This is a short synopsis of my talk entitled

## On Compactification of Drinfeld Moduli Schemes

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held at the 'Mini-Conference on Moduli Spaces, Galois Representations, and $L$-functions' on March 30, 1994, at the Research Institute for the Mathematical Sciences (RIMS) at the University of Kyoto, which was organized by Professor Ihara Yasutaka (RIMS). It was a great pleasure to attend this conference and to stay at RIMS from March 26 until May 9. Details and proofs of the statements below will appear elsewhere.

For overviews of Drinfeld modules and moduli schemes see Drinfeld [2] or Deligne-Husemoller [1]. Let $X$ be a smooth projective geometrically connected curve over a finite field $\mathbb{F}_q$ with $q$ elements. We fix a closed point $\infty \in X$ and abbreviate $U := X \smallsetminus \{\infty\}$ and $A := H^0(U, \mathcal{O}_U)$. Any non-zero ideal $I \subset A$ corresponds to a finite subscheme $D_I \subset U$ of degree $\deg(I) := \dim_{\mathbb{F}_q} A/I$. We abbreviate $U_I := U \smallsetminus D_I$. The degree of an element $0 \neq a \in A$ is defined as $\deg(a) := \deg((a))$.

For any commutative $\mathbb{F}_q$-algebra $R$ the ring of $\mathbb{F}_q$-linear endomorphisms of the additive group $\mathbb{G}_{a,R}$ consists of all polynomials of the form

$$u_0 X + u_1 X^q + u_2 X^{q^2} + \ldots \in R[X].$$

Composition of endomorphisms corresponds to substitution of polynomials. Thus the identity element is the polynomial $X$, and the endomorphism ring is generated over $R$ by the element $\tau := X^q$ subject to the relations $\tau u = u^q \tau$ for all $u \in R$. We denote this ring by $R\{\tau\}$.

Consider an $\mathbb{F}_q$-algebra homomorphism $\varphi : A \to R\{\tau\}$.

**Definition.** $\varphi$ *is called a* Drinfeld module of rank $r > 0$ *if and only if for all* $0 \neq a \in A$

$$\varphi(a) = \varphi_0(a) + \varphi_1(a) \cdot \tau + \ldots + \varphi_n(a) \cdot \tau^n$$

*with* $n = r \deg(a)$ *and* $\varphi_n(a) \in R^*$.

One sees easily that the coefficient $\varphi_0$ determines an algebra homomorphism $A \to R$, and therefore a morphism $\varphi_0^* : \operatorname{Spec} R \to U$.

For any non-zero ideal $I \subset A$ the intersection

$$\ker \varphi(I) := \bigcap_{0 \neq a \in I} \ker \varphi(a)$$

is a finite flat group scheme over $R$ of degree $q^{r \deg(I)}$. In the following we assume that it is étale; this is equivalent to the assertion that $\varphi_0^*$ factors through $U_I \subset U$. As usual, a level structure is a trivialization of this étale group scheme. Taking into account the natural action of $A$ (through $\varphi$) on $\ker \varphi(I)$, this leads to the following definition.

**Definition.** *A level structure of level $I$ is an $A$-linear homomorphism*

$$\lambda : V := (A/I)^r \longrightarrow R$$

*such that $\lambda(v) \in R^*$ for all $0 \neq v \in V$.*

Given a Drinfeld module together with a level structure $(\varphi, \lambda)$, we can obtain another one by conjugating with a unit $u \in R^*$, yielding $(u\varphi u^{-1}, u\lambda)$. We say that two pairs $(\varphi, \lambda)$ are isomorphic if and only if they can be obtained from each other by this process. The following fact is well-known:

**Theorem.** *If $I \neq A$, there exists an affine scheme $M_{A,I}^r$, smooth of relative dimension $r - 1$ over $U_I$, and a bijection*

$$M_{A,I}^r(\operatorname{Spec} R) \cong \{(\varphi, \lambda) \text{ over } R \text{ up to isomorphy}\}$$

