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The 5-cycle relation for iterated integrals

Monodromy of iterated integrals

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§0. Introduction.

0.1. In this lecture we give a proof that the element, whch describes the monodromy of iterated integrals (and also non-abelian unipotent periods) on $P^1(C) \setminus \{0,1,\infty\}$ satisfies the Drinfeld- Ihara 5-cycle relation. This is in fact the same element that one finds in the paper of Drinfeld (see [Dr]). The proof presented here is different from the Drinfeld proof. The proof given here imitates Deligne's proof of 2 and 3-cycle relations and it should be analogous to Ihara's proof of 5-cycle relation. We point out that the main point in our proof is the functoriality of the universal unipotent connection. This property can be shortly written as $f_*\omega = f^*\omega$ and it is also fundamental in our results on functional equations of polylogarithms and iterated integrals (see [W]).

Below we shall give a brief description of the paper. In Section 1 are established some general properties of the canonical unipotent connection. In Sections 2 and 3 we present a "naive" approach to the Deligne tangential base point (see [D2]),which is sufficient for our applications. In Sections 4 and 5 we describe the monodromy of iterated integrals on $P^1(C) \setminus \{a_1, \ldots, a_{n+1}\}$ and in more details on $P^1(C) \setminus \{0,1,\infty\}$. In Sections 6 and 7 we study monodromy of iterated integrals on configuration spaces.

§1. Canonical connection with logarithmic singularities.

Let X be a smooth, projective scheme of finite type over a field k of characteristic zero. Let D be a divisor with normal crossings in X and let $V = X \setminus D$. Let

$$A^*(V) := \Gamma(X, \Omega_X^*\langle \log D\rangle)$$

be a differential algebra of global sections of the algebraic De Rham complex on X with logarithmic singularities along D.

1.1. It follows from [D1] Corollaire 3.2.14 that each element of $A^*(V)$ is closed and the natural map $A^*(V) \to H^*_{DR}(V)$ is injective.

We shall denote by $\wedge^2(A^1(V))$ the exterior product of the vector space $A^1(V)$ with itself and by $A^1(V) \wedge A^1(V)$ the image of $\wedge^2(A^1(V))$ in $A^2(V)$.

Let $H(V) := (A^1(V))^*$ and $R(V) := (A^1(V) \wedge A^1(V))^*$ be dual vector spaces. The map $\wedge^2(A^1(V)) \to A^1(V) \wedge A^1(V)$ induces a map $R(V) \to \wedge^2(H(V))$.

Let $\mathrm{Lie}\,(H(V))$ be a free Lie algebra over k on $H(V)$. Observe that $R(V)$ is contained in degree 2 terms of $\mathrm{Lie}\,(H(V))$. Let $(R(V))$ be a Lie ideal generated by $R(V)$. We set

$$\mathrm{Lie}\,(V) := \mathrm{Lie}\,(H(V))/(R(V))$$

and

$$L(V) := \varprojlim_n \left(\mathrm{Lie}\,(V)/\Gamma^n \mathrm{Lie}\,(V)\right).$$

The Lie algebra $L(V)$ we equip with the multiplication given by the Baker-Campbell-Hausdorff formula and the obtained group we shall denote by $\pi(V)$. Its Lie algebra can be identified with $L(V)$. We define a one form ω_V on V with values in the Lie algebra $L(V)$ in the following way. The form ω_V corresponds to the identity homomorphism $\mathrm{id}_{A^1(V)}$ under the natural isomorphism

$$A^1(V) \otimes H(V) = A^1(V) \otimes (A^1(V))^* \approx \mathrm{Hom}\,(A^1(V), A^1(V)).$$

Lemma 1.2. *The one-form ω_V is integrable.*

Proof. It is sufficient to show that $d\omega_V + \frac{1}{2}[\omega_V, \omega_V] = 0$. It follows from 1.1 that $d\omega_V = 0$. We have exact sequences

$$0 \to K \to \wedge^2(A^1(V)) \to A^1(V) \wedge A^1(V) \to 0$$

and

$$0 \leftarrow K^* \leftarrow \wedge^2(H(V)) \leftarrow R(V) \leftarrow 0.$$

The two-form $[\omega_V, \omega_V] \in A^1(V) \wedge A^1(V) \otimes {}^{\wedge^2 H(V)}\!/_{R(V)} \approx \left({}^{\wedge^2(A^1(V))}\!/_{K}\right) \otimes K^*$ is represented by a map $K \to \wedge^2(A^1(V)) \to A^1(V) \wedge A^1(V)$, hence it is zero.

Let $T[H(V)]$ be a tensor algebra over k on $H(V)$ and let $(R(V))$ be an ideal of $T[H(V)]$ generated by $R(V)$. Let $Q(V) := {}^{T[H(V)]}\!/_{R(V)}$ be the quotient algebra and let $\widehat{Q}(V)$ be its completion with repect to the augmentation ideal $I := \ker(Q(V) \to k)$, i.e. $\widehat{Q}(V) := \varprojlim_n \left({}^{Q(V)}\!/_{I^n}\right)$. Let $P(V)$ be the group of invertible elements in $\widehat{Q}(V)$, whose constant terms are equal 1.

The elements of $L(V)$ we identify with Lie elements (can be of infinite length) in $P(V)$. The exponential series defines an injective homomorphism

$$\exp : \pi(V) \to P(V).$$

The inverse of exp is defined on the subgroup $\exp(\pi(V))$ of $P(V)$ and it is given by the formula

$$\log z = (z-1) - {}^1\!/_2(z-1)^2 + {}^1\!/_3(z-1)^3 - {}^1\!/_4(z-1)^4 + \dots.$$

Let us assume that k is the field of complex numbers C. Then V is a complex variety with the standard complex topology.

Let $x, z \in V$ be two points in V and let γ be a smooth path in V from x to z.

Let $\Lambda_V(z; x, \gamma)$ (resp. $L_V(z; x, \gamma)$) be a horizontal section along γ of the principal $P(V)$ (resp. $\pi(V)$)-bundle

$$V \times P(V) \to V \quad (\text{resp. } V \times \pi(V) \to V)$$

equipped with the connection given by ω_V and such that $\Lambda_V(x;x,\gamma) = 1$ (resp. $L_V(x;x,\gamma) = 0$).

Definition 1.3. Let $x \in V$ and let $\alpha \in \pi_1(V,x)$ be a loop. We shall define a homomorphism

$$\theta_{x,V} : \pi_1(V,x) \to P(V) \quad (\text{resp. } \theta_{x,V} : \pi_1(V,x) \to \pi(V))$$

by the formula

$$\theta_{x,V}(\alpha) := \Lambda_V(\alpha(1);x,\alpha) \quad (\text{resp. } \theta_{x,V}(\alpha) := L_V(\alpha(1);x,\alpha))$$

and we call it the monodromy homomorphism of the form ω_V (at the point x).

Proposition 1.4. *Let $x_1, x_2 \in V$. Then the monodromy homomorphisms $\theta_{x_1,V}$ and $\theta_{x_2,V}$ of the form ω_V are conjugated.*

Proof. It is a property of a connection on a principal fiber bundle.

Proposition 1.5. *Assume that*

$$A^1(V) \to H^1_{DR}(V)$$

is an isomorphism. Then the Lie algebra Lie (V) *is isomorphic to the Lie algebra of the fundamental group of V.*

Proof. Let $\Omega^*(V)$ be a differential algebra of complex valued, global, smooth, differential forms on V. The inclusion $A^*(V) \to \Omega^*(V)$ induces an isomorphism of the "stage one" minimal models. Moreover the "stage one" minimal model of $A^*(V)$ is formal as $d_1 = d_2 = 0$. This implies the statement of the proposition.

Proposition 1.6. *If $A^1(V) \to H^1_{DR}(V)$ is an isomorphism then the monodromy homomorphism $\theta_{x,V} : \pi_1(V,x) \to \pi(V)$ induces an isomorphism of the Malcev C-completion of $\pi_1(V,x)$ into $\pi(V)$.*

Proof. The Sullivan theory of minimal models recovers the Malcev Q-completion $\pi_1(V,x)_0(Q)$ of $\pi_1(V,x)$. The formality of the "state one" minimal model of V implies that the Malcev C-completion of $\pi_1(V,x)$ is (isomorphic to) $\pi(V)$. Hence the groups $\pi_1(V,x)_0(C)$ and

$\pi(V)$ are isomorphic. By the universal property of the Malcev C-completion there is an isomorphism $\bar{\theta}_{x,V} : \pi_1(V,x)_0(C) \xrightarrow{\approx} \pi(V)$ of affine, pro-nilpotent groups such that the diagram

$$\begin{tikzcd} & \pi_1(V,x) \arrow[dl, "r_C"'] \arrow[dr, "\theta_{x,V}"] & \\ \pi_1(V,x)_0(C) \arrow[rr, "\approx", "\bar{\theta}_{x,V}"'] & & \pi(V) \end{tikzcd}$$

commutes.

Let X_i (for $i = 1,2$) be smooth, projective schemes of finite type over k. Let D_i be divisors with normal crossings in X_i (for $i = 1,2$). Let $V_i = X_i \setminus D_i$ (for $i = 1,2$) and let $f : X_1 \to X_2$ be a morphism such that $f^{-1}(D_2) = D_1$. Then f induces $f^* : A^1(V_2) \to A^1(V_1)$.

Let $f_* : H(V_1) \to H(V_2)$ be the dual map. This map induces group homomorphisms

$$f_* : P(V_1) \to P(V_2)$$

and

$$f_* : \pi(V_1) \to \pi(V_2)$$

Lemma 1.7. *We have*

$$f_*(\omega_{V_1}) = f^*(\omega_{V_2}).$$

Corollary 1.8. *We have*

$$f_*(\Lambda_{V_1}(z; x, \gamma)) = \Lambda_{V_2}(f(z); f(x), f(\gamma)).$$

and

$$f_*(L_{V_1}(z; x, \gamma)) = L_{V_2}(f(z); f(x), f(\gamma)).$$

The lemma follows from the definition of ω_{V_i} as $\mathrm{id}_{A^1(V_i)}$.

