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ON THE RATIONALITY OF THE DETERMINANT OF PEROID INTEGRALS

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§0. INTRODUCTION

In this report we will explain some results for the rationality of the determinant of periods for local systems. Let me explain the motivation of the problem.

Let k be an algebraic number field in the complex number field $\mathbf{C}$ and let X be an algebraic variety defined over k. By de Rham's theorem and Grothendieck's comparison theorem of algebraic and analytic de Rham cohomologies, we have a functorial isomorphism between the de Rham cohomology and the singular cohomology of X after tensoring with $\mathbf{C}$:

$$H^i_{DR}(X/k) \otimes_k \mathbf{C} \simeq H^i_B(X, \mathbf{Q}) \otimes_{\mathbf{Q}} \mathbf{C}.$$

The space $H^i_{DR}(X/k) \otimes_k \mathbf{C}$ has a natural k-structure and $H^i_B(X, \mathbf{Q}) \otimes_{\mathbf{Q}} \mathbf{C}$ has a natural $\mathbf{Q}$-structure. The matrix of changing basis with respect to these two basis is called the period matrix. In this report we are interested in the determinant of this matrix. But it is easy to see that the square of this matrix is contained in the power of $2\pi i$ times some rational numbers. To get much interesting quantity, we will consider the period matrix for local systems. We will explain important examples.

Example 0 Let X and k be as above and suppose that there is an μ_d action on X defined over k. Let χ be the character of μ_d and $H^i_{DR}(X/k)(\chi)$ and $H^i_B(X, \mathbf{Q}(\mu_d))(\chi)$ be the χ part of the cohomologies $H^i_{DR}(X/k)$ and $H^i_B(X, \mathbf{Q}(\mu_d))$ respectively. Since the above comparison isomorphism is functorial, we have an identity:

$$H^i_{DR}(X/k)(\chi) \otimes_K \mathbf{C} \simeq H^i_B(X, \mathbf{Q})(\chi) \otimes_{\mathbf{Q}} \mathbf{C}.$$

If we take bases $\{v_1, \dots, v_b\}$ and $\{e_1, \dots, e_b\}$ of $H^i_{DR}(X/k)(\chi)$ and $H^i_B(X, \mathbf{Q}(\mu_d))(\chi)$ respectively, then we have $v_i = \sum_j a_{i,j} e_j$ and the determinant $det(a_{i,j})$ is a non-trivial transcendental number. To illustrate this fact, we will see the following most easiest case.

Example 1 Let d be a positive interger and a, b be integers such that $0 < a, b < d$ and $a + b$ is not divisible by d. Let $k = \mathbf{Q}(\zeta_d)$ and X be the curve over k corresponding to the function field $\mathbf{Q}(\zeta_d, x, y \mid y^d = x^a(1-x)^b)$. The action of μ_d is defined by the multiplication to the y-coordinate. We define the caracter χ as the natural inclusion of μ_d. In this case, bases of $H^1_{DR}(X/k)(\chi)$ and $H^1_B(X, \mathbf{Q}(\mu_d))(\chi)$ is given by the χ-projection v of the dual base of the homology $[0,1]$ and the differential form $e = x^{\frac{a}{d}-1}(1-x)^{\frac{b}{d}-1}dx$. In this case, if we evaluate both sides of the relation $v = ae$ by $[0,1]$, we get $a = \Gamma(\frac{a+b}{d})/\Gamma(\frac{a}{d})\Gamma(\frac{b}{d})$.

We prove that this determinant $det(a_{i,j})$ is a product of some algebraic number relating to the Chow group (with boundary) and some transcendental nuber comming from the Gamma factor relating to the monodromy of the local system.

Let us refer some works relating to this subject. Anderson and Loeser-Sabbah gave formulae for the determinant of twisted de Rham cohomology group of a variety X which corresponds to a Kummer character ${f_1}^{s_1} \cdots {f_n}^{s_n}$, where f_i are rational functions on X ($n = 1$ in [And] and $n \geq 1$ in [L-S]). They study the beheavior of the determinant of the twisted de Rham cohomology as funcion of s_i. The dependence of the above determinant on the parameters when the rational functions f_i and variety X are members of an algebraic family is studied in [Ter2]. In the case of arrangements of hyperplanes in a projective space, Varchenko gave an exact formula ([Var] see also [Loe].). For l-adic cohomology over finite field, T.Saito give quite general formula [S]. In the case of curve with finite monodomy coefficient see also [Ter1]. In the next section we will discuss for the period of general variation of realizations.