*that is functorial in $R$.*

The fact that this moduli scheme is in general not 'compact', i.e. not proper over $U_I$, can be seen easily by looking at the expansion

$$u\varphi(a)u^{-1} = \varphi_0(a) + u^{1-q}\varphi_1(a) + \ldots + u^{1-q^n}\varphi_n(a),$$

where the coefficients lie in the quotient field $K$ of a discrete valuation ring $R$. Assuming that the constant coefficient lies in $R$, we can certainly find $u \in K^*$ so that all remaining coefficients come to lie in $R$ as well. But to obtain an extension to a Drinfeld module over $R$ (i.e. one of constant rank) it is necessary that the highest coefficient be a unit in $R^*$, and in general (if $r \geq 2$) both conditions cannot be satisfied at the same time.

For certain applications to the Langlands-program for function fields, relating automorphic forms on the group $GL_r$ with $\ell$-adic representations of the absolute Galois group of the function field of $X$, it is very useful to dispose of good compactifications of this moduli space. One type of application is to the evaluation of the Lefschetz trace formula for Frobenius-Hecke-correspondences, in particular using a conjecture of Deligne. This conjecture had been proved before assuming the existence of a smooth toroidal compactification by the author [5] and independently by Shpiz [6] (with earlier results by Illusie and Zink), and has recently been proved in full generality by Fujiwara [3]. For other questions concerning local cohomology at the boundary it seems, however, still indispensable to have good explicit local and global information about a distinguished class of smooth toroidal compactifications of $M_{A,I}^r$. These matters will be explained elsewhere.

The main idea behind the following construction is to use the level structure as canonical global coordinates, with respect to which local charts for the desired compactifications can be defined and analyzed. In some ways this approach is similar to Mumford's theory of 'equations defining abelian varieties' [4] using algebraic theta-functions.

First we observe that the level structure alone determines a point of a certain moduli space, namely that of injective $\mathbb{F}_q$-linear homomorphisms from $V$ to the additive group, up to scalars. In explicit terms this moduli space is the closed subscheme

$$\Omega_V \subset T_V := \mathbb{G}_m^{V \smallsetminus \{0\}} / \operatorname{diag}(\mathbb{G}_m)$$

given by the equations $x_{v+v'} = x_v + x_{v'}$ for all $v, v' \in V \smallsetminus \{0\}$ such that $v + v' \neq 0$, and $x_{\alpha v} = \alpha x_v$ for all $v \in V \smallsetminus \{0\}$ and $\alpha \in \mathbb{F}_q^*$. We shall use equivariant embeddings of the torus $T_V$.

**Definition.** *For any fan $\Sigma \subset Y_*(T_V)_{\mathbb{R}}$ we let $\Omega_{V,\Sigma}$ be the scheme-theoretic closure of $\Omega_V$ in the torus embedding $T_{V,\Sigma}$.*

Since families of points in $\Omega_V$ can approach the boundary of $T_{V,\Sigma}$ only in certain ways, it is natural to focus attention on fans which reflect this restriction. Suppose that $\lambda$ defines a point of $\Omega_V$ over the quotient field $K$ of a discrete valuation ring $R$, with normalized valuation ord. Then the behavior of this 'family' (equivalently, the degeneration behavior of the injective homomorphism $\lambda$) is determined by the cocharacter

$$y_\lambda := (\operatorname{ord}(\lambda_v))_v \in \mathbb{Z}^{V \smallsetminus \{0\}} / \operatorname{diag}(\mathbb{Z}) = Y_*(T_V).$$

**Definition.** *An element $y \in Y_*(T_V)_{\mathbb{R}}$ is called $\mathbb{F}_q$-adapted if and only if, for every $v_0 \in V \smallsetminus \{0\}$, the subset $\{0\} \cup \{ v \in V \smallsetminus \{0\} \mid y_v \geq y_{v_0} \}$ is an $\mathbb{F}_q$-subspace of $V$. We let $C(V) \subset Y_*(T_V)_{\mathbb{R}}$ denote the set of all $\mathbb{F}_q$-adapted elements.*

It is easy to show that a cocharacter is $\mathbb{F}_q$-adapted if and only if it arises as $y_\lambda$ in the above way. The following result is elementary and not difficult.

