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Nonlinear Filtering, Bellman Equations, and Schrödinger Equations

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Abstract—In this paper, the relations between theories of the quantum mechanics and the stochastic control are briefly surveyed based on the recent works of the author (Ohsumi, 1989a, 1989b, 1992). Concretely, two types of (nonlinear) Schrödinger eqations from the stochsatic control theory by introducing wavefunctions which connect two solutions of nonlinear filtering equation and stochastic optimal control problem. Furthermore, an inverse problem is investigated to derive the optimal control problem from a given Schrödinger equation.

1. Introduction

Consider the nonlinear filtering problem for which the signal and the observation processes are described by (scalar) stochastic differential equations:

$$dx(t) = f(x(t))dt + G(x(t))dw(t), \quad x(0) = x_0 \tag{1}$$

$$dy(t) = h(x(t))dt + dv(t), \quad y(0) = 0, \tag{2}$$

where $f(\cdot), G(\cdot), h(\cdot)$ are nonlinear functions; and $w(t), v(t)$ are mutually independent standard Wiener processes. The filtering problem is to obtain the conditional probability density of $x(t)$ based on the observation σ-algebra $Y_t := \sigma\{y(s), 0 \leq s \leq t\}$. The conditional probability density function is the normalization of $q(t,x)$ satisfying the Duncan-Mortensen-Zakai equation (Davis and Marcus, 1981)

$$dq(t,x) = (L_0 + c(x))q(t,x)dt + h(x)q(t,x)dy(t), \quad q(0,x) = p_0(x) \tag{3}$$

where $(L_0 + c(x)) := a(x)\partial^2/\partial x^2 + b(x)\partial/\partial x + c(x)$ is the formal adjoint of the differential generator of the signal process $x(t)$, and $p_0(x)$ is the initial condition..

Let

$$p(t,x) = \exp\{-h(x)y(t)\}q(t,x). \tag{4}$$

Then $p(t,x)$ satisfies the following partial differential equation called the pathwise-robust equation of the nonlinear filtering problem (Davis and Marcus, 1981),

$$\frac{\partial p(t,x)}{\partial t} = \tilde{L}_0 p(t,x) + \tilde{c}(t,x)p(t,x), \quad p(0,x) = p_0(x) \tag{5}$$

where $\tilde{L}_0 := a(x)\partial^2/\partial x^2 + \tilde{b}(t,x)\partial/\partial x$, and $\tilde{b}(t,x)$ and $\tilde{c}(t,x)$ are the functions of $y(t)$.

On the other hand, consider the optimal control problem minimizing the cost functional

$$J(t, x_0; u) = E_{x_0}\left\{S_0[\xi(t)] + \int_0^t L[\tau, \xi(\tau), u(\tau)]d\tau\right\} \tag{6}$$

(where $E_{x_0}\{\cdot\}$ is the conditional expectation conditioned on $\xi(0) = x_0$) for either the process

$$d\xi(\tau) = \tilde{b}(\tau, \xi(\tau))d\tau + u(\tau)d\tau + G(\xi(\tau))dw(\tau), \quad 0 \leq \tau \leq t \tag{7}$$

with the feedback control $u(\tau) = u(\tau, \xi(\tau))$ and the cost rate function $L(\tau, x, u) = (1/4\nu N(x))u^2 - \nu_0\tilde{c}(\tau, x)$, or the process

$$d\xi(\tau) = u(\tau, \xi(\tau))d\tau + G(\xi(\tau))dw(\tau) \tag{8}$$

with $L(\tau, x, u) = (1/4\nu N(x))[u - \tilde{b}(\tau, x)]^2 - \nu_0\tilde{c}(\tau, x)$. For both processes the Bellman-Hamilton-Jacobi equation which the minimum cost functional to (6) satisfies is given by

$$\frac{\partial S(t,x)}{\partial t} = \tilde{L}_0 S(t,x) - \nu_0\tilde{c}(t,x) - \nu N(x)\left(\frac{\partial S(t,x)}{\partial x}\right)^2. \tag{9}$$

If we set $\nu_0 = 1, \nu = 1/2$ and $N(x) = 2a(x)$ in the control problems, then the two solutions of (5) and (9) are related by the Cole-Hopf logarithmic transformation as (Fleming and Mitter, 1982)

$$S(t,x) = -\ln p(t,x). \tag{10}$$

For the nonlinear filtering problems, see Bucy and Joseph (1968), Jazwinski (1970), Lipter and Shiryaev (1977), Kallianpur (1980); while for the stochastic optimal control, refer Wonham (1970), Fleming and Rishell (1975).

2. Derivation of Schrödinger Equations

By combining the solutions of (5) and (9), introduce the function by

$$\psi(t,x) = p^{1/2}(t,x)\exp\{iS(t,x)\} \quad (i = \sqrt{-1}). \tag{11}$$

Then we have the following theorem.