§1 General setting

Let k, F be subfields of the complex number field $\mathbf{C}$ and X be a projective smooth scheme over k containing U as a dense open subscheme such that the complement $D = X - U$ is a divisor with simple normal crossings. A divisor is said to have simple normal crossings if its irreducible components D_i are smooth and their intersections are transversal. An integrable connection $\nabla : \mathcal{E} \to \mathcal{E} \otimes \Omega^1_U$ is said to be regular singular along the boundary if there exists a locally free $\mathcal{O}_X$-module $\mathcal{E}_X$ and a logarithmic integrable connection $\nabla_X : \mathcal{E}_X \to \mathcal{E}_X \otimes \Omega^1_X(\log D)$ extending $(\mathcal{E}, \nabla)$. It is independent of the choice of compactification X. The complex manifold of the $\mathbf{C}$-valued points of U is denoted by U^{an} and the algebraic connection ∇ induces an analytic connection ∇^{an} on U^{an}. We consider the category $M_k(U, F)$ consisting of triples $\mathcal{M} = ((\mathcal{E}, \nabla), V, \rho)$ as follows

(1) A locally free $\mathcal{O}_U$-module $\mathcal{E}$ of finite rank with an integrable connection $\nabla : \mathcal{E} \to \mathcal{E} \otimes \Omega^1_U$ which is regular singular along the boundary.
(2) A local system V of F-vector spaces on the complex manifold U^{an}.
(3) An isomorphism $\rho : V \otimes_F \mathbf{C} \xrightarrow{\sim} Ker\nabla^{an}$ of local systems of $\mathbf{C}$-vector spaces on U^{an}.

We define the determinant of the period

$$p(H^*(U, \mathcal{M})) \in k^\times \backslash \mathbf{C}^\times / F^\times$$

for an object $\mathcal{M} \in M_k(U, F)$. Let $MPic_k(U, F)$ be the group of isomorphism class of the objects of $M_k(U, F)$ of rank 1 with respect to the tensor product. For $U = \text{Spec } k$, we identify $MPic_k(\text{Spec } k, F)$ with $k^\times \backslash \mathbf{C}^\times / F^\times$ by $[\mathcal{M}] \to \rho(v)/e$ for $\mathcal{M} \in M_k(\text{Spec } k, F)$ of rank 1 with basis $e \in \mathcal{E}$ and $v \in V$. For $\mathcal{M} \in M_k(U, F)$, we define $p(H^*(U,\mathcal{M})) \in k^\times\backslash\mathbf{C}^\times/F^\times$ as $[\det R\Gamma(U,\mathcal{M})] \in MPic_k(\mathrm{Spec}\ k, F)$ defined below. Let $DR(\mathcal{E})$ be the de Rham complex

$$[\mathcal{E} \xrightarrow{\nabla} \mathcal{E}\otimes\Omega^1_U \xrightarrow{\nabla} \mathcal{E}\otimes\Omega^2_U \xrightarrow{\nabla} \cdots \xrightarrow{\nabla} \mathcal{E}\otimes\Omega^n_U].$$

Since $H^q(U, DR(\mathcal{E}))\otimes_k \mathbf{C} \simeq H^q(U^{an}, DR(\mathcal{E})^{an})$ by GAGA, the isomorphism ρ induces $H^q(\rho) : H^q(U, DR(\mathcal{E}))\otimes_k \mathbf{C} \simeq H^q(U^{an}, V)\otimes_F \mathbf{C}$. In other words, the triple

$$H^q(U,\mathcal{M}) = (H^q(U, DR(\mathcal{E})), H^q(U^{an}, V), H^q(\rho))$$

is an object of $M_k(\mathrm{Spec}\ k, F)$. Taking the alternating tensor product of the determinant, we obtain an object $\det R\Gamma(U,\mathcal{M}) \in M_k(\mathrm{Spec}\ k, F)$:

$$\begin{aligned}\det R\Gamma(U,\mathcal{M}) =&(\otimes_q \det H^q(U, DR(\mathcal{E}))^{\otimes(-1)^q},\\ &\otimes_q \det H^q(U^{an},V)^{\otimes(-1)^q}, \otimes_q \det H^q(\rho)^{\otimes(-1)^q}).\end{aligned}$$

In this report we are interested in the period $p(H^*(U,\mathcal{M})) \in k^\times\backslash\mathbf{C}^\times/F^\times$. We define the dual object $H^*_c(U,\mathcal{M})$ and $p(H^*_c(U,\mathcal{M}))$ in the same way.

