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Degenerate Bifurcation in Stably Stratified Plane Poiseuille Flow

by

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1. Introduction

Weakly nonlinear evolution of a monochromatic wave disturbance is governed by the Stuart-Landau equation if the parameter set for the disturbance is in slightly supercritical state whereas all the higher harmonics including the zeroth harmonic are damping. In the ordinary situation, the Stuart-Landau equation is truncated at the cubic order approximation:

$$\frac{dA}{dt} = \sigma A + \lambda |A|^2 A,$$

where σ denotes the linear growth rate of the fundamental mode and λ is called the Landau constant. According to the cubic equation, qualitative behavior of the disturbance has been classified into four categories based on the signs of the real part of σ and λ: $|A| \to 0$ for Re $\sigma < 0$ and Re $\lambda < 0$; $|A| \to \infty$ for Re $\sigma > 0$ and Re $\lambda > 0$; $|A|^2 \to -\mathrm{Re}\ \sigma/\mathrm{Re}\ \lambda$ for Re $\sigma > 0$ and Re $\lambda < 0$ (supercritical bifurcation); $|A|^2 \to 0$ or ∞ depending respectively on whether $|A| > -\mathrm{Re}\ \sigma/\mathrm{Re}\ \lambda$ or not (subcritical bifurcation) for Re $\sigma < 0$ and Re $\lambda > 0$. Therefore the sign of Re λ is especially important in order for the classification. It is known that quasi-critical disturbances in plane Poiseuille flow and Blasius boundary-layer flow exhibit subcritical bifurcation whereas quasi-critical disturbances in plane jet, plane wake, free shear layer, Rayleigh-Bénard convection, Taylor-Couette flow, and natural convection in a vertical slot exhibit supercritical bifurcation, among others. It is also known that Re λ changes its sign along the lower branch of the neutral stability curve for plane Poiseuille flow (Pekeris and Shkoller) and for Blasius boundary layer (Herbert). We have to proceed to at least fifth order approximation in such a degenerate case with Re $\lambda = 0$ in order to unfold the bifurcation characteristics.

Recently, Eckhaus and Iooss investigated such nonlinear degenerate bifurcation problems in detail and classified the bifurcation characteristics. Moreover, they examined the stability of periodic solutions subject to general perturbations. They found strong selection or rejection of spatially periodic patterns. They referred to Sen and Vashist's numerical work on the higher order Stuart-Landau equation for Blasius boundary layer flow as the typical example of such degenerate bifurcation problems. Degenerate bifurcation is known to occur at the criticality in couple of flow fields. Among them, the Taylor-Couette flow between counter-rotating cylinders has been investigated by Laure and Demay. In the problem, with specific ratio of outer/inner rotation speeds, the critical disturbance is suffering from cubic order degeneracy. They applied the center manifold reduction technique and derived the Stuart-Landau equation with fifth order nonlinear term. The double diffusive convection has been investigated by Knobloch. Under free-free boundary conditions, we encounter degenerate Hopf bifurcation in the presence of O(2) symmetry. Knobloch derived coupled amplitude equations with fifth order nonlinear terms and classified the bifurcation characteristics. But the last case is rather artifice because the degeneracy does not set in in the system having rigid-rigid boundaries.

In the present paper, we show that the cubic order nonlinear degeneracy sets in for the critical mode in stably stratified plane Poiseuille flow. We derive the Stuart-Landau equation with the quintic order nonlinear term and classify the bifurcation characteristics especially for mercury whose Prandtl number is 0.025. We further show a possibility of hyper degenerate situation in which the cubic as well as the quintic order Landau constants lose their real parts, simultaneously.

2. Mathematical Formulation

We assume a plane Poiseuille flow in a channel whose top and bottom walls located at $z^* = \pm H$ are respectively heated and cooled at uniform temperatures $T_0 + \Delta T$ and $T_0 - \Delta T$ where $\Delta T > 0$. The flow is in x direction. The motion of fluid and temperature are governed by

$$\begin{aligned} \rho[\vec{V}^*_{t^*} + (\vec{V}^* \cdot \nabla^*)\vec{V}^*] &= -\nabla^* p^* - \rho g[1 - \beta(T^* - T_0)]\vec{e}^*_z + \mu\nabla^{*2}\vec{V}^*, \\ T^*_{t^*} + (\vec{V}^* \cdot \nabla^*)T^* &= \kappa\nabla^{*2}T^*, \\ \nabla^* \cdot \vec{V}^* &= 0, \end{aligned} \tag{1}$$