Let $\omega_1, \dots, \omega_n$ be a base of $A^1(V)$. Let $X_1, \dots, X_n$ be a dual base of $H(V)$. Then $P(V)$ is a multiplicative group of the algebra of formal power series in non-commuting variables $X_1, \dots, X_n$ divided by the ideal generated by $R(V)$.

If $\alpha(X_1, \dots, X_n)$ is a formal power series in non-commuting variables $X_1, \dots, X_n$ we shall denote by $'\alpha(X_1, \dots, X_n)$ its image in $P(V)$.

Proposition 1.9. *We have*

i) $(z, \Lambda_V(z; x, \gamma)) = (z, '1 + \sum((-1)^k \int_{x,\gamma}^{z} \omega_{i_1}, \ldots, \omega_{i_k})X_{i_k} \cdot \ldots \cdot X_{i_1}) \in V \times P(V);$

(the summation is over all non-commutative monomials in variables $X_1, \ldots, X_n$, the interated integrals are calculated along the path γ)

ii) $L_V(z; x, \gamma) = \log(\Lambda_V(z; x, \gamma))$

Proof. The principal bundle $V \times C\{\{X_1, \ldots, X_n\}\}^* \to V$ equipped with the connection given by $\sum_{i=1}^{n} \omega_i \otimes X_i$ has horizontal sections given by

$$z \to (z, 1 + \sum((-1)^k \int_{x,\gamma}^{z} \omega_{i_1}, \ldots, \omega_{i_k})X_{i_k} \cdot \ldots \cdot X_{i_1}).$$

Hence the point i) follows. Observe that $\exp : \pi(V) \to P(V)$ identifies $\omega_V \in A^1(V) \otimes \mathrm{Lie}\,(\pi(V))$ with $\omega_V \in A^1(V) \otimes \mathrm{Lie}\,(P(V))$. Hence the point ii) follows.

§2. Homotopy relative tangential base points on $P^1(C)\backslash\{a_1, \cdots, a_{n+1}\}$.

2.1. Let $X = P^1(C)\backslash\{a_1, \cdots, a_{n+1}\}$. Let $T_x(P^1(C))$ be the tangent space to $P^1(C)$ in x. Let us set $\hat{X} = X \cup \bigcup_{i=1}^{n+1} (T_{a_i}(P^1(C))\backslash\{0\})$.

Let $J'(\hat{X})$ be the set of all continous maps from the closed unit interval $[0; 1]$ to $P^1(C)$ such that

i) $\varphi((0, 1)) \subset X$;

ii) if $\varphi(0) = a_i$ then φ is smooth near a_i and $\dot{\varphi}(0) \neq 0$, and if $\varphi(1) = a_k$ then φ is smooth near a_k and $\dot{\varphi}(1) \neq 0$.

In the sequel we shall identify $\dot{\varphi}(0)$ (resp. $\dot{\varphi}(1)$) with a tangent vector to φ in $T_{a_i}(P^1(C))$ (resp. $T_{a_k}(P^1(C))$. This tangent vector we shall denote also by $\dot{\varphi}(0) \in T_{a_i}(P^1(C))$ (resp. $\dot{\varphi}(1) \in T_{a_k}(P^1(C))$.

If $\varphi(0) = x \in X$ and $\varphi(1) = y \in X$ then we say that φ is a path from x to y.

If $\varphi(0) = a_i$ (resp. $\varphi(1) = a_k$) then we say that φ is a path from $\dot{\varphi}(0) \in T_{a_i}(P^1(C))$ (resp. to $-\dot{\varphi}(1) \in T_{a_k}(P^1(C))$ and we shall write $\varphi(0) = \dot{\varphi}(0)$ (resp. $\varphi(1) = -\dot{\varphi}(1)$).

To $J'(\hat{X})$ we joint all constant maps from [0,1] to $\hat{X}$ and the resulting set we denote by $J(\hat{X})$.

We shall define a relation of homotopy in the set $J(\hat{X})$. Let $\varphi, \psi \in J(\hat{X})$. If $\varphi(0) = \psi(0) = x \in X$ and $\varphi(1) = \psi(1) = y \in X$ then we say that φ and ψ are homotopic if they are homotopic maps in the space $\mathrm{map}([0,1], 0, 1; X, x, y)$.

If $\varphi(0) = \psi(0) = v \in T_{a_i}(P^1(C))$ and $\varphi(1) = \psi(1) = y \in X$ then we say that φ and ψ are homotopic if there is a homotopy

$H_s \in \mathrm{map}([0,1], 0, 1; X \cup \{a_i\}, a_i, y)$ such that

i) $H_s \in J(\hat{X})$ and $H_s(0) = v$ for all $s \in [0,1]$;

ii) $H_s((0,1)) \subset X$ for all $s \in [0,1]$;

iii) $H_0 = \varphi$ and $H_1 = \psi$.

We left to the reader the cases when $\varphi(0) = \psi(0) = x \in X, \varphi(1) = \psi(1) = w \in T_{a_k}(P^1(C))$ and $\varphi(0) = \psi(0) = v \in T_{a_i}(P^1(C)), \varphi(1) = \psi(1) = w \in T_{a_k}(P^1(C))$.

Let $\varphi \in J(\hat{X})$ be such that $\varphi(0) = \varphi(1) = v \in T_{a_i}(P^1(C))$ and let $\psi \in J(\hat{X})$ be a constant map equal to v. We say that φ and ψ are homotopic if there is a homotopy

$$H_s \in \mathrm{map}([0,1], 0, 1; X \cup \{a_i\}, a_i, a_i)$$

such that

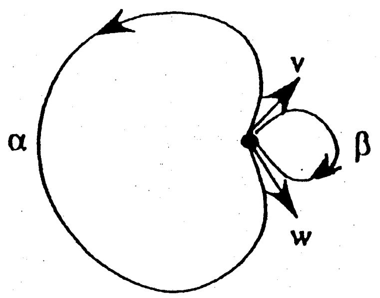
i) $H_s \in J(\hat{X})$ and $H_s(0) = H_s(1) = v$ for all $s \in [0,1]$;

ii) $H_s((0,1)) \subset X$ for all $s \in [0,1]$;

iii) $H_0 = \varphi$ and $H_1(t) = a_i$ for $t \in [0,1]$.

Observe that $G_t := H_{1-t}$ defines a homotopy between ψ and φ.

With the definition given above paths $\alpha, \beta \in J(\widehat{C \setminus \{0\}})$ are not homotopic.



We shall write $\varphi \sim \psi$ if φ and ψ are homotopic. The relation $\sim$ is an equivalence relation on the set $J(\hat{X})$. Let $\iota(\hat{X}) := J(\hat{X})/\sim$ be the set of equivalence classes.

We define a partial composition in $\iota(\hat{X})$ in the following way. Let $\phi, \Psi \in \iota(\hat{X})$ and let $\varphi, \psi \in J(\hat{X})$ be its representatives.

If $\varphi(1) = \psi(0) = y \in X$ then we set $\Psi \circ \phi := [\psi \circ \varphi]$, the class of $\psi \circ \varphi$ in $\pi(\hat{X})$.

If $\varphi(1) = \psi(0) = v \in T_{a_i}P^1(C)$ then we can assume that φ and ψ coinside near a_i and we define $\Psi \circ \phi := [\psi^\varepsilon \cdot \varphi_\eta]$, where $\varphi_{1-\varepsilon} := \varphi_{|[0,1-\varepsilon]}, \psi_\eta := \psi_{|[\eta,1]}$ and $\varphi(1-\varepsilon) = \psi(\eta)$.

The map pr:$J(\hat{X}) \to \hat{X} \times \hat{X}/\mathrm{pr}(\varphi) = (\varphi(0), \varphi(1))$ which associates to a path its beginning $(\varphi(0))$ and its end $(\varphi(1))$ agrees with the relation $\sim$ and it defines $p : \iota(\hat{X}) \to \hat{X} \times \hat{X}$. The partial composition o makes $p : \iota(\hat{X}) \to \hat{X} \times \hat{X}$ into a groupoid over $\hat{X} \times \hat{X}$.

Let $x \in \hat{X}$. We set $\pi_1(X, x) := p^{-1}(x, x)$. This is a fundamental group with a base point in $x \in \hat{X}$.

2.2. We shall construct a family of horizontal sections of ω_x, where a base point x is replace by a tangent vector.

Let us set $V = C\backslash\{a_1, \cdots, a_n\}$. Let $x_0 \in C$ and let $\delta : [0,1] \ni t \to a_i + t.(x_0 - a_i)$ be an interval joining a_i and x_0. Let γ be a path from a_i to $z \in V$ (not passing through any $a_k, k = 1, \cdots n$) tangent to δ in a_i. We assume that in a small neighbourhood of a_i the path γ coincides with δ.

Observe that $v = x_0 - a_i$ can be canonically identified with a tangent vector to C in a_i. Let $\omega_1 = \frac{dz}{z-a_1}, \cdots, \omega_n = \frac{dz}{z-a_n}$. We set

$$\Lambda_{a_i,v}(\alpha_1, \cdots, \alpha_k)(z) := \int_{a_i,\gamma}^{z} \omega_{\alpha_1}, \cdots, \omega_{\alpha_k} \quad \text{if} \quad \alpha_1 \neq i.$$

Let $\varepsilon \in im(\delta)$ be near a_i. Let γ_ε be a part of γ from ε to z, and let δ_ε be a part of δ from ε to x_0. We set

$$\Lambda_{a_i,v}(i, \cdots, i, \alpha_{k+1} \cdots, \alpha_l)(z) :=$$
$$\lim_{\varepsilon \to a_i} \int_{\varepsilon,\gamma_\varepsilon}^{z} \left(\int_{x_0,\gamma_\varepsilon + (\delta_\varepsilon)^{-1}}^{z} \frac{dz}{x - a_i}, \cdots, \frac{dz}{z - a_i} \right) \omega_{\alpha_{k+1}}, \cdots, \omega_{\alpha_l}$$

if $\alpha_{k+1} \neq i$,

and

$$\Lambda_{a_i,v}(i, \cdots, i)(z) = \int_{x_0,\gamma_\varepsilon + \delta_\varepsilon^{-1}}^{z} \frac{dz}{z - a_i}, \cdots, \frac{dz}{z - a_i}.$$

Lemma 2.2.1. *The integrals $\Lambda_{a_i,v}(\alpha_1,\cdots,\alpha_k)(z)$ exist and they are analytic, multivalued functions on V.*

Proof. Assume that $\alpha_t \neq i$ for $t \leq l$ and $\alpha_{l+1} = i$. The function $g(z) := \int_{a_i}^{z} \omega_{\alpha_1},\cdots,\omega_{\alpha_l}$ is analytic, multivalued on $V \cup \{a_i\}$ and vanishes in a_i. Hence the integral $g_1(z) := \int_{a_i}^{z} g(z)\frac{dz}{z-a_i}$ exists, the function $g_1(z)$ is analytic, multivalued on $V \cup \{a_i\}$ and vanishes in a_i. Hence by induction we get that $\Lambda_{a_i,v}(\alpha_1,\cdots,\alpha_n)(z)$ exists, and it is analytic, multivalued on $V \cup \{a_i\}$.