**Theorem.** *(a) $C(V)$ is a finite union of convex rational polyhedral cones.*
*(b) For any smooth fan $\Sigma$ with support equal to $C(V)$, the scheme $\Omega_{V,\Sigma}$ is smooth and proper, and the complement $\Omega_{V,\Sigma} \smallsetminus \Omega_V$ is a union of smooth divisors with at most normal crossings.*

The method for constructing smooth toroidal compactifications of Drinfeld moduli schemes follows the same pattern as for $\Omega_V$. The global coordinates are provided by the following proposition:

**Proposition.** *If $\deg(I) \gg 0$, then the morphism*

$$M_{A,I}^r \longrightarrow U_I \times \Omega_{V_I^r}, \quad (\varphi, \lambda) \mapsto (\varphi_0^*, \lambda)$$

*is a closed embedding.*

Next there is a notion of '$A$-adapted' elements of $Y_*(T_V)_\mathbb{R}$, whose definition is too cumbersome to explain here. Some of the important properties of the set $D(V)$ of all $A$-adapted elements (which also characterize it uniquely), are:

**Theorem.** *(a)* $D(V)$ *is a subset of* $C(V)$.
*(b)* $D(V)$ *is a finite union of convex rational polyhedral cones.*
*(c) A cocharacter is $A$-adapted if and only if some positive integral multiple arises from a point of $M_{A,I}^r$ over the quotient field of some discrete valuation ring.*

In analogy to the previous simpler situation we let $M_{A,I,\Sigma}^r$ denote the scheme-theoretic closure of $M_{A,I}^r$ in $U_I \times \Omega_{V_I^r,\Sigma}$, for any fan $\Sigma$. The central result is:

**Main Theorem.** *Suppose that $\Sigma$ is a fan with support equal to $D(V)$. Then*
*(a)* $M_{A,I}^r$ *is proper over* $U_I$.
*(b) If* $\deg(I) \gg 0$ *and $\Sigma$ is smooth, then the normalization of $M_{A,I,\Sigma}^r$ has only quotient singularities, and the complement $M_{A,I,\Sigma}^r \setminus M_{A,I}^r$ is the quotient by a finite group of a union of smooth divisors with at most normal crossings.*
*(c)* **Special case:** *Suppose that $A = \mathbb{F}_q[t]$. If $\deg(I) > 0$ and $\Sigma$ is smooth, then $M_{A,I,\Sigma}^r$ is smooth and the complement $M_{A,I,\Sigma}^r \setminus M_{A,I}^r$ is a union of smooth divisors with at most normal crossings.*

Since the construction is a priori global, the bulk of the proof of the Main Theorem consists of the local analysis of the resulting scheme. The necessary considerations are intimately related to those that lead to a uniformization theorem for degenerating Drinfeld modules over higher dimensional complete local rings. In another form, the ideas explained below had originally been conceived by Fujiwara in 1991.

Let $R$ be a normal noetherian complete local integral domain with quotient field $K$. A homomorphism $\varphi : A \to R\{\tau\}$ is called a *Drinfeld module in stable reduction form* if and only if its reduction modulo the maximal ideal $\mathfrak{m}$ of $R$ defines a Drinfeld module of positive rank. (In that case, it also induces a Drinfeld module over $K$, of rank at least that of the reduction.) Via the action of $R\{\tau\}$, the homomorphism $\varphi$ defines a structure of $A$-module on $K$.