where $\vec{V}^*$ is the velocity, T^* is the temperature, p^* is the pressure, ρ is the density, g is the acceleration due to gravity, β is the thermal expansion coefficient, μ is the viscous coefficient, and κ is the thermal diffusivity. We nondimensionalize all the quantities as

$$\vec{V}^* = \bar{u}_0\vec{v}, \quad \vec{x}^* = H\vec{x}, \quad T^* = \Delta T \cdot T, \quad t^* = H\bar{u}_0^{-1}t, \quad \text{and} \quad p^* = \rho_0\bar{u}_0^2 p,$$

where $\bar{u}_0$ is the maximum velocity on the centerline of a channel and ρ_0 is the density evaluated at a reference temperature T_0.

Split $\vec{v}$, T, and p into the basic field with overbar and the disturbance with overhat as

$$\vec{v} = \bar{\mathbf{v}} + \hat{\mathbf{v}}, \quad T = \bar{T} + \hat{T}, \quad \text{and} \quad p = \bar{p} + \hat{p}.$$

The basic field is easily obtained as

$$\bar{\mathbf{v}} = (\bar{u}, 0, 0) = (1 - z^2, 0, 0), \quad \bar{T} = z. \tag{2}$$

In the present paper, we focus ourselves on two-dimensional disturbances added to the two-dimensional basic field. We introduce the stream function $\hat{\psi}$ such that

$$\hat{u} = \partial\hat{\psi}/\partial z, \quad \hat{w} = -\partial\hat{\psi}/\partial x.$$

The disturbance components are thus described by the disturbance equations of the form of

$$\begin{aligned} \partial_t \nabla^2 \hat{\psi} + \bar{u}\partial_x \nabla^2 \hat{\psi} - \bar{u}'' \hat{\psi}_x &= Re^{-1}\nabla^4 \hat{\psi} - Ri \hat{T}_x + J(\hat{\psi}, \nabla^2 \hat{\psi}), \\ \hat{T}_t + \bar{u}\hat{T}_x - \hat{\psi}_x &= Re^{-1}P^{-1}\nabla^2 \hat{T} + J(\hat{\psi}, \hat{T}), \end{aligned} \tag{3}$$

where we have three nondimensional parameters: i.e., $Re = \rho_0 u_o H/\mu$ is the Reynolds number, $P = \mu/(\rho_0 \kappa)$ is the Prandtl number, and $Ri = RaRe^{-2}P^{-1}$ is the Richardson number. Here $Ra = \beta g \Delta T H^3/(\mu\kappa)$ is the Rayleigh number. $J(f, g)$ is the Jacobian defined by $\partial(f, g)/\partial(x, z)$.

The boundary conditions for $\hat{\psi}$ and $\hat{T}$ are imposed as

$$\hat{\psi} = \partial\hat{\psi}/\partial z = 0 \ \text{ at } \ z = \pm 1, \ \text{ and } \ \hat{T} = 0 \ \text{ at } \ z = \pm 1. \tag{4}$$

3. Weakly Nonlinear Reduction

Set $(\hat{\psi}, \hat{T})^T \equiv \vec{\Psi}$. We expand $\vec{\Psi}$ in powers of ϵ and E where ϵ is a measure of the supercriticality defined by $Re_c^{-1} - Re^{-1} \equiv \epsilon^2 \tilde{Re}$ with $\tilde{Re} \sim O(1)$ and E is the neutral wave component defined by $E \equiv \exp[i\alpha(x - ct)]$ with the wavenumber α and the real phase velocity c. The result is

$$\begin{aligned} \vec{\Psi} &= \epsilon(\vec{\Psi}_{11}E + c.c.) + \epsilon^2(\vec{\Psi}_{22}E^2 + c.c. + \vec{\Psi}_{02}) + \epsilon^3(\vec{\Psi}_{33}E^3 + \vec{\Psi}_{13}E^1 + c.c.) \\ &+ \epsilon^4(\vec{\Psi}_{44}E^2 + \vec{\Psi}_{24}E^2 + c.c. + \vec{\Psi}_{04}) + \epsilon^5(\Psi_{15}E + c.c.) + O(\epsilon^5). \end{aligned} \tag{5}$$

