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ON A CONJECTURE OF SHIMURA CONCERNING PERIODS OF HILBERT MODULAR FORMS

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Introduction. In this paper, we shall give an affirmative answer to an essential part of the conjecture of Shimura on P-invariants of Hilbert modular forms.

Let F be a totally real algebraic number field of degree n and J_F be the set of all isomorphisms of F into $\mathbf{C}$. Let F_A (resp. $F_A^\times$) be the adele ring (resp. the idele group) of F and $F_\infty^\times$ be the archimedean part of $F_A^\times$. Let χ be a primitive system of eigenvalues of Hecke operators which occurs in the space of holomorphic Hilbert modular cusp forms on $GL(2, F_A)$ of weight k and $\mathbf{f}$ be the primitive form which belongs to χ. In [S1], Shimura introduced an invariant $u(\epsilon, \mathbf{f}) \in \mathbf{C}^\times$ for every $\epsilon \in (\mathbf{Z}/2\mathbf{Z})^{J_F}$ such that

$$D(m, \mathbf{f}, \varphi) \sim \pi^{mn} u(\epsilon, \mathbf{f}) \tag{0}$$

for certain critical values $m \in \mathbf{Z}$ whenever a Hecke character φ of $F_A^\times$ satisfies $\varphi_\infty(x) = \prod_{\tau \in J_F} (\operatorname{sgn} x_\tau)^{m+\epsilon(\tau)}$ for $x = (x_\tau) \in F_\infty^\times$. Here $D(m, \mathbf{f}, \varphi)$ is the standard L-function attached to $\mathbf{f}$ twisted by φ and we write $a \sim b$ for a, $b \in \mathbf{C}$ if $b \neq 0$ and $a/b \in \overline{\mathbf{Q}}$. Put $U(\chi, \epsilon) = u(\epsilon, \mathbf{f})$.

In [S4], Shimura introduced another invariant $Q(\chi, \delta) \in \mathbf{C}^\times$ for every subset δ of J_F when χ occurs in the space of holomorphic automorphic forms on a quaternion algebra over F of signature $(\delta, J_F \backslash \delta)$ and showed that this invariant appears in critical values of the Rankin-Selberg convolution of two Hilbert modular forms. He conjectured further the following (Conjecture 5.12 of [S4], cf. also [S5], p. 293, (C1), (C2), (C3), (C4) and (C9))

Conjecture P. *Assume $k(\tau) \geq 2$ for all $\tau \in J_F$ and $k(\tau) \bmod 2$ is independent of τ. Put $k_0 = \max_{\tau \in J_F}(k(\tau))$. Then for every subset δ of J_F and every $\epsilon \in (\mathbf{Z}/2\mathbf{Z})^\delta$, there exists a constant $P(\chi, \delta, \epsilon) \in \mathbf{C}^\times/\overline{\mathbf{Q}}^\times$ which satisfies the following properties.*

$$\pi^{(k_0-2)n/2 - \sum_{\tau \in J_F} k(\tau)/2} U(\chi, \epsilon) \sim P(\chi, J_F, \epsilon). \tag{P1}$$

$$Q(\chi, \delta) \sim \pi^{|\delta|} P(\chi, \delta, \epsilon_1) P(\chi, \delta, \epsilon_2) \quad \text{if} \quad \epsilon_1(\tau) + \epsilon_2(\tau) \equiv 1 \bmod 2 \quad \text{for every} \quad \tau \in \delta. \tag{P2}$$

$$P(\chi, \delta_1 \cup \delta_2, \epsilon_1 \cup \epsilon_2) \sim P(\chi, \delta_1, \epsilon_1) P(\chi, \delta_2, \epsilon_2) \quad \text{if} \quad \delta_1 \cap \delta_2 = \emptyset, \quad \text{where} \quad (\epsilon_1 \cup \epsilon_2)(\tau) = \begin{cases} \epsilon_1(\tau) & \text{if} \quad \tau \in \delta_1, \\ \epsilon_2(\tau) & \text{if} \quad \tau \in \delta_2. \end{cases} \tag{P3}$$

(P4) *When χ is of CM-type, $P(\chi, \delta, \epsilon) \sim \pi^{-|\delta|} p_K(\xi, \eta)$ holds, where p_K stands for the symbol of CM-periods introduced in [S2].*

The principal result of this paper is:

Main Theorem. *Assume $k(\tau) \geq 3$ for all $\tau \in J_F$ and $k(\tau) \mod 2$ is independent of τ. Then, for every $\tau \in J_F$, there exist constants $c_\tau^\pm(\chi) \in \mathbf{C}^\times$ determined uniquely $\mod \overline{\mathbf{Q}}^\times$ such that*

(1) $$U(\chi, \epsilon) \sim \prod_{\tau \in J_F} c_\tau^{\epsilon(\tau)}(\chi),$$

(2) $$Q(\chi, \delta) \sim \pi^{(k_0-1)|\delta| - \sum_{\tau\in\delta} k(\tau)} \prod_{\tau\in\delta} c_\tau^+(\chi) c_\tau^-(\chi).$$

Here we understand that $c_\tau^0(\chi) = c_\tau^+(\chi)$, $c_\tau^1(\chi) = c_\tau^-(\chi)$ identifying $\mathbf{Z}/2\mathbf{Z}$ with $\{0,1\}$. By this theorem, it is clear that $P(\chi, \delta, \epsilon)$ satisfying (P1) $\sim$ (P3) is given by

(3) $$P(\chi, \delta, \epsilon) \sim \pi^{(k_0-2)|\delta|/2} \pi^{-\sum_{\tau\in\delta} k(\tau)/2} \prod_{\tau\in\delta} c_\tau^{\epsilon(\tau)}(\chi).$$

We note that in [Y], §6, we have defined $Q(\chi, \delta) \mod \overline{\mathbf{Q}}^\times$ assuming only $k(\tau) \geq 3$ for all $\tau \in \delta$.

Let us now outline our ideas of the proof and contents of each section. In §1, we shall review known properties of two basic period invariants $Q(\chi, \delta)$ and $U(\chi, \epsilon)$. In §2, Lemma 1, we shall show that a necessary and sufficient condition for the existence $c_\tau^\pm(\chi)$ as in Main Theorem is the following relations (R1) $\sim$ (R3).

(R1) $$U(\chi, \epsilon_1) U(\chi, \epsilon_2) \sim \pi^{n(1-k_0) + \sum_{\tau\in J_F} k(\tau)} Q(\chi, J_F) \quad \text{if} \quad \epsilon_1(\tau) + \epsilon_1(\tau) \equiv 1 \mod 2 \quad \text{for every } \tau.$$

(R2) $$Q(\chi, \delta_1) Q(\chi, \delta_2) \sim Q(\chi, \delta_1 \cup \delta_2) \quad \text{if} \quad \delta_1 \cap \delta_2 = \emptyset.$$

(R3) $$U(\chi, \epsilon_1) U(\chi, \epsilon_2) \sim U(\chi, \mu_1) U(\chi, \mu_2) \quad \text{if} \quad \{\epsilon_1(\tau), \epsilon_2(\tau)\} = \{\mu_1(\tau), \mu_2(\tau)\} \quad \text{for every } \tau.$$

We shall also prove (P4) in §2.

Now (R1) is already proved in [S1], Theorem 4.3. Harris [Ha3] proved (R2) under certain conditions, in particular when n, $|\delta_1|$ and $|\delta_2|$ are all even. In §3, using a base change lift of χ to a totally real quadratic extension of F, we shall remove this parity condition and obtain (R2) (Theorem 2).

In §4, we shall prove (R3). By (0), we see that (R3) follows if

(4) $$D(m, \mathbf{f}, \varphi_1) D(m, \mathbf{f}, \varphi_2) \sim D(m, \mathbf{f}, \psi_1) D(m, \mathbf{f}, \psi_2)$$

holds for one choice of a non-vanishing critical value m and of Hecke characters φ_1, φ_2, ψ_1, ψ_2 of $F_A^\times$ whose infinity types correspond to ϵ_1, ϵ_2, μ_1, μ_2 respectively. Let K be a quadratic extension of F such that the Hecke character η of $F_A^\times$ corresponding to K/F satisfies $\eta_\infty = (\varphi_1\varphi_2)_\infty = (\psi_1\psi_2)_\infty$. Again by (0), (4) reduces to

$$(5) \qquad D(m, \tilde{\mathbf{f}}, \varphi_1 \circ N_{K/F}) \sim D(m, \tilde{\mathbf{f}}, \psi_1 \circ N_{K/F}),$$

where $\tilde{\mathbf{f}}$ is the base change lift of $\mathbf{f}$ to K. By our choice of K, $(\varphi_1 \circ N_{K/F})_\infty = (\psi_1 \circ N_{K/F})_\infty$ holds and we obtain (5) from a result of Hida [Hi] (§4, Theorem 3).