§2 The case for $\mathbf{P}^1$

In this section we investigate the case $X = \mathbf{P}^1$ explicitly. From now on we fix an embedding of k into the complex number field. For simplicily we assume that $\Sigma = \{\lambda_1,\ldots,\lambda_n\}$ is a finite subset of $\mathbf{P}^1(k)$. Let W be a vector space of dimension r over k and $e_1,\ldots,e_r$ be a basis of W. By using this basis, we identify W as a column vector $W = \{v =^t (v_1,\ldots,v_r) \mid v_i \in k\}$ by the identification $e_i =^t (0,\ldots,1,\ldots,0)$. Let $B^{(i)} \in End(W)$ $(i = 1,\ldots,n)$ be endomorphisms of W. Under the identification $W = k^r$, $B^{(i)}$ can be considered as an element of $M(r,k)$. The set of eigenvalues of $B^{(i)}$ is denoted by $Spec(B^{(i)})$ and if all the real parts of the eigenvalues of $B^{(i)}$ are positive, we write $\Re Spec(B^{(i)}) > 0$. We define an $End(W)$ valued differential form P with logarithmic poles by

$$P = \sum_{i=1}^n \frac{B^{(i)}}{x-\lambda_i}dx.$$

We write $B^{(\infty)} = -\sum_{i=1}^n B^{(i)}$. If $B^{(i)}$ satisfies the condition: The difference of two eigenvalues of $B^{(i)}$ is never an integer. (In particular $B^{(i)}$ has no multiple eigenvalue.) for all $i = 1,\ldots,n,\infty$, then P is said to be generic.

The ordinary differential equation defined as

$$(*)\qquad \frac{df}{dx} = \sum_{i=1}^n \frac{B^{(i)}}{x-\lambda_i}f$$

for a column vector $f =^t (f_1,\ldots,f_r)$ defines a local system over $\mathbf{A}^1(\mathbf{C}) - \Sigma$. This differential equation can be written also as $df = Pf$. We assume that there is an F-base for the fundamental solution of the equation (*). This local system is denoted by V. In this section we are interested in the determinant of

$$H^1(\mathbf{A}^1, j_!\mathcal{M}) = (H^1_B(\mathbf{A}^1, j_!V), H^1_{DR}(\mathbf{A}^1, j_!\mathcal{M}))$$

where $\mathcal{M}$ is an element of $M_k(\mathbf{A}^1 - \Sigma, F)$ corresponding to the F-local system of the solution of $df = Bf$.

First we define the relative homology with the local system coefficient defined by the fundamental solution of the differential equation and de Rham cohomology with compact support. In this section, we always assume that $\Re Spec(B^{(i)}) > 0$ and $\Re Spec(-B^{(\infty)}) > 0$. Let D_i be an i-dimensional ball. First let us fix a small neibourhood U_j of λ_j. An i-simplex for the local system defined by P is a pair (f, g) of C^∞ maps $f : D_i \to \mathbf{A}^1 - \Sigma$ and $g : D_i \to W \otimes \mathbf{C}$ such that $dg = (f^*P)g$ and contained in the F-structure of the solution of (*). A formal F-linear combination of i-simplices is called an i-chain and the space of i-chains is denoted by $C_i(V)$. By the usual boundary operator $\partial : C_i(V) \to C_{i-1}(V)$, $C_*(V)$ forms a complex. The linear subspace of $C_i(V)$ generated by (f, g) where the image of D_i under f is contained in $U = \cup_{j=1}^n U_j$ is denoted by $C_i(V, U)$. Since $C_*(V, U)$ is a subcomplex of $C_*(V)$, we can think the relative cohomolgy $H_i(\mathbf{A}^1; \Sigma, V)$ as the cohomology of the quotient complex $C_*(\mathbf{A}^1; U, V) = C_*(V)/C_*(V, U)$ which is independent of the choice of sufficiently small U_j. We fix a base point $b \in \mathbf{A}^1 - U$, and a path γ_j from b to a point x_j in U_j. We define a complex C_* and a quasi-isomorphism from C_* to $C_*(\mathbf{A}^1; U, V)$ as follows. Let $V_{b,F}$ be the fiber of the space of F-rational solution at b.