Assume now that $\alpha_t = i$ for $t \leq l$ and $\alpha_{l+1} \neq i$. Without loss of generality we can assume that $a_i = 0$ and $x_0 = 1$.

Observe that $\lim_{\varepsilon\to 0} \int_{x_0,\gamma_\varepsilon\circ(\delta_\varepsilon)^{-1}}^{z} z^n(\log z)^m dz = z^{n+1}(\sum_{i=0}^{m} \beta_i(\log z)^{m-i})$ where β_i are rational numbers. The function $z^g(\log z)^p$ for g and p positive integers, is analytic, multvalued on V, continous on any small cone with a vertex in a_i (0 in this case) and it vanishes in a_i. The function $\frac{1}{z-a_j}$ for $j \neq i$ is bounded on any sufficiently small neighbourhood of a_i. Hence the integral

$$\lim_{\varepsilon\to a_i} \int_{\varepsilon,\gamma_\varepsilon}^{z} \left(\int_{x_0,\gamma_\varepsilon\circ(\delta_\varepsilon)^{-1}} \frac{dz}{z-a_i},\cdots,\frac{dz}{z-a_i} \right) \frac{dz}{z-a_j},\cdots,\frac{dz}{z-a_{i_k}}$$

is an analytic, multivalued function on V, continous, univalued on any small cone with a vertex in a_i and it vanishes in a_i.

Let us set

$$\Lambda_V(z;v,\gamma) = 1 + \sum(-1)^k \Lambda_{a_i,v}(\alpha_1,\cdots,\alpha_k)(z) X_{\alpha_k}\cdots X_{\alpha_1}.$$

We recall that $X_1,\cdots,X_n$ are duals of $\frac{dz}{z-a_1},\cdots,\frac{dz}{z-a_n}$.

Lemma 2.2.2. *The map*

$$V \ni z \to (z, \Lambda_V(z;v,\gamma)) \in V \times P(V)$$

is horizontal with respect to ω_V.

It rests to define functions $\Lambda_X(z;v,\gamma)$ if $a_i = \infty$ or all a_i are different from ∞.

Let $f : Y = C\backslash\{b_1, \cdots, b_n\} \to X = P^1(C)\backslash\{a_1, \cdots, a_{n+1}\}$ be a regular map of the form $\frac{az+b}{cz+d}$ with $\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} \neq 0$. It follows from corollary 3.8 that $f_*(\Lambda_Y(z; y, \gamma)) = \Lambda_X(f(x); f(y), f(\gamma))$ if $y \in Y$. We shall use this fact to define $\Lambda_X(z; v, \gamma)$ where v is a tangent vector to $P^1(C)$ in a_i and γ is a path from a_i to z, which is tangent to v and $f^{-1}(\gamma)$ coincides with $f^{-1}(v)$ near $f^{-1}(a_i)$.

We set

$$\Lambda_X(z; v, \gamma) := f_*(\Lambda_Y(f^{-1}(z); f_*^{-1}(v), f^{-1}(\gamma))).$$

It is clear that $\Lambda_X(z; v, \gamma)$ does not depend on the choice of f.

Lemma 2.2.3. *The map*

$$X \ni z \to (z, \Lambda_X(z; v, \gamma)) \in X \times P(X)$$

is horizontal with respect to ω_X.

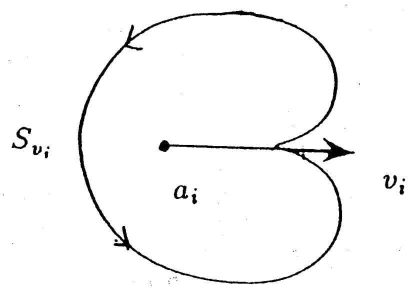
We set $L_X(z; v, \gamma) := \log\Lambda_X(z; v, \gamma)$. If we are dealing with only one space X we shall usually omit subscript X and we shall write $\Lambda(z; v, \gamma)$ and $L(z; v, \gamma)$, or even $\Lambda_v(z; \gamma)$ and $L_v(z; \gamma)$, or $\Lambda_v(z)$ and $L_v(z)$.

We summarize the constractions from 2.1 and 2.2 in the following proposition.

Proposition 2.2.4. *The functions* $\Lambda_X(z; v, \gamma)$ *and* $L_X(z; v, \gamma)$ *depend only on the homotopy class of* γ *in* $\iota(\hat{X}) = J(\hat{X})/\sim$.

§3. Generators of $\pi_1(P^1(C)\backslash\{a_1,\dots,a_{n+1}\},x)$.

Let $X = P^1(C)\backslash\{a_1,\dots,a_{n+1}\}$ and let $x \in \hat{X}$. We shall describe how to choose generators of $\pi_1(X,x)$. Let v_i be a tangent vector in a_i. Then the loop around a_i at the base point v_i is the following element S_{v_i} of $\pi_1(X,v_i)$ (see picture).



For each i let us choose a tangent vector $v_i \in T_{a_i}(P^1(C))$. Let us choose a family of paths $\Gamma = \{\gamma_i\}_{i=1}^{n+1}$ in $J(\hat{X})$ from x to each v_i such that any two paths do not intersect and no path self intersects. The indices are chosen in such a way that when we make a small circle around x (around a_k if x is a tangent vector at a_k) in the opposite clokwise direction, starting from γ_1 we meet $\gamma_2,\gamma_3,\dots,\gamma_{n+1}$. If γ_1 is a constant path equal v_1 we meet $\gamma_2,\dots,\gamma_{n+1}$. To the path γ_i we associate the following element S_i in $\pi_1(X,x)$. We move along γ_i, we make the loop S_{v_i} around a_i and we veturn along γ_i^{-1} to x. If γ_1 is a constant path equal v_1 then S_1 is S_{v_1}.

The following lemma is obvious.

Lemma 3.1. *The elements $S_1,S_2,\dots,S_{n+1}$ are generators of $\pi_1(X,x)$. We have $S_{n+1}\cdot \ldots \cdot S_2\cdot S_1 = 1$.*

Definition 3.2. *The ordered sequence $(S_1,\dots,S_{n+1})$ of elements of $\pi_1(X,x)$ obtained from the family of paths $\Gamma = \{\gamma_i\}_{i=1}^{n+1}$ we shall call a sequence of geometric generators of $\pi_1(X,x)$ associated to $\Gamma = \{\gamma_i\}_{i=1}^{n+1}$.*

§4. Monodromy of iterated integrals on $P^1(C)\backslash\{a_1,\cdots,a_{n+1}\}$.

Let $X = P^1(C)\backslash\{a_1,\cdots,a_{n+1}\}$. Let $x_1,x_2,x_3 \in \hat{X}$ and let $z_0 \in X$. Let γ_i for $i = 1,2,3$ be a path belonging to $J(\hat{X})$ from x_i to z_0. Let us set $\gamma_{ij} := \gamma_j^{-1} \circ \gamma_i$.

Proposition 4.1. *Let us prolongate each function $\Lambda_{x_i}(z)$ along γ_i to the point z_0. There exist elements $a_{x_j}^{x_i}(\gamma_{ij}) \in P(X)$ such that*

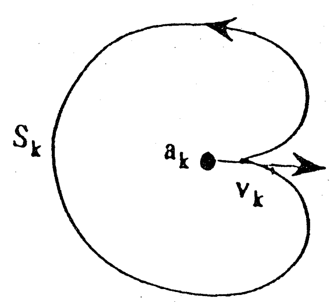
$$\Lambda_{x_i}(z) \cdot a_{x_j}^{x_i}(\gamma_{ij}) = \Lambda_{x_j}(z)$$

for all z in a small neibourhood of z_0. The elements $a_{x_j}^{x_i}(\gamma_{ij})$ satisfy the following relations

$$a_{x_i}^{x_i}(\gamma_{ii}) = 1,$$
$$a_{x_j}^{x_i}(\gamma_{ij}) \cdot a_{x_i}^{x_j}(\gamma_{ji}) = 1,$$
$$a_{x_j}^{x_i}(\gamma_{ij}) \cdot a_{x_k}^{x_j}(\gamma_{jk}) = a_{x_k}^{x_i}(\gamma_{ik}).$$

Proof. The existence of $a_{x_j}^{x_i}(\gamma_{ij})$ follows from the fact that $\Lambda_{x_i}(z)'s$ are horizontal sections. The first two relations are obvious. The last relation follows from equalities $\Lambda_{x_i}(z) \cdot a_{x_j}^{x_i}(\gamma_{ij}) = \Lambda_{x_j}(z), \Lambda_{x_j}(z) \cdot a_{x_k}^{x_j}(\gamma_{jk}) = \Lambda_{x_k}(z)$ and $\Lambda_{x_i}(z) \cdot a_{x_k}^{x_i}(\gamma_{ik}) = \Lambda_{x_k}(z)$.