In §5, we shall prove the invariance of $c_\tau^\pm(\chi)$ under the base change of χ to a totally real cyclic extension of F (Theorem 4). In §6, we shall discuss a possible generalization of Main Theorem including the case where $k(\tau) = 2$ for some τ.

Notation. Throughout the paper, we fix an algebraic closure $\overline{\mathbf{Q}}$ of $\mathbf{Q}$ as the subfield of $\mathbf{C}$. A finite extension of $\mathbf{Q}$ in $\overline{\mathbf{Q}}$ will be called an algebraic number field. For an algebraic number field F, F_v denotes the completion of F at a place v, J_F the set of all isomorphisms of F into $\mathbf{C}$ and I_F the free abelian group generated by J_F. We denote by $\mathfrak{a}_r^F$ (resp. $\mathfrak{a}_c^F$) the set of all real (resp. complex) archimedean places of F and put $\mathfrak{a}^F = \mathfrak{a}_r^F \cup \mathfrak{a}_c^F$. We shall drop the superscript F when the reference to F is clear from the context. When F is totally real, we identify $\mathfrak{a}^F$ with J_F; a totally imaginary quadratic extension of F will be called a CM-extension of F.

For an algebraic group G defined over F, G_A denotes the adelization of G, G_∞ the archimedean part of G_A and $G_{\infty+}$ the identity component of G_∞. For $x \in F_A^\times$, $|x|_A$ denotes the idele norm of x. For an irreducible automorphic representation $\pi = \otimes_v \pi_v$ of $GL(2, F_A)$, $L_f(s,\pi) = \prod_v L(s, \pi_v)$, v extending over all finite places, denotes the finite part of the Jacquet-Langlands L-function attached to π. For a, $b \in \mathbf{C}$, we denote $a \sim b$ if $b \neq 0$ and $a/b \in \overline{\mathbf{Q}}$.

§1. Review on Q-invariants and U-invariants

Let F be a totally real algebraic number field of degree n. Let B be a quaternion algebra over F such that B splits (resp. ramifies) at the archimedean places $\tau \in \delta$ (resp. δ'). We call such a B a quaternion algebra of *signature* (δ, δ'). We assume $\delta \neq \emptyset$. Put $G = \mathrm{Res}_{F/\mathbf{Q}}(B^\times)$ and call Z the center of G. We identify Z_A with $F_A^\times$. For $k = \sum_{\tau\in\delta} k(\tau)\tau$ and $\kappa = \sum_{\kappa\in\delta'} \kappa(\tau)\tau \in I_F$, we define the space of cusp forms $\mathcal{S}_{k,\kappa}(B)$ on G_A of weight (k, κ) as in [S3], II, [Y], §6.

For $\mathbf{f}$, $\mathbf{g} \in \mathcal{S}_{k,\kappa}(B)$, we define the inner product

$$(1.1) \qquad \langle \mathbf{f}, \mathbf{g} \rangle = \int_{Z_{\infty+}G_{\mathbf{Q}}\backslash G_A} {}^t\overline{\mathbf{f}(x)}\mathbf{g}(x)\, dx$$

normalizing the invariant measure so that $\mathrm{vol}(Z_{\infty+}G_{\mathbf{Q}}\backslash G_A) = 1$. If there exists $0 \neq \mathbf{f} \in \mathcal{S}_{k,\kappa}(B)$ and a Hecke character ψ of $F_A^\times$ of finite order such that

$$(1.2) \qquad \mathbf{f}|T(\mathfrak{p}) = \chi(\mathfrak{p})\mathbf{f} \quad \text{for almost all} \quad \mathfrak{p}, \qquad \mathbf{f}(zx) = \psi(z)\mathbf{f}(x), \quad z \in Z_A,\ x \in G_A,$$

$T(\mathfrak{p})$ being the Hecke operator at the prime ideal $\mathfrak{p}$, we say that a system of eigenvalues of Hecke operators χ occurs in $\mathcal{S}_{k,\kappa}(B)$. Strictly speaking, we should say that (χ, ψ) is a system of eigenvalues of Hecke operators. For simplicity, we shall drop ψ and regard χ accompanying the central character ψ. Let $\mathcal{S}_{k,\kappa}(B, \overline{\mathbf{Q}})$ be the set of all $\overline{\mathbf{Q}}$-rational elements in $\mathcal{S}_{k,\kappa}(B)$. When χ is given, we set

$$\begin{aligned} W(\chi, B) = \{\mathbf{f} \in \mathcal{S}_{k,\kappa}(B) \mid \ & \mathbf{f}|T(\mathfrak{p}) = \chi(\mathfrak{p})\mathbf{f} \quad \text{for almost all} \quad \mathfrak{p}, \\ & \text{and} \quad \mathbf{f}(zx) = \psi(z)\mathbf{f}(x), \quad z \in Z_A, \ x \in G_A\}, \\ W(\chi, B, \overline{\mathbf{Q}}) = W(\chi, B) \cap \mathcal{S}_{k,\kappa}(B). & \end{aligned}$$

By the Shimizu-Jacquet-Langlands correspondence ([JL]), if χ occurs in $\mathcal{S}_{k,\kappa}(B)$, then it also occurs in $\mathcal{S}_{m,0}(M_2(F))$, where $m(\tau) = k(\tau)$ (resp. $\kappa(\tau)+2$) if $\tau \in \delta$ (resp. $\tau \in \delta'$). If (1.2) holds for all $\mathfrak{p}$ with the primitive form (the new form) $\mathbf{f} \in \mathcal{S}_{m,0}(M_2(F))$, then we call χ *primitive* (cf. [S3], II, p. 583).

Assume that χ occurs in $\mathcal{S}_{k,0}(M_2(F))$. In [Y], §6, we have shown the following facts sharpening previous results obtained by Shimura [S4], [S5].

$$\langle \mathbf{f}, \mathbf{f} \rangle \mod \overline{\mathbf{Q}}^\times \quad \text{is independent of} \quad 0 \neq \mathbf{f} \in W(\chi, B, \overline{\mathbf{Q}}). \tag{1.3}$$

$$\begin{aligned} & \text{If } B_1 \text{ and } B_2 \text{ are of signature } (\delta, \delta') \text{ and } k(\tau) \geq 2 \text{ for all } \tau \in J_F, \\ & \text{then } \langle \mathbf{f}, \mathbf{f} \rangle \sim \langle \mathbf{g}, \mathbf{g} \rangle \text{ for } \mathbf{f} \in W(\chi, B_1, \overline{\mathbf{Q}}),\ 0 \neq \mathbf{g} \in W(\chi, B_2, \overline{\mathbf{Q}}). \end{aligned} \tag{1.4}$$

If $W(\chi, B) \neq \{0\}$ for some quaternion algebra B of signature (δ, δ'), we put

$$Q(\chi, \delta) = \langle \mathbf{f}, \mathbf{f} \rangle \tag{1.5}$$

taking some non-zero form $\mathbf{f} \in W(\chi, B, \overline{\mathbf{Q}})$. By (1.3) and (1.4), $Q(\chi, \delta) \in \mathbf{C}^\times/\overline{\mathbf{Q}}^\times$ is well defined. Let F_1 be a totally real cyclic extension of degree l of F. We exclude the case where $k(\tau) = 1$ for all $\tau \in J_F$. Then there exists a base change lift $\tilde{\chi}$ of χ which occurs in $\mathcal{S}_{\tilde{k},0}(M_2(F_1))$ where $\tilde{k}(\tau) = k(\tau|F)$, $\tau \in J_{F_1}$. We have

$$\text{If } \chi \text{ occurs in } \mathcal{S}_{m,\kappa}(B), \text{ then } \tilde{\chi} \text{ occurs in } \mathcal{S}_{\tilde{m},\tilde{\kappa}}(B \otimes_F F_1). \tag{1.6}$$

$$Q(\tilde{\chi}, \tilde{\delta}) = Q(\chi, \delta)^l \qquad \text{if} \quad k(\tau) \geq 3 \quad \text{for all} \quad \tau \in \delta. \tag{1.7}$$

Here we have assumed $k(\tau) \geq 3$ for all $\tau \in \delta$ for some technical reasons (cf. §6); $\tilde{m}(\tau) = m(\tau|J_F)$, $\tilde{\kappa}(\tau) = \kappa(\tau|J_F)$, $\tau \in J_{F_1}$ and $\tilde{\delta}$ is the full inverse image of δ under the restriction map $J_{F_1} \longrightarrow J_F$. We can use (1.7) to define $Q(\chi, \delta)$ when χ does not occur in any B of signature (δ, δ'). In other words, we can find F_1 and B_1 of signature $(\tilde{\delta}, \tilde{\delta}')$ such that $\tilde{\chi}$ occurs in $\mathcal{S}_{\tilde{m},\tilde{\kappa}}(B_1)$ and put $Q(\chi, \delta) = Q(\tilde{\chi}, \tilde{\delta})^{1/l}$. Then $Q(\chi, \delta) \in \mathbf{C}^\times/\overline{\mathbf{Q}}^\times$ is well defined and (1.7) holds for this definition. We set $Q(\chi, \emptyset) = 1 \in \mathbf{C}^\times/\overline{\mathbf{Q}}^\times$.

