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Theta functions and modular forms

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It is not easy to determine the structure of the graded ring of (Siegel) modular forms. In a series of papers beginning in 1964 [1], J. Igusa suceeded to determim structures of these rings for various levels in the case of genus 1 and 2. After that many mathmaticians attacked to this problem and succeeded. In this note, we shall determine the structure of graded ring of genus 2 Siegel modular forms of certain level, by using the structure theorem of the principally polarized abelian varieties with that level structure, which is given in the previous paper [8].

1. Thrughout this note we fix a positive integer $g(\geq 2)$. Let $\Gamma_g(2)$ denote the principal congruence subgroup, of level 2, of the modular group $\Gamma_g(1) = \mathrm{Sp}_{2g}(\mathbf{Z})$, and let

$$\Gamma_g(2,4) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_g(2) \mid \begin{array}{c} \mathrm{diag}(a^t b) \equiv 0 \\ \mathrm{diag}(c^t d) \equiv 0 \end{array} \mathrm{mod} 4 \right\}$$

. Let $\mathcal{S}_g$ denote the Siegel upper half-space of degree g on which $\Gamma_g(1)$ acts by the map : $\tau \longrightarrow (a\tau + b)(c\tau + d)^{-1}$, $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_g(1)$. We denote by $A_g(2,4)$ the quotient $\mathcal{S}_g/\Gamma_g(2,4)$, which is the moduli space of principally polarized abelian varieties with level (2,4) structure. A point $x \in A_g(2,4)$ is called an *irreducible* point if the corresponding principally polarized abelian variety is irreducible, i.e., it is not isomorphic to the product of principally polarized abelian varieties of smaller dimension. For the moduli theoretic meaning of this space, we refer to [6].

Now we recall the definition of Riemann's thta constants. Let $m = \begin{pmatrix} m' \\ m'' \end{pmatrix}$ denote a vector in $\frac{1}{2}\mathbf{Z}^{2g}(m', m" \in \frac{1}{2}\mathbf{Z}^g)$. We define the theta constant $\vartheta[m](\tau)$ of characteristic $[m]$ by

$$\vartheta[m](\tau) = \sum_{p \in \mathbf{Z}^g} \mathbf{e}(\frac{1}{2}{}^t(p+m')\tau(p+m') + {}^t(p+m')m'')$$

where τ is a variable in $\mathcal{S}_g$ and $\mathbf{e}(*) = \exp(2\pi i *)$. We say that a theta characteristic $[m]$ is even or odd according as $e(m) \stackrel{def}{=} \mathbf{e}(2^t m' m'') = \pm 1$. It is well known that $\vartheta[m](\tau) \equiv 0$ if and only if $[m]$ is odd.

Using the transformation formula of theta constants, we get a holomorphic map of $\mathcal{S}_g$ to the projective space $\mathbb{P}^N$, $N = 2^g - 1$, defined by

$$\tau \longmapsto \left(\cdots, \vartheta \begin{bmatrix} a \\ 0 \end{bmatrix} (2\tau), \cdots \right),$$

where a runs over a complete set of representatives of $2^{-1}\mathbf{Z}^g$ modulo $\mathbf{Z}^g$. It induces a holomorphic map

$$\Phi_2 : A_g(2,4) \longrightarrow \mathbb{P}^N.$$

In a previous paper [8], we showed that the following:

Theorem 1. *Φ_2 is locally embedding at irreducible points in $A_g(2,4)$ and if $x \in A_g(2,4)$ corresponds to the period matrix of a hyperelliptic curve of genus g, then $\Phi_2^{-1}(\Phi_2(x)) = \{x\}$.*

As a corollary, we have the following structure theorem of the Satake compactification $A_2(2,4)^*$ of $A_2(2,4)$ (cf. ibid. 3.Remark).

Theorem 2. *The map Φ_2 can be extended naturally to $A_2(2,4)^*$;*

$$\Phi_2 : A_2(2,4)^* \longrightarrow \mathbb{P}^3,$$

and this is an isomorphism.

2. Let Γ be a congruence subgroup of $Sp_{2g}(\mathbf{Z})$. A holomorphic function $f(\tau)$ on $\mathcal{S}_g$ satisfying

$$f(\sigma \cdot \tau) = (c\tau + d)^k f(\tau)$$

for all $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ and the finiteness at the cusp if $g = 1$, is called a Siegel modular forms of weight k relative to Γ. We denote by

$$A(\Gamma) = \bigoplus_{k=0}^{\infty} A_k(\Gamma),$$

the graded ring of modular forms relative Γ, where $A_k(\Gamma)$ is the vector space of modular forms of weight k.

For several congruence subgroups Γ, the structure theorem for $A(\Gamma)$ is known. We can find them in [1,2,3,4] when $g = 1, 2$ and in [9] for $\Gamma_3(1)$. For example, the structure theorem for $A_2(2,4)$ is given by Igusa in the following form:

Theorem 3. (Igusa)

$$A(\Gamma_2(2,4)) = \mathbf{C}[\vartheta^2[m](\tau)\vartheta^2[n](\tau)][\chi(\tau)],$$

where $[m], [n]$ *run over the set of 10 even characteristics and*

$$\chi(x) = \prod_{[m]:even} \vartheta[m](\tau).$$

We shall now give a proof for this theorem using Theorem 2.