C_* is defined by

$$\begin{array}{ccc} C_1 \simeq \oplus_{j=1}^n V_{b,F} & \xrightarrow{\partial} & C_0 \simeq V_{b,F} \\ v = (v_1, \dots, v_n) & \longrightarrow & w = \sum_{j=1}^n v_j. \end{array}$$

For $v = (v_1, \dots, v_n)$, we define an element $i_1(v)$ in $C_1(\mathbf{A}^1; U, V)$ by $i_1(v) = \sum_{j=1}^n (\gamma_j, v_j)$, where (γ_j, v_j) is the analytic continuation of v_j along γ_j. For $w \in C_0$, $i_0(w)$ is defined as $i_0(w) = (b, w)$. Then $i_* : C_* \to C_*(\mathbf{A}^1; U, V)$ is a quasi-isomorphism.

$$\begin{array}{ccc} C_1 & \xrightarrow{i_1} & C_1(\mathbf{A}^1; U, V) \\ {\scriptstyle\partial}\big\downarrow & & {\scriptstyle\partial}\big\downarrow \\ C_0 & \xrightarrow{i_0} & C_0(\mathbf{A}^1; U, V) \end{array}$$

So we have the following proposition.

Proposition 1. *Under the above notations, i_* induces an isomporphism of cohomologies:*

$$H_1(C_*) \to H_1(\mathbf{A}^1; \Sigma, V).$$

As a consequence, the dimension of $H_1(\mathbf{A}^1; \Sigma, V)$ is $(n-1)r$ and $\delta_j(e_q) = i_1(\dots, \overset{j}{e_q}, \overset{j+1}{-e_q}, \dots)$ $(i = 1, \dots, n-1, q = 1, \dots, r)$ form a basis for $H_1(\mathbf{A}^1; \Sigma, V)$.

The 1-cycle defined by $\gamma_i(v) - \gamma_{i+1}(v)$ is called a Pochhammer path and denoted by $\delta_i(v)$.

To take pairing we will construct a base of $H^1_{DR}(\mathbf{A}^1, j_!\mathcal{M})$. This can be computed by the twisted de Rham complex defined as follows. Let us define $\mathcal{O}(W^*)_{log} = k[x] \otimes W^*$ and $\Omega^1(W^*)_{log} = (\prod_{i=1}^n \frac{1}{x-\lambda_i}) \cdot k[x]dx \otimes W^*$. Since the differential form P has only logarithmic pole, the differential $\partial_P : \mathcal{O}(W^*)_{log} \to \Omega^1(W^*)_{log}$ define by $\partial_P(f) = df + fP$ is well defined. Here we identify elements of W^* with the row vectors : $W^* = \{x = (x_1, \ldots, x_r) \mid x_i \in k\}$. The complex $\partial_P : \mathcal{O}(W^*)_{log} \mapsto \Omega^1(W^*)_{log}$ is denoted by $\mathcal{M}_P$. We define $H^1(\mathbf{A}^1, j_!\mathcal{M}_P)$ by the cokernel of ∂_P. Since $k[x]dx \subset dk[x]$ and $(\prod_{i=1}^n \frac{1}{x-\lambda_i}) \cdot k[x]dx = k[x]dx \oplus \oplus_{i=1}^n \frac{dx}{x-\lambda_i} k$, by using the relation $df \equiv -\sum_{i-1}^n \frac{dx}{x-\lambda_i} fB^{(i)}$ (mod exact form), we have the natural isomorphism

$$Coker(V^* \overset{\text{right multiplication of } P}{\longrightarrow} \oplus_{i=1}^n \frac{dx}{x-\lambda_i} \otimes V^*) \to H^1(\mathbf{A}^1, j_!\mathcal{M}_P).$$