Let χ be a primitive system of eigenvalues of Hecke operators which occurs in $\mathcal{S}_{k,0}(M_2(F))$. Put

$$k_0 = \max_{\tau \in J_F}(k(\tau)), \qquad k^0 = \min_{\tau \in J_F}(k(\tau)). \tag{1.8}$$

Let $\mathbf{f} \in W(\chi, M_2(F))$ be the primitive form. We attach a Dirichlet series $D(s, \mathbf{f}) = \sum_{\mathfrak{m}} C(\mathfrak{m}, \mathbf{f}) N(\mathfrak{m})^{-s}$ by (2.25) of [S1]. For a Hecke character φ of $F_A^\times$, we put

$$D(s, \mathbf{f}, \varphi) = \sum_{\mathfrak{m}} C(\mathfrak{m}, \mathbf{f}) \varphi_*(\mathfrak{m}) N(\mathfrak{m})^{-s}$$

where φ_* denotes the ideal character associated to φ and $\mathfrak{m}$ extends over all integral ideals of F. Set $L(s, \chi, \varphi) = \sum_{\mathfrak{m}} \chi(\mathfrak{m}) \varphi_*(\mathfrak{m}) N(\mathfrak{m})^{-s}$. Then we have $L(s, \chi, \varphi) = D(s + \frac{k_0}{2} - 1, \mathbf{f}, \varphi)$. In [S1], Theorem 4.3, Shimura obtained the following result (cf. also Rohrlich [R]) which we shall recall in a crude form sufficient for our present purpose.

Theorem S. *Assume $k(\tau) \geq 2$ for all $\tau \in J_F$ and $k(\tau) \mod 2$ is independent of τ. For every $\epsilon = (\epsilon(\tau)) \in (\mathbf{Z}/2\mathbf{Z})^{J_F}$, there exists a constant $u(\epsilon, \mathbf{f}) \in \mathbf{C}^\times/\overline{\mathbf{Q}}^\times$ with the following properties.*
(I) If φ is a Hecke character of $F_A^\times$ such that

$$\varphi_\infty(x) = \prod_{\tau \in J_F} sgn(x_\tau)^{\epsilon(\tau)+m}, \qquad x = (x_\tau) \in F_\infty^\times,$$

then

$$D(m, \mathbf{f}, \varphi) \sim \pi^{mn} u(\epsilon, \mathbf{f})$$

for every integer m such that

$$\frac{k_0 - k^0}{2} < m < \frac{k_0 + k^0}{2}.$$

(II) If ϵ_1, $\epsilon_2 \in (\mathbf{Z}/2\mathbf{Z})^{J_F}$ satisfy $\epsilon_1(\tau) + \epsilon_2(\tau) \equiv 1 \mod 2$ for all τ, then

$$u(\epsilon_1, \mathbf{f}) u(\epsilon_2, \mathbf{f}) \sim \pi^{n(1-k_0) + \sum_{\tau \in J_F} k(\tau)} \langle \mathbf{f}, \mathbf{f} \rangle.$$

Put $U(\chi, \epsilon) = u(\epsilon, \mathbf{f})$ taking the primitive form $\mathbf{f} \in W(\chi, M_2(F))$.

Remark. Let $\mathbf{f}$ be as above and let $\pi = \otimes_v \pi_v$ be the irreducible automorphic representation of $GL(2, F_A)$ generated by $\mathbf{f}$. Then π is unitary.
(1) By somewhat laborious computations taking a suitable model of a local component π_v of π and letting the Hecke operator at v defined in [S1], §2 act on the new vector, we can verify the exact equality $D(s, \mathbf{f}) = L_f(s - \frac{k_0-1}{2}, \pi)$. However this is not necessarily so for $D(s, \mathbf{f}, \varphi)$ and $L_f(s - \frac{k_0-1}{2}, \pi \otimes \varphi)$. In fact, some finitely many Euler factors of $L_f(s - \frac{k_0-1}{2}, \pi \otimes \varphi)$ may not appear in $D(s, \mathbf{f}, \varphi)$. The condition for the exact coincidence

is $L(s,\pi_v\otimes\varphi_v)=1$ whenever φ ramifies at v. This condition is satisfied at v if the exponent of the conductor of φ_v is greater than the exponent of the conductor of π_v.
(2) Let ψ be a Hecke character of $F_A^\times$ such that

$$\psi_\infty(x)=\prod_{\tau\in J_F}\operatorname{sgn}(x_\tau)^{\epsilon_1(\tau)},\qquad x=(x_\tau)\in F_\infty^\times.$$

Let $\mathbf{f}_\psi$ be the primitive form which belongs to $\pi\otimes\psi$. We have

$$u(\epsilon,\mathbf{f}_\psi)\sim u(\epsilon+\epsilon_1,\mathbf{f})\qquad\text{for every}\quad \epsilon\in(\mathbf{Z}/2\mathbf{Z})^{J_F}. \tag{1.9}$$

To see this, first choose a critical value m. Take a Hecke character φ of $F_A^\times$ so that φ_∞ is given by the formula in Theorem S, (I) and that the conductor of φ is divisible by $\mathfrak{p}^{e+1}$ whenever $\mathfrak{p}^e$ divides one of the conductors of π, $\pi\otimes\psi$, ψ. Then we have

$$D(s,\mathbf{f}_\psi,\varphi)=D(s,\mathbf{f},\psi\varphi)=L_f(s-\frac{k_0-1}{2},\pi\otimes\psi\varphi).$$

By a theorem of Rohrlich, we can further impose the condition on φ that $L_f(s,\pi\otimes\psi\varphi)\neq 0$ for $s=m-\frac{k_0-1}{2}$. Then (1.9) follows from Theorem S. As a result, we see that

$$L_f(m-\frac{k_0-1}{2},\pi\otimes\varphi)\sim\pi^{mn}u(\epsilon,\mathbf{f}) \tag{1.10}$$

for a Hecke character φ and critical values m as in Theorem S.

(3) It can be shown, using the unitarity of π_v, that $D(s,\mathbf{f},\varphi)/L_f(s-\frac{k_0-1}{2},\pi\otimes\varphi)$ is an entire function which has no zeros for $\Re(s)\geq k_0/2$. We can give another proof of (1.9) and (1.10) using this fact and the functional equation of $L(s,\pi\otimes\varphi)$.

§2. Preliminary reduction of Conjecture P

Our main theorem states that 2^{n+1} quantities $U(\chi,\epsilon)$ and $Q(\chi,\delta)$ can be given by $2n$ quantities $c_\tau^\pm(\chi)$, which implies some highly non-trivial relations among $U(\chi,\epsilon)$ and $Q(\chi,\delta)$. We shall analyze these relations by the next Lemma.