By the transformation formula of theta constants, we see the $\mathbf{C}$-algebra in the right hand side is a subring of $A(\Gamma_2(2,4))$.

First we note that, by the addition formula of theta functions ([5]), we have

$$\vartheta[m](\tau)^2 = \sum_p \mathrm{e}(2m''\, {}^t p)\vartheta \begin{bmatrix} m' + p \\ 0 \end{bmatrix} (2\tau)\vartheta \begin{bmatrix} p \\ 0 \end{bmatrix} (2\tau),$$

where p runs over the complete set of representatives of $2^{-1}\mathbf{Z}^2$ modulo $\mathbf{Z}^2$.

We consider

$$\left(\vartheta \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} (2\tau), \vartheta \begin{bmatrix} 0 \\ \frac{1}{2} \\ 0 \\ 0 \end{bmatrix} (2\tau), \vartheta \begin{bmatrix} \frac{1}{2} \\ 0 \\ 0 \\ 0 \end{bmatrix} (2\tau), \vartheta \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 0 \\ 0 \end{bmatrix} (2\tau) \right)$$

as the homogeneous coordinates (X_0, X_1, X_2, X_3) of

$$A_2(2,4)^* \simeq I\!P^3.$$

If $f(\tau)$ is a modular form of even weight $2k$, then

$$f(\tau)/\vartheta[0](\tau)^{4k}$$

is a meromorphic function on $I\!P^3$, and its pole is $l \cdot Q, l \leq k$, where Q is the quadric defined by

$$P(X) = X_0^2 + X_1^2 + X_2^2 + X_3^2.$$

Comparing the divisors, we have

$$f(\tau)/\vartheta[0](\tau)^{4k} = cF(\cdots,\vartheta\begin{bmatrix} m' \\ 0 \end{bmatrix}(2\tau),\cdots)/P(\cdots,\vartheta\begin{bmatrix} m' \\ 0 \end{bmatrix}(2\tau),\cdots)^{l},$$

where c is a non-zero constant and $F(X)$ is a homogeneous polynomial of degree $2l$. Thus we have

$$f(\tau) = cF(\cdots,\vartheta\begin{bmatrix} m' \\ 0 \end{bmatrix}(2\tau),\cdots)\vartheta[0](\tau)^{4k-2l}.$$

Let $f(\tau)$ be a modular form of odd weight $2k+1$. If $2k+1 \leq 3$, i.e, $k \leq 2$, then

$$\phi_k(\tau) = f(\tau)\vartheta[0](\tau)^{8-2k}/\chi(\tau), \qquad k = 0, 1$$

is a meromorphic function on $I\!P^3$. The divisor of the square of this function is of the form

$$\sum_i A_i - \sum_j B_j,$$

where $\{B_j\}$ are *distinct* irreducible quadrics. Therefore ϕ_k can not be a non-constant meromorphic function. Since ϕ_k is obviously not a non-zero constant, it follows $f = 0$.

If $2k+1 \geq 5$, then

$$\psi_k(\tau) = f(\tau)/\chi(\tau)\vartheta[0](\tau)^{4k-8},$$

is a meromorphic function on $I\!P^3$. Then the divisor of ψ_k^2 has the form :

$$div(\psi_k^2) = \sum_i A_i - \sum_j B_j - l\cdot Q = 2div(\psi_k),$$

where $\{B_j\}, Q$ are distict irreducible quadrics. Therefore $\sum B_j$ can not occur and l is even; $l = 2l' \leq 2k-4$. Comparing the divisors, we have

$$f(\tau)/\chi(\tau)\vartheta[0](\tau)^{4k-8} = cG(\cdots,\vartheta\begin{bmatrix} m' \\ 0 \end{bmatrix}(2\tau),\cdots)/P(\cdots,\vartheta\begin{bmatrix} m' \\ 0 \end{bmatrix}(2\tau),\cdots)^{l'},$$

where c is a non-zero constant and $G(X)$ is a homogeneous polynomial of degree $l = 2l'$. Hence we have

$$f(\tau) = cG(\cdots,\vartheta\begin{bmatrix} m' \\ 0 \end{bmatrix}(2\tau),\cdots)\vartheta[0](\tau)^{4k-8-2l'}\chi(\tau).$$

Thus we completed the proof.

References

[1] J. Igusa: On Siegel modular forms of genus two, Amer. J. Math. **84** (1962),175-200.

[2] J. Igusa: On Siegel modular forms of genus two (2), Amer. J. Math. **86** (1964),392-412.

[3] J. Igusa: On the graded ring of theta-constants, Amer. J. Math. **86** (1964),219-246.

[4] J. Igusa: On the graded ring of theta-constants II, Amer. J. Math. **88** (1966),221-236.

[5] J. Igusa: Theta functions. Springer Verlag (1972).

[6] D. Mumford: Lecture no theta II. Prog. in Math.,**43**, Birkhäuser (1984).

[7] B. Runge: On Siegel modular forms, part I, J. Reine angew. Math. **436** (1993),57-85 .

[8] R. Sasaki: Some remarks on the moduli space of principally polarized abelian varieties with level (2,4) structure, Comp. Math. **85** (1993), 87-97.

[9] S. Tuyumine: On Siegel modular forms of degree three, Amer. J. Math,**108** (1987), 755-862.