Therefore the dimension of $H^1(\mathbf{A}^1, j_!\mathcal{M}_P)$ is $(n-1)r$. The image of $(\frac{1}{x-\lambda_i} - \frac{1}{x-\lambda_{i+1}})dx \otimes e_i^*$ in $H^1(\mathbf{A}^1, j_!\mathcal{M}_P)$ is denoted by $\omega_i(e_i^*)$. They are known to be a base of $H^1(\mathbf{A}^1, j_!\mathcal{M}_P)$. Let $1 \leq i, j \leq n-1$. The matrix

$$A_{i,j} = (\int_{\delta_j(e_q)} \omega_i(e_p^*))_{1 \leq p,q \leq r}$$

is called the (matrix valued) Pochhammer integral. It is easy to see that $A_{i,j}$ is a holomorphic but multi-valued matrix valued function of $\lambda_1, \ldots \lambda_n$. Then by the definition of the determinant of period, we have

$$p(H^1(\mathbf{A}^1, j_!\mathcal{M})) = det((A_{i,j})_{i,j}).$$

Then we have the following product formula for $p(H^1(\mathbf{A}^1, j_!\mathcal{M}_P))$ To state the product formula, we will define tame symbol and gamma factor. Let us fix a fundamental solution $f_1, \ldots, f_r$ of the equation $df = Pf$ which give the F-structure of the local system. The limit $\lim_{x \to \lambda_i} det(f_1, \ldots, f_r)/(x-\lambda_i)^{tr(B^{(i)})}$ exists by the theory of differential equation with regular singular and it is well defined as an element of $k^\times \backslash \mathbf{C}^\times / F^\times$ It is called the tame symbol and denoted by $(\mathcal{M}, x-\lambda_i)$. We define the tame symbol at infinity $(\mathcal{M}, 1/x)$ in the same way. For a matrix, we define the Gamma funcition as

$$\Gamma(B) = \det(\int_0^\infty x^B e^{-x} \frac{dx}{x}).$$

Theorem 2 (Product formula over $\mathbf{P}^1$)[Ter3]. *$p(H^1(\mathbf{A}^1, j_!\mathcal{M}_P))$ is equal to*

$$\prod_{i=1}^n (P, x-\lambda_i) \cdot (P, 1/x)^{-1} \prod_{i=1}^n \Gamma(B^{(i)}) \cdot \Gamma(-B^\infty)^{-1}.$$

In the next section, we generalize this result for an arbitrary dimension. Before going into the general result we reformulate the above result in terms of logarithmic canonical class using Weil reciprocity law. Let us take a rational differential form

ω such that ω has a simple pole at $\Sigma \cup \{\infty\}$ and $res_{\lambda_i}\omega = res_\infty \omega = 1$. (Of course, one must allow poles and zeros outside of $\Sigma \cup \{\infty\}$.) Then we have

$$\prod_{i=1}^{n}(P, x-\lambda_i)\cdot(P,1/x)^{-1} = \prod_{y\in supp(\omega+\Sigma+\infty)} det(f_1,\dots,f_r)(y)^{ord(y)}$$

In this way, first part of the product formula is closely related to the top chern class of logarithmic differential sheaf.

§3 N-DIMENSIONAL CASE (WITH T.SAITO)

In this section we will discuss about the rationalty of the determinant of the period integral in arbitrary dimension. As remarked in §1, first, we will define the relative Chow group and top chern class of $\Omega_X(logD)$ in this relative Chow group. Next we define a pairing of this relative Chow group and the group of rank 1 variation of realization. Using these notation, we give the statement of the determinant formula of period integral.

First we define the ralative Chow group and top chern class. Let X be a projective smooth scheme over a field k of dimension n and $D = \cup_{i\in I}D_i$ be a divisor with simple normal crossings. Let $\mathcal{K}_n(X)$ denote the sheaf of Quillen's K-group on X_{Zar}. Namely the Zariski sheafification of the presheaf $U \mapsto K_n(U)$. Let $\mathcal{K}_n(X \bmod D)$ be the complex $[\mathcal{K}_n(X) \to \oplus_i \mathcal{K}_n(D_i)]$. Here $\mathcal{K}_n(X)$ is put on degree 0 and $\mathcal{K}_n(D_i)$ denotes their direct image on X. It is the truncation at degree 1 of the complex $\mathcal{K}_{n,X,D}$ studied in [S] and there is a natural map $\mathcal{K}_{n,X,D} \to \mathcal{K}_n(X \bmod D)$. We call the hypercohomology $H^n(X, \mathcal{K}_n(X \bmod D))$ the relative Chow group of dimension 0 and write