Lemma 1. *Let $J=\{1,2,\cdots,n\}$ and let Λ_n be the set of all mappings from J to $\{\pm1\}$. Assume that for every $\epsilon\in\Lambda_n$ and every subset I of J, there are given quantities $p(\epsilon)\in\mathbf{C}^\times/\overline{\mathbf{Q}}^\times$ and $q(I)\in\mathbf{C}^\times/\overline{\mathbf{Q}}^\times$ which satisfy the following properties:*

$$p(\epsilon)p(-\epsilon)=q(J)\quad\textit{where}\quad(-\epsilon)(i)=-\epsilon(i),\quad i\in J. \tag{R1}$$

$$q(I_1\cup I_2)=q(I_1)q(I_2)\quad\textit{if}\quad I_1\cap I_2=\emptyset. \tag{R2}$$

$$p(\epsilon_1)p(\epsilon_2)=p(\mu_1)p(\mu_2)\quad\textit{if}\quad\{\epsilon_1(i),\epsilon_2(i)\}=\{\mu_1(i),\mu_2(i)\}\quad\textit{for every}\quad 1\leq i\leq n. \tag{R3}$$

Then there exist $2n$ constants $c_i^{\pm} \in \mathbf{C}^\times/\overline{\mathbf{Q}}^\times$, $1 \le i \le n$ such that

$$p(\epsilon) = \prod_{i=1}^{n} c_i^{\epsilon(i)}, \qquad \epsilon \in \Lambda_n, \tag{2.1}$$

$$q(I) = \prod_{i \in I} c_i^+ c_i^-, \qquad \text{if} \quad I \subseteq J. \tag{2.2}$$

Moreover $c_i^{\pm} \in \mathbf{C}^\times/\overline{\mathbf{Q}}^\times$, $1 \le i \le n$ are unique. In (2.1) and (2.2), we understand that $c_i^1 = c_i^+$, $c_i^{-1} = c_i^-$, $\prod_{i\in\emptyset} c_i^+ c_i^- = 1$.

Proof. By (R2), we have $q(\emptyset) = 1$. Hence (2.2) for $I = \emptyset$ holds. If $n = 1$, the assertion holds with

$$c_1^+ = p(\epsilon), \qquad c_1^- = p(-\epsilon) \quad \text{for} \quad \epsilon : 1 \longrightarrow 1.$$

Now we assume $n \ge 2$ and that the assertion holds up to $n-1$. Let $J' = \{1, 2, \cdots, n-1\} = J \backslash \{n\}$ and let Λ_{n-1} be the set of all mappings from J' to $\{\pm 1\}$. Define $\omega_\pm$, $\omega'_\pm \in \Lambda_n$ by

$$\begin{aligned}
\omega_+ &: \{1, 2, \cdots, n-1, n\} \longrightarrow \{1, 1, \cdots, 1, 1\}, \\
\omega'_+ &: \{1, 2, \cdots, n-1, n\} \longrightarrow \{-1, -1, \cdots, -1, 1\}, \\
\omega_- &: \{1, 2, \cdots, n-1, n\} \longrightarrow \{1, 1, \cdots, 1, -1\}, \\
\omega'_- &: \{1, 2, \cdots, n-1, n\} \longrightarrow \{-1, -1, \cdots, -1, -1\}.
\end{aligned}$$

By (R1), we have

$$p(\omega_+)p(\omega'_-) = p(\omega'_+)p(\omega_-) = q(J). \tag{2.3}$$

For a given $\epsilon \in \Lambda_{n-1}$, choose an extension $\epsilon^* \in \Lambda_n$ so that $\epsilon^*(i) = \epsilon(i)$, $1 \le i \le n-1$ and set

$$p'(\epsilon) = p(\epsilon^*)/\sqrt{p(\omega_{\epsilon^*(n)})p(\omega'_{\epsilon^*(n)})/q(J')} \in \mathbf{C}^\times/\overline{\mathbf{Q}}^\times. \tag{2.4}$$

By (R3), we see that $p'(\epsilon)$ does not depend on the choice of ϵ^*. For $I' \subseteq J'$, we set

$$q'(I') = q(I'). \tag{2.5}$$

Then we can verify that the quantities $p'(\epsilon)$, $\epsilon \in \Lambda_{n-1}$ and $q'(I')$ satisfy

$$p'(\epsilon)p'(-\epsilon) = q'(J'), \tag{R$'$1}$$

$$q'(I_1' \cup I_2') = q'(I_1')q'(I_2') \quad \text{if} \quad I_1' \cap I_2' = \emptyset, \tag{R$'$2}$$

$$\text{(R}'3) \quad \begin{array}{l} p'(\epsilon_1)p'(\epsilon_2) = p'(\mu_1)p'(\mu_2) \quad \text{if} \quad \{\epsilon_1(i), \epsilon_2(i)\} = \{\mu_1(i), \mu_2(i)\} \\ \text{for every} \quad 1 \le i \le n-1. \end{array}$$

Relation (R$'$2) is trivial. To see (R$'$1), we may choose an extension ϵ^* of ϵ so that $\epsilon^*(n) = 1$ and may apply (2.4). Then we have

$$\begin{aligned} p'(\epsilon)p'(-\epsilon) &= p(\epsilon^*)/\sqrt{p(\omega_+)p(\omega'_+)/q(J')} \cdot p(-\epsilon^*)/\sqrt{p(\omega_-)p(\omega'_-)/q(J')} \\ &= q(J)q(J')/\sqrt{p(\omega_+)p(\omega'_+)p(\omega_-)p(\omega'_-)} = q(J') \end{aligned}$$

by (2.3) and (R1). Similarly (R$'$3) follows from (R3).

By the hypothesis of induction, there exist $2(n-1)$ quantities $c_i^\pm \in \mathbf{C}^\times/\overline{\mathbf{Q}}^\times$, $1 \le i \le n-1$ such that

$$p'(\epsilon) = \prod_{i=1}^{n-1} c_i^{\epsilon(i)}, \qquad \epsilon \in \Lambda_{n-1}, \tag{2.6}$$

$$q(I') = q'(I') = \prod_{i \in I'} c_i^+ c_i^-, \qquad I' \subseteq J'. \tag{2.7}$$

Set

$$c_n^+ = \sqrt{p(\omega_+)p(\omega'_+)/q(J')}, \qquad c_n^- = \sqrt{p(\omega_-)p(\omega'_-)/q(J')}. \tag{2.8}$$

To see the relation (2.1), put $\epsilon = \epsilon^*|J$ for $\epsilon^* \in \Lambda_n$. By (2.4), (2.6) and (2.8), we have

$$p(\epsilon^*) = p'(\epsilon)\sqrt{p(\omega_{\epsilon^*(n)})p(\omega'_{\epsilon^*(n)})/q(J')} = (\prod_{i=1}^{n-1} c_i^{\epsilon(i)})c_n^{\epsilon^*(n)} = \prod_{i=1}^{n} c_i^{\epsilon^*(i)}.$$

Hence (2.1) is satisfied.

To see (2.2), we may assume $I \ni n$. Put $I' = I \backslash \{n\}$. By (2.8), (2.3) and (R2), we get

$$c_n^+ c_n^- = q(J)/q(J') = q(\{n\}).$$

Then we obtain

$$q(I) = q(I')q(\{n\}) = (\prod_{i \in I'} c_i^+ c_i^-)c_n^+ c_n^- = \prod_{i \in I} c_i^+ c_i^-$$

by (R2).

The uniqueness of $c_i^\pm$ is clear since we can express $c_i^\pm$ by a formula similar to (2.8) if (2.1) and (2.2) hold. This completes the proof.

Identify J_F with $\{1, 2, \cdots, n\}$ and $\mathbf{Z}/2\mathbf{Z}$ with $\{1, -1\}$. By the above Lemma, we see that our Main Theorem is reduced to (R1) $\sim$ (R3) given in the introduction. We note that (R1) follows from Theorem S, (II) in view of the definition of $Q(\chi, J_F)$.

