrak{X}_{\overline{\eta}}, *) \to \pi_1^{p'_s}(\mathfrak{X}_{\overline{s}}, *)$$

is an isomorphism, where $\pi_1^{p'_s}$ means the maximal prime-to-p_s quotient of π_1 ($\pi_1^{p'_s} = \pi_1$, if $p_s = 0$). In particular, for $l \neq p_s$, we have

$$\pi_1^l(\mathfrak{X}_{\overline{\eta}}, *) \simeq \pi_1^l(\mathfrak{X}_{\overline{s}}, *),$$

where π_1^l means the maximal pro-l quotient of π_1. Therefore, if X has good reduction at s, then the images of the inertia group at s under the outer Galois representations

$$\operatorname{Gal}(K^{\text{sep}}/K) \to \operatorname{Out}(\pi_1^{p'_s}(X_{\overline{K}}, *))$$

and

$$\operatorname{Gal}(K^{\text{sep}}/K) \to \operatorname{Out}(\pi_1^l(X_{\overline{K}}, *))$$

are trivial.

When X is a curve, the converse also holds, which has been proved by Takayuki Oda ([O]). (He states his theorem only when S is the integer ring of an algebraic number field (or its completion).)

Theorem (Oda). *Let X be a proper smooth geometrically connected curve of genus > 1 over K. Then X has good reduction at s if and only if the image of the inertia group at s in $\operatorname{Out}(\pi_1^l(X_{\overline{K}}, *))$ is trivial for some $l \neq p_s$.* □

This theorem now can be obtained also as a corollary of deep results by Asada-Matsumoto-Oda ([AMO]) on the 'universal' local monodromy, which is based on transcendental (or topological) methods and moduli theory. Our aim is to generalize Oda's theorem for not necessarily proper curves (by 'algebraic' methods).

§1. Main result.

Let S, η, and K be as in §0, and *assume* that $\kappa(s)$ is perfect for all closed point s of S. From now on, X always denotes a proper smooth geometrically connected curve over K, and D denotes a relatively étale effective divisor in X/K. Note that, when $\operatorname{char}(K) = 0$, a relatively étale divisor in X/K is just a reduced (effective) divisor in X/K. Put $U = X - D$. The divisor D is uniquely determined by U.

Definition. We say that (X, D) *has good reduction on* S, if there exist a proper smooth S-scheme $\mathfrak{X}$ and a relatively étale divisor $\mathfrak{D}$ in $\mathfrak{X}/S$ whose generic fiber $(\mathfrak{X}_\eta, \mathfrak{D}_\eta)$ is isomorphic to (X, D) over K. We say that (X, D) *has good reduction at* s, if (X, D) has good reduction on $\operatorname{Spec}(\mathcal{O}_{S,s})$.

Let g be the genus of the curve X and n the number of $D(\overline{K}) = D(K^{\text{sep}})$. Then our main theorem is as follows:

Theorem. *Assume $2g - 2 + n > 0$. (i. e. $g \geq 2$; $g = 1, n \geq 1$; or $g = 0, n \geq 3$.) Then the following conditions are equivalent:*

(a) *(X, D) has good reduction on S.*
(b) *For each closed point s of S, the image of the inertia group at s in $\operatorname{Out}(\pi_1^{p'_s}(U_{\overline{K}}, *))$ is trivial.*
(c) *For each closed point s of S and for each prime number $l \neq p_s$, the image of the inertia group at s in $\operatorname{Out}(\pi_1^l(U_{\overline{K}}, *))$ is trivial.*
(d) *For each closed point s of S, there exists a prime number $l \neq p_s$, such that the image of the inertia group at s in $\operatorname{Out}(\pi_1^l(U_{\overline{K}}, *))$ is trivial.* □

Remark. The following fact and its purely algebraic proof are known:

$$\pi_1^l(U_{\overline{K}}, *) \simeq \begin{cases} (\Pi_g)^{\wedge l}, & \text{for } n = 0, \\ (F_{2g+n-1})^{\wedge l}, & \text{for } n > 0, \end{cases}$$

where Π_g is the surface group of genus g:

$$\Pi_g = \langle \alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g \mid \alpha_1\beta_1\alpha_1^{-1}\beta_1^{-1} \ldots \alpha_g\beta_g\alpha_g^{-1}\beta_g^{-1} = 1 \rangle,$$

F_r is the free group of rank r, and $G^{\wedge l}$ means the pro-l completion of a group G.

The implication (a)⇒(b) follows from [SGA1], Exp. XIII, and the implications (b)⇒(c)⇒(d) are trivial. The proof of (d)⇒(a) goes as follows: (i) construct the 'minimal' (regular) model $(\mathfrak{X}, \mathfrak{D})$ over S of (X, D); (ii) investigate local properties of (ramified) coverings of $(\mathfrak{X}, \mathfrak{D})$, using Abhyankar's lemma, and obtain information on the substructure of the pro-l fundamental group given by the decomposition groups and the inertia groups at the irreducible components and the singular points of the special fibers; and (iii) prove that $(\mathfrak{X}, \mathfrak{D})$ is a good model, resorting to graph theory and pro-l group theory.

§2. Weight filtration.

Following the notations above, let I be the inertia group at a closed point s of S, and l a prime number $\neq p_s$. By [AK] and [K] (see also [NT]), we have the weight filtration of $\pi_1^l(U_{\overline{K}}, *)$, which induces the weight filtration of I:

$$I \supset I(0) \supset I(1) \supset I(2) \supset \cdots \supset I(\infty).$$

Here $I/I(0)$ is isomorphic to a subgroup of the symmetric group S_n, $I(0)/I(1)$ is isomorphic to a subgroup of $GSp_{2g}(\mathbb{Z}_l)$, and, for $i \geq 1$, $\operatorname{gr}^i(I) = I(i)/I(i+1)$ is a free $\mathbb{Z}_l$-module of finite rank. For simplicity, *assume* $D(\overline{K}) = D(K)$, which implies $I = I(0)$. Then:

Theorem. *One (and only one) of the following occurs:*

(1) $I \supsetneq I(1) = I(\infty)$, $I/I(1)$: *infinite;*
(2) $I \supsetneq I(1) = I(2) \supsetneq I(3) = I(\infty)$, $I/I(1)$: *finite,* $I(2)/I(3) \simeq \mathbb{Z}_l$;
(3) $I \supsetneq I(1) = I(\infty)$, $I/I(1)$: *finite;*
(4) $I = I(1) = I(2) \supsetneq I(3) = I(\infty)$, $I(2)/I(3) \simeq \mathbb{Z}_l$;
(5) $I = I(\infty)$.

In each case, the reduction at s of the Jacobian variety J of X and that of (X, D) are as follows:

(1) *Both J and (X, D) have essentially bad reduction;*
(2) *J has bad and potentially good reduction and (X, D) has essentially bad reduction;*
(3) *Both J and (X, D) have bad and potentially good reduction;*
(4) *J has good reduction and (X, D) has essentially bad reduction;*
(5) *Both J and (X, D) have good reduction.*

Here 'having bad reduction' (resp. 'having essentially bad reduction') means 'not having good reduction' (resp. 'not having potentially good reduction'). □

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