$$CH^n(X \bmod D) = H^n(X, \mathcal{K}_n(X \bmod D)).$$

In this group we define the relative canonical class

$$c_{X \bmod D} = (-1)^n c_n(\Omega^1_X(\log D), res) \in CH^n(X \bmod D).$$

Let V be the covariant vector bundle associated to the locally free $\mathcal{O}_X$-module $\Omega^1_X(\log D)$ of rank n. For each irreducible component D_i, let $\Delta_i = r_i^{-1}(1)$, where $r_i : V|_{D_i} \to \mathbf{A}^1_{D_i}$ is induced by the Poincare residue $res_i : \Omega^1_X(\log D)|_{D_i} \to \mathcal{O}_{D_i}$ and $1 \subset \mathbf{A}^1$ is the 1-section. Let $\mathcal{K}_n(V \bmod \Delta)$ be the complex $[\mathcal{K}_n(V) \to \oplus_i \mathcal{K}_n(\Delta_i)]$ defined similarly as above and $\{0\} \subset V$ be the zero section. Then we have

$$\begin{array}{l} H^n_{\{0\}}(V,\mathcal{K}_n(V \bmod \Delta)) \simeq H^n_{\{0\}}(V,\mathcal{K}_n(V)) \simeq H^0(X,\mathbf{Z}) \\ \quad\downarrow \\ H^n(V,\mathcal{K}_n(V \bmod \Delta)) \simeq H^n(X,\mathcal{K}_n(X \bmod D)) = CH^n(X \bmod D) \end{array}$$

by the purity and homotopy property of K-cohomology. The relative top chern class $c_n(\Omega^1_X(\log D), res) \in CH^n(X \bmod D)$ is defined as the image of $1 \in H^0(X,\mathbf{Z})$.

Next we define a pairing

$$\begin{aligned}(\ ,\ ) : MPic_k(U,F) \otimes CH^n(X \bmod D) &\to MPic_k(\mathrm{Spec}\ k, F)\\ &\simeq k^\times \backslash \mathbf{C}^\times / F^\times,\end{aligned}$$

where $MPic_k(U,F)$ is the class group of the rank 1 objects of $M_k(U,F)$. To define the pairing, we should rewrite the relative Chow group in terms of local cohomology. By using Gersten resolution of K-group, $CH^n(X \bmod D)$ can be expressed as a cokernel of the following homomorphism:

$$Coker(\partial : \bigoplus_{y\in X_1} H_y^{n-1} \to \bigoplus_{x\in X_0} H_x^n),$$

where the groups H_y^{n-1} and H_x^n are define as follows.

(1) The group H_x^n for $x \in X_0$. It is an extension of $\mathbf{Z}$ by $\bigoplus_{i\in I_x} \kappa(x)^\times$ with the index set $I_x = \{i; x \in D_i\}$. For $i \in I_x$, let $N_i(x)$ be the one-dimensional $\kappa(x)$-vector space $\mathcal{O}_X(-D_i) \otimes \kappa(x)$. The $\kappa(x)$-algebra $\bigoplus_{m\in\mathbf{Z}} N_i(x)^{\otimes m}$ is non-canonically isomorphic to the Laurent polynomial ring $\kappa(x)[T,T^{-1}]$. We put $H_{x,i}^n = (\bigoplus_{m\in\mathbf{Z}} N_i(x)^{\otimes m})^\times$. It is an extension of $\mathbf{Z}$ by $\kappa(x)^\times$. By pulling-back $\bigoplus_{i\in I_x} H_{x,i}^n$ by the diagonal $\mathbf{Z} \to \bigoplus_i \mathbf{Z}$, we obtain H_x^n by

$$\begin{tikzcd}
0 \arrow[r] & \bigoplus_{i\in I_x} \kappa(x)^\times \arrow[r] \arrow[d, equal] & H_x^n \arrow[r] \arrow[d] & \mathbf{Z} \arrow[r] \arrow[d, "\text{diagonal}"] & 0 \\
0 \arrow[r] & \bigoplus_{i\in I_x} \kappa(x)^\times \arrow[r] & \bigoplus_{i\in I_x} H_{x,i}^n \arrow[r] & \bigoplus_{i\in I_x} \mathbf{Z} \arrow[r] & 0.
\end{tikzcd}$$