In the rest of this section, we shall prove (P4). Let K be a CM-extension of F. For $\alpha,\ \beta \in I_K$, let $p_K(\alpha,\beta) \in \mathbf{C}^\times/\overline{\mathbf{Q}}^\times$ denote the CM-period defined in [S2]. Let Φ be a CM-type of K and set $\xi = \sum_{\tau\in\Phi} \xi_\tau \cdot \tau \in I_K$, $\xi_\tau \geq 0$ for all τ. Let Ξ be a primitive Hecke character of the ideal group of K with conductor $\mathfrak{c}$ such that

$$\Xi((a)) = a^\xi/|a^\xi| \qquad \text{if} \quad a \in K, \quad a \equiv 1 \mod^\times \mathfrak{c},$$

where $a^\xi = \prod_{\tau\in\Phi}(a^\tau)^{\xi_\tau}$. Assume $\xi_\tau > 0$ for some τ. Then there exists a primitive system of eigenvalues of Hecke operators χ occuring in $\mathcal{S}_{k,0}(M_2(F))$ such that

$$L(s,\chi) = L(s-1/2,\Xi), \tag{2.9}$$

where $k(\tau|F) = \xi_\tau + 1$, $\tau \in \Phi$ (cf. [S4], §5). If $\xi_\tau \mod 2$ is independent of τ and $\xi_\tau > 0$ for all τ, then we have

$$U(\chi,\epsilon) \sim \pi^{(\sum_{\tau\in J_F} k(\tau)-nk_0)/2} p_K(\xi,\Phi) \quad \text{for every} \quad \epsilon \in (\mathbf{Z}/2\mathbf{Z})^{J_F} \tag{2.10}$$

by [S4], Theorem 5.11, (iii). On the otherhand, we have

$$Q(\chi,\delta) \sim \pi^{-|\delta|} p_K(\xi,2\eta) \tag{2.11}$$

by [S4], Theorem 5.8, where η is the subset of Φ such that $\mathrm{Res}_{K/F}(\eta) = \delta$. Now for such a χ, (R2) follows from the bilinearity of p_K (cf. [S2], Theorem 1.1) and (R3) is trivially satisfied. We see that the solution to (1) and (2) in the introduction is given by

$$c_\tau^+(\chi) = c_\tau^-(\chi) = \pi^{(k(\tau)-k_0)/2} p_K(\xi,\tilde{\tau}), \qquad \tau \in J_F \tag{2.12}$$

from the bilinearity of p_K, where $\tilde{\tau} \in \Phi$ denotes the element such that $\tilde{\tau}|F = \tau$. By (3) in the introduction, we have

$$P(\chi,\delta,\epsilon) \sim \pi^{-|\delta|} p_K(\xi,\eta) \qquad \text{for every} \quad \epsilon \in (\mathbf{Z}/2\mathbf{Z})^\delta, \tag{2.13}$$

which is consistent with (C9) of [S5].

§3. Verification of (R2)

We shall use the following result of Harris (cf. [Ha3], §2.6).

Theorem HA. *Let χ be a primitive system of eigenvalues of Hecke operators which occurs in $\mathcal{S}_{k,0}(M_2(F))$. Assume $k(\tau) \geq 2$ for all $\tau \in J_F$ and $k(\tau) \mod 2$ is independent of τ. Let α and β be subsets of J_F such that $\alpha \cap \beta = \emptyset$. If n, $|\alpha|$ and $|\beta|$ are all even, then*

$$Q(\chi,\alpha\cup\beta) \sim Q(\chi,\alpha)Q(\chi,\beta).$$

By a base change argument, we can remove the parity condition in Theorem HA when $k(\tau) \geq 3$ for all $\tau \in \alpha\cup\beta$.

Theorem 2. *Let α and β be subsets of J_F such that $\alpha \cap \beta = \emptyset$. Assume that $k(\tau) \geq 2$ for all $\tau \in J_F$, $k(\tau) \geq 3$ for all $\tau \in \alpha \cup \beta$ and that $k(\tau) \bmod 2$ is independent of τ. Then we have*

$$Q(\chi, \alpha \cup \beta) \sim Q(\chi, \alpha) Q(\chi, \beta).$$

Proof. Let F_1 be a totally real quadratic extension of F. Let $\tilde{\alpha}$ and $\tilde{\beta}$ be the full inverse images of α and β under the restriction map $J_{F_1} \longrightarrow J_F$ respectively. Let $\tilde{\chi}$ be a base change lift of χ which occurs in $\mathcal{S}_{\tilde{k},0}(M_2(F_1))$, where $\tilde{k}(\tau) = k(\tau | J_F)$, $\tau \in J_{F_1}$. We can apply Theorem HA to $\tilde{\chi}$, $\tilde{\alpha}$, $\tilde{\beta}$ and obtain

$$Q(\tilde{\chi}, \tilde{\alpha} \cup \tilde{\beta}) \sim Q(\tilde{\chi}, \tilde{\alpha}) Q(\tilde{\chi}, \tilde{\beta}).$$

By (1.7), we have

$$Q(\chi, \alpha \cup \beta)^2 \sim Q(\chi, \alpha)^2 Q(\chi, \beta)^2.$$

Hence the assertion follows.

By Theorem 2, the condition (R2) is verified.

§4. Verification of (R3)

To present our arguments in a clear-cut way, let us first recall a few facts on representation theory of $GL(2, L)$ for an archimedean field L. Let $\mathcal{H}_L$ denote the Hecke algebra of $GL(2, L)$ defined in Jacquet-Langlands [JL], p. 153, p. 220.

First let $L = \mathbf{R}$. For a positive integer p, let

$$\mu_1(t) = |t|^{p/2}, \qquad \mu_2(t) = |t|^{-p/2} \operatorname{sgn}(t)^{\epsilon(p)}, \qquad t \in \mathbf{R}^\times$$

where $\epsilon(p) = 0$ or 1 according as p is odd or even. Consider the representation $\sigma_p = \sigma(\mu_1, \mu_2)$ described in [JL], Theorem 5.11. Then σ_p is a unitary discrete series representation of $\mathcal{H}_{\mathbf{R}}$. If an irreducible automorphic representation $\pi = \otimes_v \pi_v$ of $GL(2, F_A)$ is generated by $\mathbf{f} \in \mathcal{S}_{k,0}(M_2(F))$, then we have

$$\pi_\infty = \otimes_{\tau \in J_F} \sigma_{k(\tau)-1}$$

if $k(\tau) \geq 2$ for all $\tau \in J_F$. Let ω_p be the character of $\mathbf{C}^\times$ given by

$$\omega_p(z) = z^p (z\bar{z})^{-p/2}, \qquad z \in \mathbf{C}^\times.$$

Then we have

$$\sigma_p = \pi(\omega_p) \tag{4.1}$$

in the notation of [JL], p. 176–181. We also have

$$L(s, \sigma_p) = L(s, \omega_p) = 2(2\pi)^{-(s+p/2)} \Gamma(s + \frac{p}{2}). \tag{4.2}$$

Let $L = \mathbf{C}$. For two quasi-characters μ_1, μ_2 of $\mathbf{C}^\times$, let $\pi(\mu_1, \mu_2)$ be the representation of $\mathcal{H}_{\mathbf{C}}$ described in [JL], Theorem 6.2.

Now let $W_{\mathbf{C}} = \mathbf{C}^\times$, $W_{\mathbf{R}} = W_{\mathbf{R},\mathbf{C}}$ be the Weil groups. We may write (4.1) as $\sigma_p = \pi(\operatorname{Ind}_{W_{\mathbf{C}}}^{W_{\mathbf{R}}} \omega_p)$ in terms of the Langlands parametrization. Hence the base change lift of σ_p to $\mathcal{H}_{\mathbf{C}}$ is given by $\pi((\operatorname{Ind}_{W_{\mathbf{C}}}^{W_{\mathbf{R}}} \omega_p) | W_{\mathbf{C}}) = \pi(\omega_p, \bar{\omega}_p)$ by Langlands [L], p.16, e).

We quote Hida [Hi], Theorem 8.1 in a crude form sufficient for our present purpose.

Theorem HI. *Let K be an algebraic number field. Let $\pi = \otimes_w \pi_w$ be an irreducible unitary cuspidal automorphic representation of $GL(2, K_A)$. Assume that*

$$\pi_\infty = \otimes_{\tau\in\mathfrak{a}_r} \sigma_{k(\tau)-1} \otimes_{\tau\in\mathfrak{a}_c} \pi(\omega_{k(\tau)-1}, \bar{\omega}_{k(\tau)-1})$$

with $k(\tau) \geq 2$ for all $\tau \in \mathfrak{a}$ and $k(\tau) \mod 2$ is independent of τ. Put $k_0 = \max_{\tau\in\mathfrak{a}} k(\tau)$. Then for every $\epsilon \in (\mathbf{Z}/2\mathbf{Z})^{\mathfrak{a}_r}$, there exists a constant $U(\pi, \epsilon) \in \mathbf{C}^\times$ which satisfies the following properties. If φ is a Hecke character of $K_A^\times$ of finite order such that

$$\varphi_\infty(x) = \prod_{\tau\in\mathfrak{a}_r} sgn(x_\tau)^{\epsilon(\tau)+m}, \qquad x = (x_\tau)_{\tau\in\mathfrak{a}} \in K_\infty^\times,$$

then

$$L_f(m - \frac{k_0-1}{2}, \pi) \sim \pi^{m[K:\mathbf{Q}]} U(\pi, \epsilon)$$

for every integer m such that

$$\frac{k_0 - k(\tau)}{2} < m < \frac{k_0 + k(\tau)}{2} \qquad \text{for every} \quad \tau \in \mathfrak{a}.$$