Proposition 4.2. *Let $v_k \in T_{a_k}P^1(C)\backslash\{0\}$. Let S_k be a loop around a_k based at $v_k \in T_{a_k}(P^1(C))\backslash\{0\}$, (see picture)*



The monodromy of Λ_{v_k} along S_k is given by

$$S_k : \Lambda_{v_k}(z) \to \Lambda_{v_k}(z) \cdot e^{-2\pi i X_k}$$

Proof. The monodromy of $\Lambda_{v_k}(k^m)(z) := \Lambda_{a_k,v_k}(k,k,\cdots,k)(z)$ along S_k is given by $S_k : \Lambda_{v_k}(k^m)(z) \to \Lambda_{v_k}(k^m)(z) + \sum_{l=1}^{m} \Lambda_{v_k}(k^{m-l})(z)\frac{(-2\pi i)^l}{l!}$. This implies that the monodromy of $\Lambda_{v_k}(\alpha_1,\cdots,\alpha_p,k^m)(z)$ along S_k is given by $S_k : \Lambda_{v_k}(\alpha_1,\cdots,\alpha_p,k^m)(z) \to \Lambda_{v_k}(\alpha_1,\cdots,\alpha_p,k^m)(z) + \sum_{l=1}^{m} \Lambda_{v_k}(a_1,\cdots,a_p,k^{m-l})(z)\frac{(-2\pi i)^l}{l!}$. Hence it follows the formula for the monodromy of $\Lambda_{v_k}(z)$ along S_k.

Let $x \in \hat{X}$. Let us choose $v_i \in T_{a_i}P^1(C)\backslash\{0\}$ for $i = 1,2,\cdots,n+1$. Let $(S_1,\cdots,S_{n+1})$ be a sequence of geometric generators of $\pi_1(X,x)$ associated to $\Gamma = \{\gamma_1\}_{i=1}^{n+1}$ where Γ is a family of paths in $J'(\hat{X})$ from x to v_i for $i = 1,2,\cdots,n+1$.

Theorem 4.3. *The monodromy of the function $\Lambda_x(z)$ along the loop S_k is given by*

$$S_k : \Lambda_x(z) \to \Lambda_x(z) \cdot a^x_{v_k}(\gamma_k) \cdot e^{-2\pi i X_k} \cdot (a^x_{v_k}(\gamma_k))^{-1}.$$

Proof. It follows from Proposition 4.1 that

$$(*_1) \qquad \Lambda_x(z) \cdot a^x_{v_k}(\gamma_k) = \Lambda_{v_k}(z)$$

for z in the small neibourhood of some $\gamma(z_0)$. This equality is preserved after the monodromy transformation along S_k, hence we have

$$(*_2) \qquad (\Lambda_x(z))^{S_k} \cdot a^x_{v_k}(\gamma_k) = (\Lambda_{v_k}(z))^{S_k},$$

where $(\ )^{S_k}$ denotes the function $(\ )$ after the monodromy transformation along S_k. If follows from Proposition 4.2 that

$$(*_3) \qquad (\Lambda_{v_k}(z))^{S_k} = \Lambda_{v_k}(z) \cdot e^{-2\pi i X_k}.$$

If we substitute $(*_3)$ in $(*_2)$ and then substitute $(*_1)$ for $\Lambda_{v_k}(z)$ we get the formula for $(\Lambda_x(z))^{S_k}$.

Corollary 4.4. *The monodromy of the function $L_x(z)$ along the loop S_k is given by*

$$S_k : L_x(z) \to L_{v_k}(z) \cdot \alpha^x_{v_x}(\gamma_k) \cdot (-2\pi i X_k) \cdot \alpha^x_{v_k}(\gamma_k)^{-1}$$

where $\alpha^x_{v_k}(\gamma_k) = \log(\alpha^x_{v_k}(\gamma_k))$.

Proof. The corollary follows immediately from Proposition 1.9. ii).

4.5 It follows from above, that the definition of the monodromy homomorphism (Definition 1.3) extends to any $x \in \hat{X}$. Hence for any $v \in \hat{X}$ we have a monodromy homomorphism

$$\theta_{v,X} : \pi_1(X, v) \to P(X)$$

and if $v, v' \in \hat{X}$, then the homomorphisms $\theta_{v,X}$ and $\theta_{v',X}$ are conjugated.

Proposition 4.6. *Let $f : X = P^1(C)\backslash\{a_1, \ldots, a_{n+1}\} \to Y = P^1(C)\backslash\{b_1, \ldots, b_{m+1}\}$ be a regular map. Then for any $v, w \in \hat{X}$, a path γ from v to z and a path δ from v to w we have*

$$f_*(\Lambda_X(z; v, \gamma)) = \Lambda_Y(f(z); f(v), f(\gamma))$$

and

$$f_*(a^v_w(\delta)) = a^{f(v)}_{f(w)}(f(\delta)).$$

(notation: $f(v) := f_*(v)$ if v is a tangent vector).

Proof. The proposition follows from the definition of $\Lambda_X(z; v, \gamma)$ and a^v_w for tangent vectors and from Corollary 1.8.

§5. Calculations

Let $X = P^1(C)\setminus\{0,1,\infty\}$. The forms $\frac{dz}{z}$ and $\frac{dz}{z-1}$ form a base of $A^1(X)$. Let $X := (\frac{dz}{z})^*$ and $Y := (\frac{dz}{z-1})^*$ be the dual base of $(A^1(X))^*$. Let us set $Z := -X-Y$. The group $P(X)$ is the group of invertible power series with a constant term equal 1 in non-commuting variables X and Y.

Let us fix a path γ_1 = interval $[0,1]$ from $\overrightarrow{01}$ to $\overrightarrow{10}$. It follows from Proposition 4.1 that

$$\Lambda_{\overrightarrow{10}}(z)\cdot a_{\overrightarrow{01}}^{\overrightarrow{10}}(X,Y) = \Lambda_{\overrightarrow{01}}(z). \tag{1}$$

Let $f(z) = 1-z$. It follows from Proposition 4.6 that

$$f_*(a_{01}^{10}(X,Y)) = a_{10}^{01}(X,Y).$$

(We omit arrows over 10 and 01.) Proposition 4.1 implies

$$a_{10}^{01}(X,Y) = (a_{01}^{10}(X,Y))^{-1}.$$

Observe that $f_*(X) = Y$ and $f_*(Y) = X$. Hence we get the Deligne formula

$$a_{01}^{10}(X,Y) = (a_{01}^{10}(Y,X))^{-1}. \tag{2}$$

(The proof of (2) given here repeats essentially the Deligne proof.)

Let us fix a path γ_∞ = interval $[\infty,-\varepsilon]+$ arc from $-\varepsilon$ to ε passing by $(-i)\cdot\varepsilon$ $(\varepsilon>0)$ + interval from ε to 0) from $\overrightarrow{\infty 0}$ to $\overrightarrow{01}$. Let S_0 (around 0), S_1 (around 1) and S_∞ (around ∞) be geometric generators of $\pi_1(X,\overrightarrow{01})$ associated to the family $\{\gamma_0,\gamma_1,\gamma_\infty\}$, where γ_0 is the constant path equal $\overrightarrow{01}$. Then we have $S_0\circ S_1\circ S_\infty = 1$. The monodromy of $\Lambda_{\overrightarrow{01}}(z)$ is given by the following formulas (see Theorem 4.3)

$$\begin{aligned} &S_0 : \Lambda_{\overrightarrow{01}}(z) \to \Lambda_{\overrightarrow{01}}(z)\cdot e^{(-2\pi i)X}, \\ &S_1 : \Lambda_{\overrightarrow{01}}(z) \to \Lambda_{\overrightarrow{01}}(z)\cdot (a_{10}^{01}(X,Y))^{-1}\cdot e^{(-2\pi i)Y}\cdot a_{10}^{01}(X,Y), \\ &S_\infty : \Lambda_{\overrightarrow{01}}(z) \to \Lambda_{\overrightarrow{01}}(z)\cdot e^{-\pi i X}\cdot (a_{10}^{01}(Z,X))^{-1}\cdot e^{(-2\pi i)Z}\cdot (a_{10}^{01}(Z,X))\cdot e^{\pi i X}. \end{aligned} \tag{3}$$

The monodromy along S_∞ needs some explanations. By Theorem 4.3 it is given by the formula $S_\infty : \Lambda_{\overrightarrow{01}}(z) \to \Lambda_{\overrightarrow{01}}(z) \cdot (a_{01}^{\infty 0}(X,Y))^{-1} \cdot e^{(-2\pi i)Z} \cdot a_{01}^{\infty 0}(X,Y)$. By Proposition 4.1 $a_{01}^{\infty 0}(X,Y) = a_{0\infty}^{\infty 0}(X,Y) \cdot a_{01}^{0\infty}(X,Y)$. One calculates that $a_{01}^{0\infty}(X,Y) = e^{\pi i \cdot X}$. Let $f(z) = \frac{z-1}{z}$. Then it follows from Proposition 4.6 that $a_{10}^{01}(Z,X) = a_{0\infty}^{\infty 0}(X,Y)$. Hence we get the formula describing the monodromy along S_∞.

The Lie algebra $L(X)$ is the completion of the free Lie algebra on two generators X and Y. Let us set $\alpha(X,Y) := \alpha_{10}^{01}(X,Y) := \log a_{10}^{01}(X,Y)$. The monodromy of $L_{\overrightarrow{01}}(z)$ is given by the following formulas (see Corollary 4.4).

$$
\begin{aligned}
&S_0 : L_{\overrightarrow{01}}(z) \to L_{\overrightarrow{01}}(z) \cdot (-2\pi i)X, \\
(4) \qquad &S_1 : L_{\overrightarrow{01}}(z) \to L_{\overrightarrow{01}}(z) \cdot \alpha(X,Y)^{-1} \cdot (-2\pi i)Y \cdot \alpha(X,Y), \\
&S_\infty : L_{\overrightarrow{01}}(z) \to L_{\overrightarrow{01}}(z) \cdot (-\pi i)X \cdot \alpha(Z,X)^{-1} \cdot (-2\pi i)Z \cdot \alpha(Z,X) \cdot (\pi i) \cdot X.
\end{aligned}
$$