Theorem 1. Let $p(t,x)$ be the solution to the pathwise-robust nonlinear filtering equation (5) and $S(t,x)$ be the solution to the Bellman equation (9). Then the complex function $\psi(t,x)$ defined by (11) satisfies

$$\frac{\partial\psi(t,x)}{\partial t} = [L_0 + V(t,x;\psi)]\psi(t,x) \tag{12}$$

with the initial condition $\psi(0,x) = {p_0}^{1/2}(x)\exp\{iS_0(x)\}$, and

$$\begin{aligned} V(t,x;\psi) := {} & \frac{1}{2}\,(1-2i\nu_0)\,\tilde{c}(t,x) \\ & + a(x)\left\{\left(\frac{\partial}{\partial x}\ln\psi^*(t,x)\right)^2 + 2h_x(x)y(t)\left(\frac{\partial}{\partial x}\ln\psi(t,x)\right)\right\} \\ & - \frac{1}{2}\{a(x) - \frac{1}{2}i\nu N(x)\}\left(\frac{\partial}{\partial x}\ln\frac{\psi^*(t,x)}{\psi(t,x)}\right)^2, \end{aligned} \tag{13}$$

where $\psi^*(t,x)$ is the complex conjugate of $\psi(t,x)$.

If the time t is formally replaced by the imaginary time $t/i\hbar$ (where $\hbar$ is the Planck constant) in (12), then it is nothing but a Schrödinger equation with complex random nonlinear potential. In quantum physics, such a type of nonlinear Schrödinger equation is called the *Schrödinger-Langevin equation* and familiar in describing the collective motion of Cooper pairs causing the superconducting current (Razavy, 1978; Yasue, 1979).

Instead of $p(t,x)$, by using the unnormalized conditional density $q(t,x)$ in (11), we get another version of nonlinear Schrödinger equation.

Theorem 2. Let

$$\psi_0(t,x) = q^{1/2}(t,x)\exp\{iS(t,x)\}. \tag{14}$$

Then $\psi_0(t,x)$ satisfies the stochastic partial differential equation,

$$d\psi_0(t,x) = [L_0 + V_0(t,x;\psi_0)]\psi_0(t,x)dt + \frac{1}{2}h(x)\psi_0(t,x)dy(t) \tag{15}$$

with the same initial condition as in (12), where $V_0(t,x;\psi_0)$ is the random potential similar to (13).

Such a version similar to (15) with nonrandom potential independent of its state is known as the *Itô-Schrödinger equation* which describes wave propagation in random media (Dawson and Papanikolaou, 1984). For the proofs of Theorems 1 and 2 and the relation between the two wavefunctions, see Ohsumi (1989a, 1989b).

3. Probabilistic Interpretation of Wavefunctions

From the definition of wavefunction $\psi(t,x)$, we can readily see that the square amplitude of the wavefunction yields the conditional probability density for the nonlinear filtering problem described by (1) and (2), i.e.,

$$| \psi |^2 = \psi(t,x)\psi^*(t,x) = p(t,x), \tag{16}$$

and furthermore that its argument gives the minimum cost functional of the optimal control problem (6), i.e.,

$$\arg[\psi(t,x)] = S(t,x), \tag{17}$$

where $\arg[z]$ is an argument of $z \in C$. Furthermore, recalling the relations $| \psi_0 |^2 = q(t,x)$ and $q(t,x) = \Lambda(t)\rho(t,x)$, where $\rho(t,x)$ is the solution to Kushner equation for the conditional probability density of $x(t)$ given observation data Y_t (Kushner, 1967) and $\Lambda(t)$ is the likelihood-ratio function (Radon-Nikodym derivative) given by

$$\Lambda(t) := \exp\left\{\int_0^t \hat{h}(s,x_s)dy(s) - \frac{1}{2}\int_0^t \hat{h}^2(s,x_s)ds\right\}, \tag{18}$$

where $\hat{h}(s,x_s) := \int_{-\infty}^{\infty} h(x)\rho(s,x)dx$, we have the relation,

$$\int_{-\infty}^{\infty} | \psi_0(t,x) |^2 \, dx = \Lambda(t). \tag{19}$$

4. Control-Theoretic View of Schrödinger Equations: An Inverse Problem

Our interesting problem will be as follows: Given a Schrödinger equation, what control problems correspond to it? In this section we seek to find a clue to this problem.

To fix the idea, give the Schrödinger equation

$$i\hbar\frac{\partial\psi(t,x)}{\partial t} = [L + V(t,x)]\psi(t,x), \tag{20}$$

where $L := (-\hbar^2/2m)\partial^2/\partial x^2, m$ the effective mass of the Cooper pairs in superconducting media, and $V(t,x)$ the potential function (Yasue, 1979).