We are going to verify (R3) using this theorem. It suffices to show

Theorem 3. *Let $\mathbf{f} \in \mathcal{S}_{k,0}(M_2(F))$ be a primitive cusp form. We assume $k(\tau) \geq 3$ for all $\tau \in J_F$ and $k(\tau) \mod 2$ is independent of τ. Then we have*

$$u(\epsilon_1, \mathbf{f})u(\epsilon_2, \mathbf{f}) \sim u(\mu_1, \mathbf{f})u(\mu_1, \mathbf{f}) \tag{4.3}$$

whenever ϵ_1, ϵ_2, μ_1, $\mu_2 \in (\mathbf{Z}/2\mathbf{Z})^{J_F}$ satisfy

$$\{\epsilon_1(\tau), \epsilon_2(\tau)\} = \{\mu_1(\tau), \mu_2(\tau)\} \qquad \text{for all} \quad \tau \in J_F. \tag{4.4}$$

Proof. We choose an integer m which satisfies the condition of Theorem S, (I). Since we have assumed $k^0 \geq 3$, we can choose such an m so that $m \geq (k_0+1)/2$. We fix and denote it by m_0. Then we have $D(m_0, \mathbf{f}, \varphi) \neq 0$ for every Hecke character φ of $F_A^\times$ of finite order (cf. [S1], Prop. 4.16). Let φ_1, φ_2, ψ_1, ψ_2 be Hecke characters of $F_A^\times$ of finite order such that

$$\begin{aligned} (\varphi_i)_\infty(x) &= \prod_{\tau\in J_F} (\text{sgn}(x_\tau))^{\epsilon_i(\tau)+m_0}, \qquad i = 1, 2, \\ (\psi_i)_\infty(x) &= \prod_{\tau\in J_F} (\text{sgn}(x_\tau))^{\mu_i(\tau)+m_0}, \qquad i = 1, 2, \end{aligned} \tag{4.5}$$

for $x = (x_\tau) \in F_\infty^\times$. By Theorem S, (I), (4.3) reduces to

$$D(m_0, \mathbf{f}, \varphi_1)D(m_0, \mathbf{f}, \varphi_2) \sim D(m_0, \mathbf{f}, \psi_1)D(m_0, \mathbf{f}, \psi_2). \tag{4.6}$$

By (4.4), we have $(\varphi_1\varphi_2)_\infty = (\psi_1\psi_2)_\infty$. If $(\varphi_1\varphi_2)_\infty$ is trivial, then we have $\epsilon_1 = \epsilon_2 = \mu_1 = \mu_2$ by (4.4); hence (4.3) holds. We may assume that $(\varphi_1\varphi_2)_\infty$ is non-trivial. Choose $a \in F$ so that $\tau(a) > 0$ (resp. $\tau(a) < 0$) if $(\varphi_1\varphi_2)_{\infty_\tau}$ is trivial (resp. non-trivial). Set $K = F(\sqrt{a})$. Then K is a quadratic extension of F. Let η_K be the Hecke character of $F_A^\times$ which corresponds to the extension K/F. By the choice of a, we have $(\eta_K)_\infty = (\varphi_1\varphi_2)_\infty$.

Let $\pi = \otimes_v \pi_v$ be the irreducible automorphic representation of $GL(2, F_A)$ generated by $\mathbf{f}$ and $\tilde{\pi} = \otimes_w \tilde{\pi}_w$ be the base change lift of π to $GL(2, K_A)$. Then we have

$$(4.7)\qquad \begin{aligned} &L(s,\tilde{\pi}) = L(s,\pi)L(s,\pi\otimes\eta_K), \qquad L_f(s,\tilde{\pi}) = L_f(s,\pi)L_f(s,\pi\otimes\eta_K),\\ &\pi_\infty = \otimes_{\tau\in J_F}\sigma_{k(\tau)-1},\\ &\tilde{\pi}_\infty = (\otimes_{\tau\in\mathfrak{a}_r}\sigma_{k(\tau|F)-1})\otimes(\otimes_{\tau\in\mathfrak{a}_c}\pi(\omega_{k(\tau|F)-1},\bar{\omega}_{k(\tau|F)-1})). \end{aligned}$$

Since the base change lift of $\pi\otimes\varphi_1$ to K is $\tilde{\pi}\otimes(\varphi_1\circ N_{K/F})$, we have

$$L_f(s,\tilde{\pi}\otimes(\varphi_1\circ N_{K/F})) = L_f(s,\pi\otimes\varphi_1)L_f(s,\pi\otimes\varphi_1\eta_K).$$

We have $D(m_0,\mathbf{f},\varphi) \sim L_f(m_0 - \frac{k_0-1}{2}, \pi\otimes\varphi)$ for every Hecke character φ of $F_A^\times$ of finite order. Since $(\varphi_1\eta_K)_\infty = (\varphi_2)_\infty$, we have $L_f(m_0-\frac{k_0-1}{2},\pi\otimes\varphi_1\eta_K) \sim L_f(m_0-\frac{k_0-1}{2},\pi\otimes\varphi_2)$ by Theorem S, (I). Therefore (4.6) reduces to

$$(4.8)\qquad L_f(m_0 - \frac{k_0-1}{2}, \tilde{\pi}\otimes(\varphi_1\circ N_{K/F})) \sim L_f(m_0 - \frac{k_0-1}{2}, \tilde{\pi}\otimes(\psi_1\circ N_{K/F})).$$

Assume $\tau \in J_F$ is unramified in K. Then $(\varphi_1\varphi_2)_{\infty_\tau} = (\psi_1\psi_2)_{\infty_\tau} = 1$ and we see that $\{\epsilon_1(\tau),\epsilon_2(\tau)\}$ and $\{\mu_1(\tau),\mu_2(\tau)\}$ are either $\{0,0\}$ or $\{1,1\}$. By (4.4), we get $\epsilon_1(\tau) = \mu_1(\tau)$, $(\varphi_1)_{\infty_\tau} = (\psi_1)_{\infty_\tau}$. Therefore we obtain

$$(\varphi_1\circ N_{K/F})_\infty = (\psi_1\circ N_{K/F})_\infty.$$

By the consideration given in §2, we may assume that χ is not of CM-type. Then $\tilde{\pi}$ is cuspidal (cf. [L], Lemma 11.3). Now (4.8) follows from Theorem HI. This completes the proof.

Now we have completed our proof of Main Theorem. An identification of $c_\tau^\pm(\chi)$ with Deligne's periods of the motive attached to χ is described in [Y], §4. We note that there is a slight notational difference between [S4] and [S5]. In [S5], p. 293, (C3),

$$P(\chi,\epsilon,J_F) \sim \pi^{-n-\sum_{\tau\in J_F}k(\tau)/2}V(\chi,\epsilon) \sim \pi^{(k_0-2)n/2-\sum_{\tau\in J_F}k(\tau)/2}U(\chi,(-1)^{k_0/2}\epsilon)$$

is required when $k(\tau)$ is even for all τ. We adjusted our notation to [S4], which is simpler.

Remark. We have

$$(4.9)\qquad c_\tau^\pm(\bar{\chi}) \sim \overline{c_\tau^\pm(\chi)} \qquad \text{for every} \quad \tau\in J_F$$

where — denotes the complex conjugation. To see this, let π be the unitary automorphic representation of $GL(2,F_A)$ which corresponds to χ and call ψ the central character of π.

We have $\bar{\pi} \cong \pi \otimes \psi^{-1}$. By definition, it is obvious that $Q(\chi,\delta) \sim \overline{Q(\chi,\delta)}$ for every $\delta \subset J_F$. Since $\bar{\chi} = \chi \otimes \psi^{-1}$, we have (cf. [Y], Prop. 6.5)

$$Q(\bar{\chi},\delta) \sim \overline{Q(\chi,\delta)}. \tag{4.10}$$

As in Theorem S, choose a critical value m and a Hecke character φ of $F_A^\times$ of finite order for $\epsilon \in (\mathbf{Z}/2\mathbf{Z})^{J_F}$ so that $L_f(m - \frac{k_0-1}{2}, \pi \otimes \varphi) \neq 0$. By Theorem S, we have

$$\overline{\pi^{mn}U(\chi,\epsilon)} \sim \overline{L_f(m - \frac{k_0-1}{2}, \pi \otimes \varphi)} = L_f(m - \frac{k_0-1}{2}, \bar{\pi} \otimes \varphi^{-1}) \sim \pi^{mn}U(\bar{\chi},\epsilon).$$

Hence we get

$$U(\bar{\chi},\epsilon) \sim \overline{U(\chi,\epsilon)}. \tag{4.11}$$

By (4.10), (4.11) and Lemma 2.1, we obtain (4.9) (cf. [S5], p. 293, (C2)).