We shall calcualte coefficients of $a_{10}^{01}(X,Y)$ and $\alpha(X,Y)$. If ω is a monomial in X and $Y, a(\omega)$ is the coefficient at ω of $a_{10}^{01}(X,Y)$. Let X be the first basic Lie element and let Y be the second basic Lie element. We shall choose a base of a free Lie algebra on X and Y as in [MKS] pages 324-325. If ω is an element of this base, let $\alpha(\omega)$ be the coefficient at ω of $\alpha(X,Y)$. It follows from the formula (1) that

$$
\begin{aligned}
(5) \qquad & a(X^nY) = (-1)^n\zeta(n+1), \ a(Y^nX) = (-1)^{n+1}\zeta(n+1), \\
(6) \qquad & a(X^iY^j) = \int_0^1 (-\frac{dz}{z-1})^j, (\frac{-dz}{z})^i, a(Y^jX^i) = \int_0^1 (-\frac{dz}{z})^i, (-\frac{dz}{z-1})^j.
\end{aligned}
$$

(If ω is a one-form then $\omega^i := \omega, \omega, \cdots, \omega$ i-times.) It follows from (2) that $a(X^iY^j) + a(Y^iX^j) = 0$. It follows from [Ch] that $a(X^iY^j) + (-1)^{i+j}a(Y^jX^i) = 0$. Hence we get

$$\alpha_{i,j} := \alpha((YX)X^{i-1}Y^{j-1}) = (-1)^i a(X^iY^j) = (-1)^{j-1}a(Y^jX^i)$$

and

$$\alpha((YX)X^{j-1}Y^{i-1}) = \alpha((YX)X^{i-1}Y^{j-1}).$$

Let us set $\pi' := [\pi(X), \pi(X)]$ and $\pi'' := [\pi', \pi']$. It follows from (3) that the monodromy homomorphism

$$\theta_{\overrightarrow{01}} : \pi_1(X, \overrightarrow{01}) \to \pi_2(X) := \pi(X)/\pi''$$

is given by

$$
\begin{aligned}
(7)\qquad & S_0 \to (-2\pi i)X, \\
& S_1 \to (-2\pi i)Y + [-2\pi i Y, \alpha(X,Y)] \\
& \quad = (-2\pi i)Y + \sum_{i=0,j=0}^{\infty} (2\pi i)\alpha_{i+1,j+1}((YX)X^iY^{j+1}).
\end{aligned}
$$

The formula

$$\int_0^z F(z)\frac{dz}{z}, (\frac{dz}{z})^n = \sum_{i=0}^{n} \frac{(-1)^{n-i}}{(n-i)!i!}(\int_0^z F(z)(\log z)^{n-i}\frac{dz}{z})(\log z)^i$$

implies

$$(8)\qquad \alpha_{n+1,m} = \frac{(-1)^n(-1)^m}{n!m!}\int_0^1 (\log(1-z))^m(\log z)^n\frac{dz}{z}.$$

§6. The configuration spaces

Let $X = P^1(C)\backslash\{a_1, \cdots, a_{n+1}\}$ and $X' = P^1(C)\backslash\{a'_1, \cdots, a'_{n+1}\}$. If the sequences $(a,x) := (a_1, \cdots, a_{n+1}, x)$ and $(a', x') := (a'_1, \cdots, a'_{n+1}, x')$ are close then the groups $\pi_1(X,x)$ and $\pi_1(X',x')$ are canonically isomorphic. We shall study how the monodromy homomorphisms $\theta_{x,a}; = \theta_{x,X}$ and $\theta_{x',a'} := \theta_{x',X'}$ from sections 1 and 4

$$
\begin{array}{cccc}
\theta_{x,a}: & \pi_1(X,x) & \to & \pi(X) \\
 & \wr & & \| \\
\theta_{x',a'}: & \pi_1(X',x') & \to & \pi(X')
\end{array}
$$

depend on a and a'.

Let $X_n = \{(z_1, \cdots, z_n) \in C^n | z_i \neq z_j \text{ if } i \neq j\}$. The space of global one-forms on X_n with logarithmic singularities, $A^1(X_n)$ is spanned by $\frac{dz_i - dz_j}{z_i - z_j}$ for $i,j \in \{1, 2, \cdots, n\}$ and $i < j$. Let $X_{ij} = (\frac{dz_i - dz_j}{z_i - z_j})^*$ be their formal duals. We set $X_{ji} = X_{ij} = 0$. Dualizing the map

$$\bigwedge^2 (A^1(X_n)) \to A^1(X_n) \wedge A^1(X_n)$$

we get that $R(X_n)$ is generated by

$$[X_{ij}, X_{ik} + X_{jk}] \text{ with } i,j,k \text{ different}$$

and

$$[X_{ij}, X_{kl}] \text{ with } i,j,k,l \text{ different.}$$

Let $x = (x_1, \cdots, x_n, x_{n+1}) \in X_{n+1}$ be a base point. Let $p_i : X_{n+1} \to X_n$ $(i = 1, \cdots, n+1)$ be a projection $p_i(z_1, \cdots, z_{n+1}) = (z_1, \cdots, \hat{z}_i, \cdots, z_{n+1})$, let $x(i) :=$ $(x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_{n+1})$ and let $X(i,x) := p_i^{-1}(x(i)) = C \backslash \{x_1, \cdots, \hat{x}_i, \cdots, x_{n+1}\}$. ($\hat{z}$ means z is omitted). Let $k_i : X(i,x) \to X_{n+1}$ be given by $k_i(z) = (x_1, \cdots, x_{i-1}, z, x_{i+1},$ $\cdots x_{n+1})$. The inclusion k_i induces

$$(k_i)_* : P(X(i,x)) \to P(X_{n+1})$$

and

$$(k_i)_* : \pi(X(i,x)) \to \pi(X_{n+1})$$

where $(k_i)_*(X_j) = X_{ij}$ and X_j is the formal dual of $\frac{dz}{z-a_j}$ on $X(i,x)$. The map $(k_i)_*$ is injective and its image, $(k_i)_*(\pi(X(i,x))$ is a normal subgroup of $\pi(X_{n+1})$.

Let $x = (x_1, \cdots, x_n, x_{n+1}) \in X_{n+1}$ and $x' = (x'_1, \cdots, x'_n, x'_{n+1}) \in X_{n+1}$. Let us set $X := X(n+1, x)$ and $X' = X(n+1, x')$. We choose a family of non-intersecting paths $\gamma_1, \cdots, \gamma_n, \gamma_{n+1}$ in C from x_1 to $x'_1, \cdots, x_n$ to x'_n and x_{n+1} to x'_{n+1}. We shall identify $\pi_1(X, x_{n+1})$ and $\pi_1(X', x'_{n+1})$ in the following way. Observe that $\gamma = (\gamma_1, \cdots, \gamma_n, \gamma_{n+1})$ is a path in X_{n+1} from x to x'. The identification isomorphism $\gamma. : \pi_1(X, x_{n+1}) \to$ $\pi_1(X', x'_{n+1})$ is the unique isomorphism making the following diagram commutative

$$\begin{array}{ccc}
\pi_1(X, x_{n+1}) & \xrightarrow{(k_{n+1})_*} & \pi_1(X_{n+1}, x) \\
\vdots\downarrow \gamma. & & \downarrow \gamma_\# \\
\pi_1(X', x'_{n+1}) & \xrightarrow{(k_{n+1})_*} & \pi_1(X_{n+1}, x').
\end{array}$$

($\gamma_\#$ is induced by the path γ is a standard way).

Proposition 6.1. *After the identification of the fundamental groups of* $X = C \backslash \{x_1, \cdots, x_n\}$ *and* $X' = C \backslash \{x'_1, \cdots, x'_n\}$ *by* $\gamma.$ *the monodromy homomorphisms*

$$\theta_{x_{n+1}, X} : \pi_1(X, x_{n+1}) \to \pi(X) \textit{ and } \theta_{x'_{n+1}, X'} : \pi_1(X', x'_{n+1}) \to \pi(X')$$

$(\pi(X) = \pi(X'))$ are conjugated by an element of the group $\pi(X_{n+1})$. (The group $(k_{n+1})_*\pi(X)$ is a normal subgroup of $\pi(X_{n+1})$ so $\pi(X_{n+1})$ acts on $\pi(X)$ by conjugations.)

Proof. The corollary follows from the commutative diagram

$$\begin{tikzcd}
\pi_1(X, x_{n+1}) \arrow[rr, "\theta_{x_{n+1},X}"] \arrow[dr, "(k_{n+1})_*"'] \arrow[ddd, "\gamma\cdot"'] \arrow[drr, phantom, "(3)"] \arrow[ddr, phantom, "(4)"] & & \pi(X) \arrow[dr, "(k_{n+1})_*"] & \\
& \pi_1(X_{n+1}, x) \arrow[rr, "\theta_{x,X_{n+1}}"] \arrow[d, "\gamma_\#"] \arrow[drr, phantom, "(1)"] & & \pi(X_{n+1}) \arrow[d, "c_{L_{X_{n+1}}(x';x,\gamma)}"] \\
& \pi_1(X_{n+1}, x') \arrow[rr, "\theta_{x',X_{n+1}}"] \arrow[d, phantom, "(2)"] & & \pi(X_{n+1}) \\
\pi_1(X', x'_{n+1}) \arrow[ur, "(k_{n+1})_*"] \arrow[rr, "\theta_{x'_{n+1},X'}"] & & \pi(X') = \pi(X) \arrow[ur, "(k_{n+1})_*"'] &
\end{tikzcd}$$

where $c_{L_{X_{n+1}}(x';x,\gamma)}$ is a conjugation by the element $L_{X_{n+1}}(x'; x, \gamma)$. It follows from Proposition 1.4 that the square (1) commutes. Corollary 1.8 implies that (2) and (3) commutes. The square (4) commutes by the construction.

Corollary 6.2. *Let $x = (x_1, \cdots, x_{n+1}) \in X_{n+1}$. Let us set $X(i) := X(i, x)$. Let a_{ij} be the following element of $\pi_1(X(i), x_i)$- a geometric generator of $\pi_1(X(i), x_i)$, which is a loop around the point x_j. Let A_{ij} be its image in $\pi_1(X_{n+1}, x)$. Then $\theta_{x,X_{n+1}}(A_{ij})$ is conjugated to $(-2\pi i)X_{ij}$ in the group $\pi(X_{n+1})$.*

Proof. It follows from Proposition 4.2 that $\theta_{x_i,X(i)}(a_{ij})$ is conjugated to $(-2\pi i)X_{ij}$ in the group $\pi(X(i))$. Hence the statement follows from Corollary 1.8.