Let

$$\psi(t,x) = \exp\left\{-\frac{i}{\hbar}S(t,x)\right\}. \tag{21}$$

This is just the complex Cole-Hopf transformation, and conversely we have

$$S(t,x) = i\hbar\ln\psi(t,x) \tag{22a}$$

$$= -i\hbar\ln\psi^*(t,x). \tag{22b}$$

Differentiating (22a) with respect to t, we obtain

$$\frac{\partial S(t,x)}{\partial t} = \frac{1}{\psi(t,x)}\, L\psi(t,x) + V(t,x). \tag{23}$$

Now

$$L\psi(t,x) = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\, e^{-\frac{i}{\hbar}S(t,x)}$$

$$= \frac{i\hbar}{2m}\,\psi(t,x)\left[\frac{\partial^2 S(t,x)}{\partial x^2} - \frac{i}{\hbar}\left(\frac{\partial S(t,x)}{\partial x}\right)^2\right], \tag{24}$$

so that, by substituting (24) into (23), we get

$$-\frac{\partial S(t,x)}{\partial t} = -\frac{i\hbar}{2m}\frac{\partial^2 S(t,x)}{\partial x^2} - V(t,x) - \frac{1}{2m}\left(\frac{\partial S(t,x)}{\partial x}\right)^2. \tag{25}$$

Noting the relation

$$-\frac{1}{2m}\left(\frac{\partial S}{\partial x}\right)^2 = \min_u \left\{\frac{\partial S}{\partial x}u + \frac{1}{2}mu^2\right\}, \tag{26}$$

we can rewrite (25) as

$$-\frac{\partial S(t,x)}{\partial t} = \min_u \left\{L(t,x,u) + \frac{\partial S(t,x)}{\partial x}u - \frac{i\hbar}{2m}\frac{\partial^2 S(t,x)}{\partial x^2}\right\}, \tag{27}$$

where $L(t,x,u) := (m/2)u^2 - V(t,x)$ is the Lagrangian.

For (27) we try to give a control-theoretic interpretation. To do this, consider a controlled stochastic process $\xi(t)$ generated by

$$d\xi(t) = u(t)dt + Gdw(t), \tag{28}$$

and the cost functional to be minimized,

$$J(u) = E_{x_0}\left\{\tilde{S}_0[\xi(T)] + \int_0^T L[t,\xi(t),u]dt\right\}, \tag{29}$$

where $\tilde{S}_0(\cdot)$ is a some properly given function (which depends on the boundary conditions to the Schrödinger equation (20)). Let $\tilde{S}(t,x)$ be the minimum cost functional for (29); then this satisfies the Bellman equation,

$$-\frac{\partial \tilde{S}(t,x)}{\partial t} = \min_u \left\{L(t,x,u) + \frac{\partial \tilde{S}(t,x)}{\partial x}u + \frac{1}{2}G^2\frac{\partial^2 \tilde{S}(t,x)}{\partial x^2}\right\}. \tag{30}$$

Comparing (30) with (27), we know that these two equations coincide each other under the correspondence

$$G^2 \longleftrightarrow -\frac{i\hbar}{m}. \tag{31}$$

Such a correspondence similar to (31) was first suggested by Schrödinger (1931) and Métadier (1931) in the analogy between Schrödinger equation for a free particle and the diffusion equation appearing in the Einstein's theory of Brownian motions, and later pointed out in the analogy between stochastic optimal control and quantum mechanics by Papiez (1982).

Under the correspondence (31) we may set as $\tilde{S}(t,x) \equiv S(t,x)$ with proper terminal function $\tilde{S}_0$; so that the optimal control is given by

$$u = u(t,x) = -\frac{1}{m}\frac{\partial S(t,x)}{\partial x}, \tag{32}$$

or, in terms of the wavefunction $\psi(t,x)$,

$$u = -\frac{i\hbar}{m}\frac{1}{\psi(t,x)}\frac{\partial \psi(t,x)}{\partial x}. \tag{33}$$

In other words, for the stochastic optimal control problem described by (28) and (29), the control $u(t)$ can be obtained by solving the "linear" Schrödinger equation (20) instead of solving the backward "nonlinear" Bellman equation (30).

5. Conclusions

In this paper, it has been shown that the (nonlinear) Schrödinger equation can be derived from the stochastic control theory, and an inverse problem has also been investigated. The form of wavefunction (11) was first introduced by Nelson (1966) to derive the (linear) Schrödinger equation in which $S(t,x)$ was taken as a Newtonian potential. Since the wavefunction keeps all information about the optimal control and/or filtering as discussed in Sec.3, we may say that the present paper will give us a new aspect of stochastic control theory.

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