§5. The invariance of $c_\tau^\pm(\chi)$ under a base change

Theorem 4. *Let F_1 be a totally real cyclic extension of F. Let χ be a primitive system of eigenvalues of Hecke operators which occurs in $\mathcal{S}_{k,0}(M_2(F))$. We assume that $k(\tau) \geq 3$ for all $\tau \in J_F$ and that $k(\tau) \mod 2$ is independent of τ. Let $\tilde{\chi}$ be the base change lift of χ such that $\tilde{\chi}$ occurs in $\mathcal{S}_{\tilde{k},0}(M_2(F_1))$ and that $\tilde{\chi}$ is primitive, where $\tilde{k}(\tau) = k(\tau|F)$, $\tau \in J_{F_1}$. Then we have*

$$c_\tau^\pm(\tilde{\chi}) = c_{\tau|F}^\pm(\chi) \qquad \text{for every} \quad \tau \in J_{F_1}. \tag{5.1}$$

Proof. Let $\tilde{\mathbf{f}} \in W(\tilde{\chi}, M_2(F_1), \overline{\mathbf{Q}})$ and $\mathbf{f} \in W(\chi, M_2(F), \overline{\mathbf{Q}})$ be primitive forms. Let $\tilde{\pi}$ (resp. π) be the irreducible automorphic representation of $GL(2,(F_1)_A)$ (resp. $GL(2,F_A)$) generated by $\tilde{\mathbf{f}}$ (resp. $\mathbf{f}$). Then we have

$$L_f(s, \tilde{\pi} \otimes \varphi^\sigma) = L_f(s, \tilde{\pi} \otimes \varphi) \tag{5.2}$$

for every $\sigma \in \mathrm{Gal}(F_1/F)$ and every Hecke character φ of $(F_1)_A^\times$. Here $\varphi^\sigma(x) = \varphi(x^\sigma)$, $x \in (F_1)_A^\times$. Take $m \in \mathbf{Z}$ so that $(k_0 - k^0)/2 < m < (k_0 + k^0)/2$. By a theorem of Rohrlich [R], for every $\tilde{\epsilon} \in (\mathbf{Z}/2\mathbf{Z})^{J_{F_1}}$, we can find a Hecke character φ of $(F_1)_A^\times$ such that

$$L_f(m - \frac{k_0-1}{2}, \tilde{\pi} \otimes \varphi) \neq 0, \qquad \varphi_\infty(x) = \prod_{\tau \in J_{F_1}} \mathrm{sgn}\,(x_\tau)^{m+\tilde{\epsilon}(\tau)}, \qquad x = (x_\tau) \in (F_1)_\infty^\times.$$

Applying Theorem S, (I) to (5.2) taking $s = m - \frac{k_0-1}{2}$, we obtain

$$u(\tilde{\epsilon}^\sigma, \tilde{\mathbf{f}}) \sim u(\tilde{\epsilon}, \tilde{\mathbf{f}}) \qquad \text{for every} \quad \sigma \in \mathrm{Gal}(F_1/F), \tag{5.3}$$

where $\tilde{\epsilon}^{\sigma}(y) = \tilde{\epsilon}(\sigma y)$, $y \in J_{F_1}$. In a similar way, using Theorem 6.8 of [Y], we can derive the relation

$$(5.4) \qquad Q(\tilde{\chi}, \sigma\tilde{\delta}) \sim Q(\tilde{\chi}, \tilde{\delta}) \qquad \text{for every} \quad \emptyset \neq \tilde{\delta} \subseteq J_{F_1}.$$

By (5.3) and (5.4), we get

$$(5.5) \qquad c^{\pm}_{\sigma\tau}(\tilde{\chi}) \sim c^{\pm}_{\tau}(\tilde{\chi}) \qquad \text{for every} \quad \sigma \in \mathrm{Gal}(F_1/F), \quad \tau \in J_{F_1}$$

in view of the uniqueness of the solution to (1) and (2) in the introduction. Taking $\delta = \{\tau|F\}$, $\tau \in J_{F_1}$ in (1.7) and applying (5.5), we get

$$(5.6) \qquad c^{+}_{\tau}(\tilde{\chi})c^{-}_{\tau}(\tilde{\chi}) \sim c^{+}_{\tau|F}(\chi)c^{-}_{\tau|F}(\chi), \qquad \tau \in J_{F_1}.$$

On the other hand, we have

$$L_f(s, \tilde{\pi} \otimes (\varphi \circ N_{F_1/F})) = \prod_{\eta} L_f(s, \pi \otimes \varphi\eta)$$

for every Hecke character φ of $F^{\times}_{A}$. Here η extends over l Hecke characters of $F^{\times}_{A}$ which are trivial on $F^{\times}N_{F_1/F}((F_1)^{\times}_{A})$, l being the degree of F_1 over F. Since $k(\tau) \geq 3$ for all τ, we can apply Theorem S, (I) to this relation in a similar manner to the above and obtain

$$(5.7) \qquad u(\tilde{\epsilon}, \tilde{\mathbf{f}}) \sim u(\epsilon, \mathbf{f})^{l} \qquad \text{for every} \quad \epsilon \in (\mathbf{Z}/2\mathbf{Z})^{J_F},$$

where $\tilde{\epsilon}(y) = \epsilon(y|F)$, $y \in J_{F_1}$. By (5.7) and (5.5), we get

$$(5.8) \qquad \prod_{\tau \in J_F} c^{\epsilon(\tau)}_{\tilde{\tau}}(\tilde{\chi}) \sim \prod_{\tau \in J_F} c^{\epsilon(\tau)}_{\tau}(\chi), \qquad \text{for every} \quad \epsilon \in (\mathbf{Z}/2\mathbf{Z})^{J_F},$$

where $\tilde{\tau}$ denotes an arbitrary extension of τ to J_{F_1}.

Take any $\tau_0 \in J_F$ and its extension $\tilde{\tau}_0$ to J_{F_1}. Take any $\epsilon \in (\mathbf{Z}/2\mathbf{Z})^{J_F}$ and define $\epsilon' \in (\mathbf{Z}/2\mathbf{Z})^{J_F}$ by

$$\epsilon'(\tau) = -\epsilon(\tau) \quad \text{if } \tau \neq \tau_0, \qquad \epsilon'(\tau_0) = \epsilon(\tau_0).$$

We have

$$\prod_{\tau \in J_F} c^{\epsilon(\tau)}_{\tilde{\tau}}(\tilde{\chi})c^{\epsilon'(\tau)}_{\tilde{\tau}}(\tilde{\chi}) \sim (\prod_{\tau \in J_F \setminus \{\tau_0\}} c^{+}_{\tau}(\chi)c^{-}_{\tau}(\chi))c^{\epsilon(\tau_0)}_{\tilde{\tau}_0}(\tilde{\chi})^2$$

by (5.6) and

$$\prod_{\tau \in J_F} c^{\epsilon(\tau)}_{\tilde{\tau}}(\tilde{\chi})c^{\epsilon'(\tau)}_{\tilde{\tau}}(\tilde{\chi}) \sim (\prod_{\tau \in J_F \setminus \{\tau_0\}} c^{+}_{\tau}(\chi)c^{-}_{\tau}(\chi))c^{\epsilon(\tau_0)}_{\tau_0}(\chi)^2$$

by (5.8). Hence we get

$$c^{\epsilon(\tau_0)}_{\tilde{\tau}_0}(\tilde{\chi})^2 \sim c^{\epsilon(\tau_0)}_{\tau_0}(\chi)^2.$$

This completes the proof.