Now we shall study the relation between the monodromy representation for the configuration spaces $(C\backslash\{0,1\})^n_*$ and $(C\backslash\{0,1\})^m_*$. We shall use the Ihara result (see [I1] The Injectivity Theorem (i)). Let $Y_n := (P^1(C))^n_*$. The group $PGL_2(C)$ acts diagonally on Y_n and let $\mathcal{Y}_n := Y_n/PGL_2(C)$. Let $\psi_k : X_{n-1} \to \mathcal{Y}_n$ be the composition of the map $(x_1, \cdots, x_{k-1}, x_{k+1}, \cdots, x_n) \to (x_1, \cdots, x_{k-1}, \infty, x_k, \cdots, x_n)$ and the projection $Y_n \to \mathcal{Y}_n$. The map ψ_k induces $(\psi_k)_* : H(X_{n-1}) \to H(\mathcal{Y}_n)$. Let us set $X_{ij} = (\psi_k)_*(X_{ij})$ where $X_{ij} = (\frac{dx_i - dx_j}{x_i - x_j})^* \in H(X_{n-1})$. (We use the same notation for $X_{ij} \in H(X_{n-1})$ and its image in $H(\mathcal{Y}_n)$. Notice also that X_{ij} in $H(\mathcal{Y}_n)$ does not depend on the choice of ψ_k.)

Let $A_{ij} \in \pi_1(X_{n-1}, x)$ be such as in Corollary 6.2. The image of A_{ij} in $\mathcal{Y}_n$ we shall also denote by A_{ij}.

Corollary 6.3. *The element $\theta_{y,\mathcal{Y}_n}(A_{ij})$ is conjugated to $(-2\pi i)X_{ij}$ in $\pi(\mathcal{Y}_n)$.*

Proof. It follows from Corollary 6.2 and the commutative diagram

$$,\theta_{x,X_{n-1}}:\pi_1(X_{n-1},x)\to\pi(X_{n-1})\downarrow(\psi_k)_*\downarrow(\psi_k)_*\theta_{y,\mathcal{Y}_n}:\pi_1(\mathcal{Y}_n,y=\psi_k(x))\to\pi(\mathcal{Y}_n).$$

Let $\mathrm{Aut}^*(\pi(\mathcal{Y}_n))$ be a subgroup of $\mathrm{Aut}_C(\pi(\mathcal{Y}_n))$ defined in the following way:

$$\mathrm{Aut}^*(\pi(\mathcal{Y}_n))=\{f\in\mathrm{Aut}_C(\pi(\mathcal{Y}_n))|\exists\alpha_f\in C^*,f(X_{ij})\sim\alpha_f\cdot X_{ij}\}.$$

($\mathrm{Aut}_C(\ )$ dentoes C-linear automorphisms and $\sim$ means conjugated.)

Let us set

$$T^n(C)=\{\varphi\in\mathrm{Hom}(\pi_1(\mathcal{Y}_{n,y});\pi(\mathcal{Y}_n))|\exists\alpha\in C^*,\forall A_{ij},\varphi(A_{ij})\sim\alpha X_{ij}\}$$

($A_{ij}\in\pi_1(\mathcal{Y}_n,y)$ are as in Corollary 6.3.). Observe that $T^n(C)$ is an $\mathrm{Aut}^*(\pi(\mathcal{Y}_n))$-torsor. The subgroup of inner automorphisms $\mathrm{Inn}(\pi(\mathcal{Y}_n))$ is a normal subgroup of $\mathrm{Aut}^*(\pi(\mathcal{Y}_n))$. Hence $t^n(C):=T^n(C)/\mathrm{Inn}(\pi(\mathcal{Y}_n))$ is a $\mathrm{Out}^*(\pi(\mathcal{Y}_n)):=\mathrm{Aut}^*(\pi(\mathcal{Y}_n))/\mathrm{Inn}(\pi(\mathcal{Y}_n))$-torsor.

The following result is an analog of the Ihara Injectivity Theorem (see [I1] page 4).

Proposition 6.4. *The canonical map* $\mathrm{Out}^*(\pi(\mathcal{Y}_n))\to\mathrm{Out}^*(\pi(\mathcal{Y}_{n-1}))$ *is injective for* $n\geq 5$.

Proof. Let $\mathrm{Out}_1^*(\pi(\mathcal{Y}_n)):=\ker(\mathrm{Out}^*(\pi(\mathcal{Y}_n))\xrightarrow{N}C^*)$, where $N(f)=\alpha_f$. The Lie algebra of $\mathrm{Out}^*(\pi(\mathcal{Y}_n))$ is the Lie algebra of special derivations of $L(\pi(\mathcal{Y}_n))$ modullo inner derivations. The Lie version is proved in [I1] page 12. Because $\mathrm{Out}_1^*(\pi(\mathcal{Y}_n))$ is pro-nilpotent, the Lie version implies the result for $\mathrm{Out}_1^*(\pi(\mathcal{Y}_n))$, and then also for $\mathrm{Out}^*(\pi(\mathcal{Y}_n))$.

The surjective homomorphisms $(p_{n+1})_*:\pi_1(\mathcal{Y}_{n+1},y)\to\pi_1(\mathcal{Y}_n,y')$ and $(p_{n+1})_*:\pi(\mathcal{Y}_{n+1})\to\pi(\mathcal{Y}_n)$ induce the morphism of torsors

$$t^{n+1}(C)\to t^n(C)$$

compatible with $\mathrm{Out}^*(\pi(\mathcal{Y}_{n+1}))\to\mathrm{Out}^*(\pi(\mathcal{Y}_n))$.

Lemma 6.5. *The canonical morphism of torsors $t^{n+1}(C)\to t^n(C)$ is injective for $n\geq 4$.*

This follows immediately from Proposition 6.4.

Corollary 6.6. *The monodromy homomorphism $\theta_{y,\mathcal{Y}_n} : \pi_1(\mathcal{Y}_n, y) \to \pi(\mathcal{Y}_n)$ is determined (up to conjugacy by an element of $\pi(\mathcal{Y}_n)$) by the homomorphism $\theta_{y',\mathcal{Y}_4} : \pi_1(\mathcal{Y}_4, y') \to \pi(\mathcal{Y}_4)$.*

Proof. Observe that $\theta_{y,\mathcal{Y}_n} \in t^n(C)$ and $\theta_{y',\mathcal{Y}_4}$ is the image of $\theta_{y,\mathcal{Y}_n}$ under the canonical morphism $t^n(C) \to t^4(C)$.

Let $a := (a_1, \cdots, a_n, a_{n+1})$ be a sequence of $n+1$ different points in $P^1(C)$ and let $X_a := P^1(C)\backslash\{a_1, \cdots, a_n, a_{n+1}\}$. The vector space $H(X_a)$ is spanned by $X_i := (\frac{dz}{z-a_i} - \frac{dz}{z-a_{n+1}})^*$ $i = 1, \cdots, n$. Let us set $X_{n+1} := -\sum_{i=1}^n X_i$. Let A_k denotes a geometric generator of X_a, which is a loop around a_k. Let us set

$$T_a(C) := \{f \in \mathrm{Hom}(\pi_1(X_a, x) \to \pi(X_a)) | \exists \alpha_f \in C^*, \forall A_k \;\; f(A_k) \sim \alpha_f X_k\}.$$

Assume that $a = (a_i)_{i=1}^{n+1}$ is such that $a_1 = 0, a_2 = 1, a_3 = \infty$. The fibration

$$X_a \xrightarrow{k_{n+2}} \mathcal{Y}_{n+2} \xrightarrow{p_{n+2}} \mathcal{Y}_{n+1} \qquad (X_a = (p_{n+2})^{-1}(a_1, \cdots, a_{n+1}))$$

realizes $\pi(X_a)$ as a normal subgroup of $\pi(\mathcal{Y}_{n+2})$ ($(k_{n+2})_*(X_i) = X_{i,n+2}$). Hence the group $\pi(\mathcal{Y}_{n+2})$ acts on $T_a(C)$ and let

$$t_a(C) := T_a(C)/\pi(\mathcal{Y}_{n+2}).$$

Observe that any $\pi(\mathcal{Y}_{n+2})$-conjugate of $X_{i,n+2}$ is in the image of $\pi(X_a)$. Hence the restriction map

$$(k_{n+2})^* : t^{n+2}(C) \to t_a(C)$$

given by $f \to f_{|\pi_1(X_a,x)}$ is defined. We set

$$\tau_a(C) := \mathrm{im}\; (t^{n+2}(C) \to t_a(C)).$$

Observe that the diagram

$$\begin{array}{ccc} t^{n+2}(C) & \xrightarrow{(k_{n+2})^*} & \tau_a(C) \\ \downarrow pr & & \downarrow pr_1 \\ t^4(C) & \xrightarrow[\approx]{(k_4)_*} & \tau_{0,1,\infty}(C) \end{array}$$

commutes where the map pr_1 is induced by the inclusion $X_a \hookrightarrow P^1(C)\backslash\{0,1,\infty\}$. The map $(k_4)_*$ is bijective because $\mathcal{Y}_4 = P^1(C)\backslash\{0,1,\infty\}$. Lemma 6.5 implies that the map pr is injective. Hence both maps, $(k_{n+2})^*$ and pr_1 are injective. Hence we have proved the following result.

Proposition 6.7. *i) The $\pi(\mathcal{Y}_{n+2})$-conjugacy class of the monodromy homomorphism $\theta_{x,\mathcal{Y}_{n+2}}: \pi_1(\mathcal{Y}_{n+2},x) \to \pi(\mathcal{Y}_{n+2})$ is determined by its restriction to $\pi_1(X_a,x')$.*
ii) The $\pi(\mathcal{Y}_{n+2})$-conjugacy class of the monodromy homomorphism $\theta_{x,X_a}: \pi_1(X_a,x) \to \pi(X_a)$ is determined by the monodromy homomorphism

$$\theta_{x',P^1(C)\backslash\{0,1,\infty\}}: \pi_1(P^1(C)\backslash\{0,1,\infty\},x') \to \pi(P^1(C)\backslash\{0,1,\infty\}).$$

§7. The Drinfeld-Ihara $Z/5$-cycle relation

In this section we show that the element which describes the monodromy of all iterated integrals on $P^1(C)\setminus\{0,1,\infty\}$ satisfies the Drinfeld-Ihara relation.