(2) The group H_y^{n-1} for $y \in X_1$. It is an extension of $\kappa(y)^\times$ by $\oplus_{i\in I_y} K_2(\kappa(y))$ with the index set $I_y = \{i; y \in D_i\}$. In the same way as above, we define an extension H_y' (resp. $H_{y,i}'$) of $\mathbf{Z}$ by $\oplus_{i\in I_y}\kappa(y)^\times$ (resp. by $\kappa(y)^\times$ for $i \in I_y$). The tensor product $H_y' \otimes \kappa(y)^\times$ is an extension of $\kappa(y)^\times$ by $\oplus_{i\in I_y}(\kappa(y)^\times \otimes \kappa(y)^\times)$. By pushing it by the symbol map $\kappa(y)^\times \otimes \kappa(y)^\times \to K_2(\kappa(y))$ we obtain H_y^{n-1} by

$$\begin{tikzcd}
0 \arrow[r] & \bigoplus_{i\in I_y}(\kappa(y)^\times \otimes \kappa(y)^\times) \arrow[r] \arrow[d, "\text{symbol}"'] & H_y' \otimes \kappa(y)^\times \arrow[r] \arrow[d] & \kappa(y)^\times \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \bigoplus_{i\in I_y} K_2(\kappa(y)) \arrow[r] & H_y^{n-1} \arrow[r] & \kappa(y)^\times \arrow[r] & 0.
\end{tikzcd}$$

(3) The homomorphism ∂. It is the direct sum of the (x,y)-component $\partial_{x,y} : H_y^{n-1} \to H_x^n$ for $x \in X_0$ and $y \in X_1$. This fits in the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \bigoplus_{i\in I_y} K_2(\kappa(y)^\times) \arrow[r] \arrow[d, "\bigoplus_{i\in I_y}(\ ,\ )_x"'] & H_y^{n-1} \arrow[r] \arrow[d, "\partial_{x,y}"] & \kappa(y)^\times \arrow[r] \arrow[d, "\mathrm{ord}_x"] & 0 \\
0 \arrow[r] & \bigoplus_{i\in I_x} \kappa(x)^\times \arrow[r] & H_x^n \arrow[r] & \mathbf{Z} \arrow[r] & 0
\end{tikzcd}$$

and is 0 unless x is not in the closure Y of $\{y\}$. Here $\mathrm{ord}_x : \kappa(y)^\times \to \mathbf{Z}$ is the usual order and $(\ ,\ )_x : K_2(\kappa(y)) \to \kappa(x)^\times$ is the tame symbol. If $\{\tilde{x}_j\}_j$ denote the inverse image of x in the normalization of Y, they are defined by $\mathrm{ord}_x(f) = \sum_j [\kappa(\tilde{x}_j) : \kappa(x)] \cdot \mathrm{ord}_{\tilde{x}_i}(f)$ and $(f,g)_x = \prod_j N_{\kappa(\tilde{x}_j)/\kappa(x)}(f,g)_{\tilde{x}_j}$ for $f,g \in \kappa(y)^\times$. Here $\mathrm{ord}_{\tilde{x}_j}$ is the valuation, $(f,g)_{\tilde{x}} = ((-1)^{\mathrm{ord}_{\tilde{x}_j}(f)\mathrm{ord}_{\tilde{x}_j}(g)} f^{\mathrm{ord}_{\tilde{x}_j}(g)} g^{-\mathrm{ord}_{\tilde{x}_j}(f)})(\tilde{x}_j)$ is the usual tame symbol and N denotes the norm.

Using this expression for $CH^n(X \bmod D)$, the definition of the required pairing is reduced to the pairing of the local pairing;

$$\begin{aligned}(\ , \ )_x : MPic_k(U,F) \otimes H^n_x \to & MPic_k(x,F) \\ & \simeq (\kappa(x) \otimes 1)^\times \backslash (\kappa(x) \otimes_k \mathbf{C})^\times / \prod_j F^\times\end{aligned}$$

This is defined using the local horizontal section for the connection. See [ST] for the details.

Under the above notation, we have the following determinant formula.

Theorem 3 [ST]. *Let $\mathcal{M}$ be the rank r object in $M_k(U,F)$. Then we have*

$$p(H^*_c(U,\mathcal{M}))/p(H^*_c(U,F))^r = (\det \mathcal{M}, c_{X \bmod D}) \times \prod_{i \in I} \Gamma(\nabla_i)^{c_i} \in k^\times \backslash \mathbf{C}^\times / F^\times,$$

where ∇_i is the residue of ∇ along D_i and c_i be the Eular characteristic of the regular part of D_i.

To prove the above theorem, take a Lefschetz pencil and use the relation of the canonical calss of the base and fiber and that of the total space. We reduce the theorem to the case of curve.

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