**Definition.** *A strict $\varphi$-lattice is a finitely generated $A$-submodule $M \subset K$ with the properties*
*(a)* $1/m \in R$ *for all* $0 \neq m \in M$,
*(b)* $m/m' \in R$ *or* $m'/m \in R$ *for any* $0 \neq m, m' \in M$, *and*
*(c) for any integer $i \geq 0$ there are at most finitely many $0 \neq m \in M$ with $1/m \notin \mathfrak{m}^i$.*

Here (c) is a kind of discreteness condition, (a) says that the lattice should not be too dense, and condition (b) ensures that we can work with $M$ as if the ring $R$ were a valuation ring. For any strict $\varphi$-lattice $M$ the following product converges to an $\mathbb{F}_q$-linear power series:

$$e_M(x) := x \cdot \sum_{0 \neq m \in M} \left(1 - \frac{x}{m}\right) \in R\{\{\tau\}\} \subset R[[x]].$$

This is called the *exponential function* of $M$.

**Proposition.** *If $M$ is a strict $\varphi$-lattice, there exists a unique Drinfeld module in stable reduction form $\psi : A \to R\{\tau\}$ such that*

$$\psi(a) \circ e_M = e_M \circ \varphi(a)$$

*in $R\{\{\tau\}\}$ for all $a \in A$. Moreover the rank of $\psi$ over $K$ is equal to the rank of $\varphi$ over $K$ plus the generic rank of $M$.*

One can interpret the Drinfeld module $\psi$ as the 'quotient of $\varphi$ by $M$'. When $R$ is a valuation ring this is well-known and belongs under the heading 'Tate-uniformization'. Namely, in that case let $\hat{\bar{K}}$ denote the completion of an algebraic closure of $K$. Then $e_M$ defines an analytic function on all of $\hat{\bar{K}}$ which is known to be surjective. Since $M$ is, by assumption, an $A$-submodule, the defining property of $\psi$ can be expressed by the commutative diagram with exact rows

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & M & \longrightarrow & \hat{\bar{K}} & \longrightarrow & \hat{\bar{K}} & \longrightarrow & 0 \\
 & & \downarrow \varphi(a) & & \downarrow \varphi(a) & & \downarrow \psi(a) & & \\
0 & \longrightarrow & M & \longrightarrow & \hat{\bar{K}} & \longrightarrow & \hat{\bar{K}} & \longrightarrow & 0
\end{array}
$$

Coming back to general $R$ observe that the definition of a strict $\varphi$-lattice applies to finite $A$-submodules and hence in particular to level structures. Thus a level structure $\lambda : V \to K$ of $\varphi$ over $K$ is called *strict* if and only if the image of $\lambda$ is a strict $\varphi$-lattice.

**Theorem.** *Let $\psi : A \to R\{\tau\}$ be a Drinfeld module in stable reduction form which has not good reduction, i.e. it does not define a Drinfeld module of constant rank over $R$. Suppose that $\psi$ possesses a strict level structure of level $I$, where $\deg(I) \gg 0$ (the bound depending on $A$ and the rank of $\psi$ over $K$). Then there exist a Drinfeld module in stable reduction form $\varphi : A \to R\{\tau\}$ and a strict $\varphi$-lattice $M$ such that*

$$\psi(a) \circ e_M = e_M \circ \varphi(a)$$

*for all $a \in A$ (i.e. $\psi$ is the quotient of $\varphi$ by $M$), and the rank of $\varphi$ over $K$ is strictly smaller than that of $\psi$. Moreover $\varphi$ again possesses a strict level structure of level $I$.*

Note that it is not asserted that $\varphi$ has good reduction; in fact, as Fujiwara observed, this cannot be achieved in general. But the last assertion of the theorem assures that it can be applied inductively, so that one always reaches a good reduction Drinfeld module in a finite number of steps.

This uniformization theorem leads to a good understanding of a certain class of degenerating Drinfeld modules over local rings of arbitrary dimension. The global construction of the compactification of $M^r_{A,I}$ is made such that the crucial strictness assumption holds locally on $M^r_{A,I,\Sigma}$. Thus, at least in heuristic sense, the uniformization theorem is the essential tool in the local analysis of our compactification.

# REFERENCES

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