7.1. Configuration spaces

If T is a topological space we set $T^n_* = \{(t_1,\cdots,t_n) \in T^n \mid t_i \neq t_j \text{ if } i \neq j\}$. The group Σ^n acts on T^n_* by permutations.

Let us set $Y_n = \left(P^1(C)\right)^n_*$ and $\mathcal{Y}_n = \left(P^1(C)\backslash\{0,1,\infty\}\right)^{n-3}_*$. Let $a,b,c \in P^1(C)$ be three different points and let $\varphi_{a,b,c}(z) = \frac{b-c}{b-a}\cdot\frac{z-a}{z-c}$. The map $\Phi_{4,5}: Y_5 \to \mathcal{Y}_5$ given by $\Phi_{4,5}(x_1,x_2,x_3,x_4,x_5) = \left(\varphi_{x_1,x_2,x_3}(x_4), \varphi_{x_1,x_2,x_3}(x_5)\right)$ induces a bijection

$$\varphi_{4,5}: Y_5/\mathrm{PGL}_2(C) \to \mathcal{Y}_5.$$

The action of Σ_5 on Y_5 induces an action of Σ_5 on $\mathcal{Y}_5$. The map $\sigma: \mathcal{Y}_5 \to \mathcal{Y}_5$, $\sigma(s,t) = \left(\frac{t-1}{t-s}, \frac{1}{s}\right)$ corresponds to the permutation $\tilde{\sigma}$ of Y_s given by

$$\tilde{\sigma}(x_1,x_2,x_3,x_4,x_5) = (x_2,x_3,x_4,x_5,x_1).$$

Observe that the points $A = \left(\frac{\sqrt{5}-1}{2}, \frac{\sqrt{5}+1}{2}\right) \in \mathcal{Y}_5$ and $B = \left(\frac{-\sqrt{5}-1}{2}, \frac{-\sqrt{5}+1}{2}\right) \in \mathcal{Y}_5$ are fixed by σ.

The one-forms $\frac{ds}{s}, \frac{ds}{s-1}, \frac{dt}{t}, \frac{dt}{t-1}, \frac{ds-dt}{s-t}$ generate $A^1(\mathcal{Y}_5)$ and $H^1_{\mathrm{DR}}(\mathcal{Y}_5)$. Let S_0, S_1, T_0, T_1 and N be their formal duals. The subspace $R(\mathcal{Y}_5)$ of $H(\mathcal{Y}_5)^{\otimes 2}$ is generated by

$$[S_i,N]+[T_i,N] \quad i=0,1;$$

$$[S_i,T_i]+[S_i,N] \quad i=0,1;$$

$$[T_i,S_i]+[T_i,N] \quad i=0,1;$$

$$[S_0,T_1] \quad \text{and} \quad [S_1,T_0]$$

where $[A,B] = A\otimes B - B\otimes A$.

Let $G := P(\mathcal{Y}_5)$ i.e. G is a multiplicative group of the algebra of formal power series in non-commuting variables S_0, S_1, T_0, T_1 and N divided by the ideal generated by $R(\mathcal{Y}_5)$.

The principal fibration

$$\mathcal{Y}_5 \times G \to \mathcal{Y}_5$$

we equipped with the integrable connection given by the one form

$$\begin{aligned}\omega_{\mathcal{Y}_5} = &\Big(\frac{dt}{t-1} - \frac{dt}{t}\Big)\otimes T_1 + \Big(-\frac{dt}{t}\Big)\otimes T_\infty \\ &+ \Big(\frac{ds-dt}{s-t} - \frac{dt}{t}\Big)\otimes N + \frac{ds}{s}\otimes S_0 + \frac{ds}{s-1}\otimes S_1\end{aligned}$$

where $T_\infty = -T_0 - T_1 - N$. We shall write shortly ω instead of $\omega_{\mathcal{Y}_5}$.

7.2. Integration of w

We recall that on $P^1(C)\backslash\{0,1,\infty\}$ we have

7.2.0 $$\Lambda_{\overrightarrow{\infty 1}}(z)\cdot a_{\overrightarrow{1\infty}}^{\overrightarrow{\infty 1}} = \Lambda_{\overrightarrow{1\infty}}(z) \quad \text{(see Proposition 4.1)}.$$

The monodromy of $\Lambda_{\overrightarrow{\infty 1}}(z)$ is given by:

$$\begin{aligned}&(\text{around } \infty): \Lambda_{\overrightarrow{\infty 1}}(z) \to \Lambda_{\overrightarrow{\infty 1}}(z)\cdot e^{-2\pi i T_\infty};\\ &(\text{around } 1): \Lambda_{\overrightarrow{\infty 1}}(z) \to \Lambda_{\overrightarrow{\infty 1}}(z)\cdot a_{1\infty}^{\infty 1}\cdot e^{-2\pi i T_1}\cdot (a_{1\infty}^{\infty 1})^{-1},\end{aligned}$$

(see Theorem 4.3). We have $f_*\big(a_{10}^{01}(T_0,T_1)\big) = a_{1\infty}^{\infty 1}(T_\infty, T_1)$ where $f_*(T_0) = T_\infty$, $f_*(T_1) = T_1$ and $f(z) = {}^1\!/\!_z$.

We have assymptotically at ∞

7.2.1 $$\Lambda_{\overrightarrow{\infty 1}}(z) \underset{z=\infty}{\sim} e^{(\int_1^z \frac{dt}{t})T_\infty}$$

$$\text{i.e. } \lim_{\substack{z\to\infty\\ z>1}} \big(\Lambda_{\overrightarrow{\infty 1}}(z)\cdot e^{-(\int_1^z \frac{dt}{t})T_\infty}\big) = 1.$$

Let $P_\varepsilon = (\varepsilon, 1+\varepsilon) \in \mathcal{Y}_5$ where $\varepsilon > 0$ and small. Let $\Lambda_{P_\varepsilon}(z;\text{ path})$ be a horizontal section of ω such that $\Lambda_{P_\varepsilon}(P_\varepsilon) = 1$. Let γ be a path in $\mathcal{Y}_5$ from P_ε to $\sigma(P_\varepsilon) = (\varepsilon, {}^1\!/\!_\varepsilon)$ which is constant ($=\varepsilon$) on the first coordinate.

Assuming $s = \text{constant}\ (= \varepsilon)$ we have

$$\Lambda_{P_\varepsilon}(z) \cdot a^{1+\varepsilon}_{\overrightarrow{1\infty}} = \Lambda_{\overrightarrow{1\infty}}(z).$$

Hence we have assymptotically for positive, small ε

7.2.2 $$a^{1+\varepsilon}_{\overrightarrow{1\infty}} \underset{\varepsilon=0}{\sim} e^{\left(-\int_\infty^{1+\varepsilon}\left(\frac{dt}{t-1}-\frac{dt}{t}\right)T_1\right)}.$$

It follows from 7.2.0, 7.2.1 and 7.2.2 and Proposition 4.1 that for ε positive, near 0

7.2.3 $$\Lambda_{P_\varepsilon}(\sigma(P_\varepsilon);\gamma) \underset{\varepsilon=0}{\sim} e^{\left(\int_1^{1/\varepsilon}\frac{dz}{z}\right)T_\infty} \cdot a^{\infty 1}_{1\infty}(T_\infty, T_1) \cdot e^{\left(\int_\infty^{1+\varepsilon}\frac{dt}{t-1}-\frac{dt}{t}\right)T_1}.$$

Let $p = \gamma + \sigma(\gamma) + \sigma^2(\gamma) + \sigma^3(\gamma) + \sigma^4(\gamma)$. Then $\Lambda_{P_\varepsilon}(P_\varepsilon; p) = 1$ because the path p is contractible in $\mathcal{Y}_5$. On the other hand

$$1 = \Lambda_{P_\varepsilon}(P_\varepsilon, p) = \Lambda_{\sigma^4(P_\varepsilon)}\big(P_\varepsilon; \sigma^4(\gamma)\big) \cdot \Lambda_{\sigma^3(P_\varepsilon)}\big(\sigma^4(P_\varepsilon); \sigma^3(\gamma)\big)$$
$$\cdot \Lambda_{\sigma^2(P_\varepsilon)}\big(\sigma^3(P_\varepsilon); \sigma^2(\gamma)\big) \cdot \Lambda_{\sigma(P_\varepsilon)}\big(\sigma^2(P_\varepsilon); \sigma(\gamma)\big) \cdot \Lambda_{P_\varepsilon}\big(\sigma(P_\varepsilon); \gamma\big).$$

The formula

$$(\sigma^i)_*\big(\Lambda_{P_\varepsilon}(\sigma(P_\varepsilon), \gamma)\big) = \Lambda_{\sigma^i(P_\varepsilon)}\big(\sigma^{i+1}(P_\varepsilon), \sigma^i(\gamma)\big)$$

(see Corollary 1.8) implies that

$$1 = \sigma^4_*(L) \cdot \sigma^3_*(L) \cdot \sigma^2_*(L) \cdot \sigma_*(L) \cdot L$$

where $L = \Lambda_{P_\varepsilon}(\sigma(P_\varepsilon), \gamma)$. Let

$$\text{Ł} = e^{\left(\int_1^{1/\varepsilon}\frac{dz}{z}\right)T_\infty} \cdot a^{\infty 1}_{1\infty}(T_\infty, T_1) \cdot e^{\left(\int_\infty^{1+\varepsilon}\frac{dt}{t-1}-\frac{dt}{t}\right)T_1}.$$

It follows from 7.2.3 that

$$1 \underset{\varepsilon=0}{\sim} \sigma^4_*(\text{Ł}) \cdot \sigma^3_*(\text{Ł}) \cdot \sigma^2_*(\text{Ł}) \cdot \sigma_*(\text{Ł}) \cdot \text{Ł}.$$