Remark. In this remark, we use left action of the automorphism group. For $\sigma \in \mathrm{Aut}(\mathbf{C})$, let $\sigma(B)$ be the quaternion algebra over $\sigma(F)$ obtained from B by transporting the algebra structure by the isomorphism $\sigma : F \longrightarrow \sigma(F)$. If $B = \sum_{i=1}^{4} Fe_i$ with $e_ie_j = \sum_{k=1}^{4} c_{ijk}e_k$, then $\sigma(B) = \sum_{i=1}^{4} \sigma(F)e'_i$ with $e'_ie'_j = \sum_{k=1}^{4} \sigma(c_{ijk})e'_k$. We have the isomorphism of $\mathbf{Q}$-algebras $\sigma : B \ni \sum a_ie_i \longrightarrow \sum \sigma(a_i)e'_i \in \sigma(B)$. If B is of signature (δ, δ'), then $\sigma(B)$ is of signature $(\delta\sigma^{-1}, \delta'\sigma^{-1})$. This isomorphism extends to the isomorphism (we use the same letter) from $G = \mathrm{Res}_{F/\mathbf{Q}}(B^\times)$ to $\sigma(G) = \mathrm{Res}_{\sigma(F)/\mathbf{Q}}(\sigma(B)^\times)$ and also from G_A to $\sigma(G)_A$. For $\mathbf{f} \in \mathcal{S}_{k,\kappa}(B)$, put $\sigma(\mathbf{f})(\sigma x) = \mathbf{f}(x)$, $x \in G_A$. Then $\sigma(\mathbf{f}) \in \mathcal{S}_{k',\kappa'}(\sigma(B))$, where $k'(\tau) = k(\tau\sigma)$, $\kappa'(\tau) = \kappa(\tau\sigma)$, $\tau \in J_{\sigma(F)}$.

If $\mathbf{f} \in W(\chi, B)$, then we see that $\sigma(\mathbf{f}) \in W(\sigma(\chi), \sigma(B))$, where $\sigma(\chi)(\sigma(\mathfrak{m})) = \chi(\mathfrak{m})$ for an integral ideal $\mathfrak{m}$ of F. We can check easily that $\langle \mathbf{f}, \mathbf{f} \rangle = \langle \sigma(\mathbf{f}), \sigma(\mathbf{f}) \rangle$. We can verify that if $\mathbf{f}$ is $\overline{\mathbf{Q}}$-rational, then $\sigma(\mathbf{f})$ is $\overline{\mathbf{Q}}$-rational. Therefore, both (5.3) and (5.4) hold under the condition $k_0 \geq 2$.

§6. Comments on the case where $k(\tau) = 2$ for some τ

We expect that our Main Theorem remains true under the weaker condition that $k(\tau) \geq 2$ for all $\tau \in J_F$ and that $k(\tau) \mod 2$ is independent of τ. Let us first state necessary ingredients to prove Main Theorem in this generality by our method in this paper. Let π be the irreducible unitary cuspidal automorphic representation of $GL(2, F_A)$ which corresponds to χ.

To prove Theorem 2 in this case by base change argument, it suffices to generalize (1.7) for any totally real *quadratic* extension F_1 of F. For this purpose, the following Hypothesis is sufficient, as remarked in §6.4 of [Y].

Hypothesis 1. *There exist a CM-extension K of F and a unitary Hecke character ψ of $K_A^\times$ which satisfy the following conditions.*

$$(1) \qquad \psi_v(x) = (x/|x|)^{l_v - 1}, \qquad x \in K_v^\times \cong \mathbf{C}^\times \quad \textit{for} \quad v \in \mathfrak{a}^K,$$

where l_v is a positive integer such that $l_\tau < k_\tau$ if $\tau \in \delta$, $l_\tau > k_\tau$ if $\tau \in J_F \backslash \delta$ and that $k_\tau - l_\tau \mod 2$ is independent of τ. Here we put $l_\tau = l_v$ taking $v \in \mathfrak{a}^K$ such that $v|F = \tau$.
(2) Let π' be the irreducible unitary automorphic representation of $GL(2, F_A)$ which corresponds to ψ. Then

$$L(\frac{1}{2}, \pi \times \pi')L(\frac{1}{2}, \pi \times \pi' \otimes \eta_{F_1}) \neq 0.$$

Here $L(s, \pi \times \pi')$ denotes the L-function obtained by the convolution of π and π'; η_{F_1} is the Hecke character of $F_A^\times$ which corresponds to the extension F_1/F.

Similarly Theorem 3 can be proved in this generality if the following Hypothesis is valid. Assume $k(\tau) = 2$ for some $\tau \in J_F$.

Hypothesis 2. *We use the same notation as in the proof of Theorem 3. There exist a quadratic extension K of F and a Hecke characters φ_1, ψ_1 of $F_A^\times$ which satisfy the following conditions.*

$$(1) \qquad (\varphi_1)_\infty \textit{ and } (\psi_1)_\infty \textit{ are given by (4.5) with } m_0 = k_0/2.$$

(2) *K is ramified at $\tau \in J_F$ if and only if* $\{\epsilon_1(\tau), \epsilon_2(\tau)\} = \{0, 1\}$.

$$(3)\qquad L(\frac{1}{2}, \pi \otimes \varphi_1)L(\frac{1}{2}, \pi \otimes \varphi_1\eta_K) \neq 0, \qquad L(\frac{1}{2}, \pi \otimes \psi_1)L(\frac{1}{2}, \pi \otimes \psi_1\eta_K) \neq 0,$$

where η_K is the Hecke character of $F_A^\times$ which corresponds to the extension K/F.

If these two Hypotheses are valid, Main Theorem holds under the weaker condition stated above. These hypotheses, in which we require *simultaneous* non-vanishing, are somewhat beyond our present knowledge. We only mention Harris [Ha4], Rohrlich [R] and Waldspurger [W] as papers treating related subjects.

When $k(\tau) = 2$ for all τ, Shimura proposed a construction of an abelian variety from critical values of $D(s, \chi, \varphi)$ in [S5], §11. If it were shown that that these abelian varieties have models over $\overline{\mathbf{Q}}$, as is well known when $F = \mathbf{Q}$, this construction would imply a still deeper assertion on the nature of critical values. If we could prove Main Theorem also in this case, Shimura's periods in [S5], §11 essentially coincide with $c_\tau^\pm(\chi)$, since $P(\chi, \{\tau\}, \epsilon) \sim \pi^{-1} c_\tau^{\epsilon(\tau)}(\chi)$.

References

[Ha1] M. Harris, Period invariants of Hilbert modular forms I, Lecture notes in Math. 1447, 155–202, 1990, Springer Verlag.

[Ha2] M. Harris, Period invariants of Hilbert modular forms II, preprint.

[Ha3] M. Harris, L-functions of 2×2 unitary groups and factorization of periods of Hilbert modular forms, preprint, 1991.

[Ha4] M. Harris, Non-vanishing of L-functions of 2×2 unitary groups, Forum Math. 5(1993), 405–419.

[Hi] H. Hida, On the critical values of L-functions of $GL(2)$ and $GL(2) \times GL(2)$, preprint , 1993 (to appear in Duke Math J.).

[JL] H. Jacquet and R.P. Langlands, Automorphic forms on $GL(2)$, Lecture notes in Math. 114, 1970, Springer Verlag.

[L] R.P. Langlands, Base change for $GL(2)$, Ann. of Math. Studies No. 96, Princeton University Press, 1980.

[R] D.E. Rohrlich, Nonvanishing of L-functions for $GL(2)$, Inv. Math. 97(1989), 381–403.

[S1] G. Shimura, The special values of the zeta functions associated with Hilbert modular forms, Duke Math. J. 45(1978), 637–679.

[S2] G. Shimura, The arithmetic of certain zeta functions and automorphic forms on orthogonal groups, Ann. of Math. 111(1980), 313–375.

[S3] G. Shimura, On certain zeta functions attached to two Hilbert modular forms I, II, Ann. of Math. 114(1981), 127–164, 569–607.

[S4] G. Shimura, Algebraic relations between critical values of zeta functions and inner products, Amer. J. Math. 104(1983), 253–285.

[S5] G. Shimura, On the critical values of certain Dirichlet series and the periods of automorphic forms, Inv. Math. 94(1988), 245–305.

[S6] G. Shimura, On the fundamental periods of automorphic forms of arithmetic type, Inv. Math. 102(1990), 399–428.

[S7] G. Shimura, The critical values of certain Dirichlet series attached to Hilbert modular forms, Duke Math. J. 63(1991), 557–613.

[W] J.-L. Waldspurger, Correspondances de Shimura et quaternions, Forum Math. 3(1991), 219–317.

[Y] H. Yoshida, On the zeta functions of Shimura varieties and periods of Hilbert modular forms, preprint, 1992 (to appear in Duke Math. J.).

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