The factors $e^{\int_{\infty}^{1+\varepsilon}\left(\frac{dt}{t-1}-\frac{dt}{t}\right)T_\infty\left(=\sigma_*^2(T_1)\right)}$ and $e^{\left(\int_1^{1/\varepsilon}\frac{dz}{z}\right)T_\infty}$ can be placed together in the product $\sigma_*^4(\mathrm{L})\cdot\ldots\cdot\mathrm{L}$ because $T_\infty = \sigma_*^2(T_1)$ commutes with $\sigma_*(T_1) = S_0$ and $\sigma_*(T_\infty) = S_1$. After the calculations we get

$$\int_{\infty}^{1+\varepsilon}\left(\frac{dt}{t-1}-\frac{dt}{t}\right)-\int_1^{1/\varepsilon}\frac{dt}{t} = -\log(1+\varepsilon).$$

Repeating the same argument for S_1, S_1+T_1+N, T_1 and S_0 and passing to the limit with ε we get

$$\sigma_*^4(\mathsf{a})\cdot\sigma_*^3(\mathsf{a})\cdot\sigma_*^2(\mathsf{a})\cdot\sigma_*(\mathsf{a})\cdot\mathsf{a} = 1$$

where $\mathsf{a} = a_{1\infty}^{\infty 1}(T_\infty, T_1)$. The last formula we can write in the form

$$\mathsf{a}(S_0, S_1+T_1+N)\cdot\mathsf{a}(T_1,S_1)\cdot\mathsf{a}(S_1+T_1+N,T_\infty)\cdot\mathsf{a}(S_1,S_0)\cdot\mathsf{a}(T_\infty,T_1) = 1$$

because $\sigma_*(S_0) = T_\infty$, $\sigma_*(S_1) = S_1+T_1+N$, $\sigma_*(T_0) = N$, $\sigma_*(T_1) = S_0$ and $\sigma_*(N) = -S_0-S_1-N$.

Let $\psi_5 : C_*^4 \to \mathcal{Y}_5$ be given by $\psi_5(z_1,z_2,z_3,z_4) = \Phi_{4,5}(z_1,z_2,z_3,z_4,\infty)$. Let $(A_{ij})_{i,j}$ be formal duals of $\left(\frac{dz_i-dz_j}{z_i-z_j}\right)_{i,j}$. Then we have

$$\psi_{5*}(A_{12}) = -S_0-S_1-T_0-T_1-N,$$

$$\psi_{5*}(A_{13}) = S_1+T_1+N,$$

$$\psi_{5*}(A_{14}) = S_0,$$

$$\psi_{5*}(A_{23}) = S_0+T_0+N,$$

$$\psi_{5*}(A_{24}) = S_1,$$

$$\psi_{5*}(A_{34}) = -S_0-S_1-N.$$

Using $\psi_1 : C_*^4 \to \mathcal{Y}_5$ given by $\psi_1(z_2,z_3,z_4,z_5) = \Phi_{4,5}(\infty,z_2,z_3,z_4,z_5)$ we get

$$\psi_{1*}(A_{23}) = S_0+T_0+N,$$

$$\psi_{1*}(A_{24}) = S_1,$$

$$\psi_{1*}(A_{25}) = T_1,$$

$$\psi_{1*}(A_{34}) = -S_0-S_1-N,$$

$$\psi_{1*}(A_{35}) = -T_0-T_1-N,$$

$$\psi_{1*}(A_{45}) = N.$$

We set $X_{ij} := \psi_{\varepsilon *}(A_{ij})$ $\varepsilon = 1, 5$. Then $X_{15} = T_0$. Hence finally we get a formula

7.2.4 $\quad \mathsf{a}(X_{14}, X_{13}) \cdot \mathsf{a}(X_{25}, X_{24}) \cdot \mathsf{a}(X_{13}, X_{35}) \cdot \mathsf{a}(X_{24}, X_{14}) \cdot \mathsf{a}(X_{35}, X_{25}) = 1.$

If we use $\Phi_{2,4} : X_*^5 \to \mathcal{Y}_5$ given by $\Phi_{2,4}(0, s, 1, t, \infty) = (s, t)$ and repeat the calculations in $\mathcal{Y}_5$ we get the same formula as before, but the X_{ij}'s names of $S_0, S_1, \ldots$ are now different and the resulting formula is:

7.2.5 $\quad \mathsf{a}(X_{12}, X_{15}) \cdot \mathsf{a}(X_{34}, X_{23}) \cdot \mathsf{a}(X_{15}, X_{45}) \cdot \mathsf{a}(X_{23}, X_{12}) \cdot \mathsf{a}(X_{45}, X_{34}) = 1.$

This is exactly the formula which appears in [I2] page 106 if we replace a( ) by a( )$^{-1}$.

Proposition 7.3. *For any permutation σ of five letters we have*

i)
$$\mathsf{a}(X_{\sigma(14)}, X_{\sigma(13)}) \cdot \mathsf{a}(X_{\sigma(25)}, X_{\sigma(24)}) \cdot \mathsf{a}(X_{\sigma(13)}, X_{\sigma(35)}) \cdot \mathsf{a}(X_{\sigma(24)}, X_{\sigma(14)}) \cdot \mathsf{a}(X_{\sigma(35)}, X_{\sigma(25)}) = 1,$$

ii)
$$\mathsf{a}(X_{\sigma(12)}, X_{\sigma(15)}) \cdot \mathsf{a}(X_{\sigma(34)}, X_{\sigma(23)}) \cdot \mathsf{a}(X_{\sigma(15)}, X_{\sigma(45)}) \cdot \mathsf{a}(X_{\sigma(23)}, X_{\sigma(12)}) \cdot \mathsf{a}(X_{\sigma(45)}, X_{\sigma(34)}) = 1,$$

where $\sigma(ij) = \sigma(i)\sigma(j)$.

Proof. It follows from 7.2.4, 7.2.5 and Corollary 1.8.

Remark. The formulas of Proposition 7.3 are in the group $P(\mathcal{Y}_5)$. If we apply log we get formulas in the group $\pi(\mathcal{Y}_5)$.

In the sequel we shall work in the group $\pi(\mathcal{Y}_5)$.

We finish this section with a formula from which the Deligne $Z/3$-cycle relation can be obtained. The proof is an imitation of the Deligne proof.

Proposition 7.4. *Let $\alpha :=$* loga. *In the group $\pi(\mathcal{Y}_5)$ we have*

i) $\quad \alpha(X_{25}, X_{23})(-\pi i X_{23})\alpha(X_{23}, X_{35})(-\pi i X_{35})\alpha(X_{35}, X_{25})(-\pi i X_{25}) = -\pi i X_{14}$

and

ii) $\quad \alpha(X_{25}, X_{23})(\pi i X_{23})\alpha(X_{23}, X_{35})(\pi i X_{35})\alpha(X_{35}, X_{25})(\pi i X_{25}) = \pi i X_{14}.$

Proof. Let $\tilde{\sigma}(x_1, x_2, x_3, x_4, x_5) = (x_1, x_5, x_2, x_4, x_3)$. Then the induced map $\sigma : \mathcal{Y}_5 \to \mathcal{Y}_5$ is given by $\sigma(s,t) = (\frac{t-1}{t} : \frac{s}{s-1}, \frac{t-1}{t})$ and $\sigma^2(s,t) = (\frac{s}{s-t}, \frac{1}{1-t})$. Let $P_- = (r, 1-r)$ and $P_+ = (r, 1+r)$ where r is positive and small. Let $Q_- = (-r, 1-r)$ and $Q_+ = (-r, 1+r)$. Let γ be a path from $P_+ = (r, 1+r)$ to $\sigma^2(Q_-) = (r, 1/r)$, which is constant on the first coordinate. Let γ' be a path from Q_+ to $\sigma^2(P_-)$ passing through the point $(\frac{r}{2r-1}, 1+r)$ which is piecewise constant, first on the second coordinate, next on the first coordinate.

Let S be a path $[0,\pi] \ni \varphi \to (r, 1 + re^{i(\varphi+\pi)})$ and let S' be a path $[0,\pi] \ni \varphi \to (-r, 1 + re^{i(\varphi+\pi)})$. Let us consider the composition $p = \sigma(\gamma') \circ \sigma(S') \circ \sigma^2(\gamma) \circ \sigma^2(S) \circ \gamma' \circ S' \circ \sigma(\gamma) \circ \sigma(S) \circ \sigma^2(\gamma') \circ \sigma^2(S') \circ \gamma \circ S$. If we integrate the form ω along this path and pass to the limit if $r \to 0$ we get the square of the left hand side of the expression i).

Let α be a loop in the opposite clockwise direction around $(0,0)$ in the plane $P = \{(s,t) \in C^2 \mid \alpha s + \beta t = 0\}$. The integration of the form ω along α gives $(-2\pi i)(S_0 + N + T_0) = (-2\pi i)X_{23}$. In the model of $Y_*^5/PGL_2(C)$ in which the subspace $\{(x_1, x_2, x_3, x_4, x_5) \mid x_1 = x_4\}$ of $(P^1(C))^5$ degenerates to a point (for example for $\Phi_{2,5}(0, s, 1, \infty, t) = (s,t)$), the path p is homotopic to a loop around one of the points $(0,0)$, $(1,1)$ or (∞,∞) in the plane passing through the corresponding point $(0,0)$, $(1,1)$ or (∞,∞) (the point $(1,1)$ in the case of the model $\Phi_{2,5}$). Hence the left hand side of the expression i) is also $(-2\pi i) \cdot X_{14}$. The proof of the second equality is similar.

Corollary 7.5. *For any permutation σ of five letters $1,2,3,4,5$ we have formulas i') and ii'), which are obtained from formulas i) and ii) by replacing indices $1,2,3,4,5$ by $\sigma(1)$, $\sigma(2)$, $\sigma(3)$, $\sigma(4)$, $\sigma(5)$.*

Proof. One consider the map of Y_5 given by $(x_i)_{i=1,\ldots,5} \to (x_{\sigma(i)})_{i=1,\ldots,5}$. The induced map $\sigma : \mathcal{Y}_5 \to \mathcal{Y}_5$ satisfies $\sigma^*\omega = \sigma_*\omega$. This implies formulas i') and ii').

Remark. We have $X_{23} + X_{25} + X_{35} = X_{14}$ in the Lie algebra Lie($\mathcal{Y}_5$). If we set $X_{14} = 0$ then the formulas i) and ii) reduce to the Deligne formula.

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