Moreover, we assume that $Ri = Ri_c + \epsilon^2 \tilde{Ri}$ with $\tilde{Ri} \sim O(1)$ and $\alpha = \alpha_c + \epsilon^2 \tilde{\alpha}$ with $\tilde{\alpha} \sim O(1)$. Let us apply the method of multiple scales by introducing the derivative expansions

$$\partial_t = \sum_{j=0} \epsilon^{2j} \partial_{t_j}, \quad t_j \equiv \epsilon^j t. \tag{6}$$

For later convenience, we introduce some linear operators:

$$M_j \equiv \begin{pmatrix} S_j & 0 \\ 0 & 1 \end{pmatrix},$$

$$L_j \equiv \begin{pmatrix} ij\alpha_c\bar{u}S_j - ij\alpha_c\bar{u}'' - Re_c^{-1}S_j^2 & ij\alpha_c Ri_c \\ -ij\alpha_c & ij\alpha_c\bar{u} - Re_c^{-1}P^{-1}S_j \end{pmatrix},$$

where $S_j \equiv D^2 - j^2\alpha_c^2$ and $D \equiv \partial/\partial z$.

Substitute (5) as well as (6) into (3) and equate the same powers of $\epsilon^k E^l$ to zero. Then we obtain the following system of equations: at ϵE, we have

$$[-i\alpha_c c M_1 + L_1]\vec{\Psi}_{11} = 0, \tag{7}$$

where the solution is expressed as

$$\vec{\Psi}_{11} = A_1(t_1, t_2, \cdots)\vec{\Phi}_{11}(z), \quad \vec{\Phi}_{11} = \begin{pmatrix} \phi_{11}(z) \\ \theta_{11}(z) \end{pmatrix}. \tag{8}$$

In (8), $A_1(t_1, t_2, \cdots)$ represents an amplitude function whose temporal evolution will be determined in the course of the reduction. Equation (7) subject to (4) consist of the eigenvalue problem and Φ_{11} corresponds to the eigenfunction.

By carrying out straightforward manipulation, we finally obtain the quintic Stuart-Landau equation

$$\begin{aligned} da/dt = {} & (\beta\lambda_{11} + \gamma\lambda_{12} + \delta\lambda_{13} + \beta^2\lambda_{21} + \beta\gamma\lambda_{22} + \beta\delta\lambda_{23} + \gamma^2\lambda_{24} + \gamma\delta\lambda_{25} + \delta^2\lambda_{26})a \\ & + (\lambda_{14} + \beta\lambda_{27} + \gamma\lambda_{28} + \delta\lambda_{29})|a|^2 a + \lambda_{210}|a|^4 a, \end{aligned} \tag{9}$$

where $a \equiv \epsilon A_1 + \epsilon^3 A_2 + \cdots$, A_2 is an amplitude function appeared at the cubic order approximation, $\beta \equiv \epsilon^2\tilde{\alpha}$, $\gamma \equiv \epsilon^2\tilde{Re}$, and $\delta \equiv \epsilon^2 Ri$.

4.1. Behavior of The Cubic Landau Constant λ_{13}

We evaluate the critical conditions (α_c, Re_c) and the coefficients involved in eq.(9) for different values of P, Ri. For simplicity of analysis, the derivation of the Stuart-Landau equation (9) in §3 is based on the constant mass flux condition. The constant pressure gradient condition is also important when we try to compare theoretical results with experimental ones. There is, however, no qualitative difference upon the behavior of λ_{13} depending on which condition is imposed as has been already pointed out by Craik. In this paper, therefore, we evaluate the first Landau constant under the constant mass flux condition. We plot the distribution of Re_c as a function of Ra in Fig.1 for different values of P. In Fig.2, we plot the corresponding values of Re λ_{13} in the same manner. Now we find that the real part of the first Landau constant becomes negative beyond some critical value on Ra for $P < 0.17$ while is always positive for $P > 0.17$. Subcritical feature thus changes to the supercritical when the Prandtl number is decreased. These figures are based on the normalization for the eigenfunction that $i\alpha\phi_1(z = 0) = 1$. Different normalization causes different values on the Landau constants. The change from subcritical to supercritical is not affected, however, by different normalization conditions.

The critical Richardson number beyond which supercritical bifurcation occurs is plotted as a function of the Prandtl number in Fig.3. From the figure, we find that the critical number has an asymptotic behavior for small P as $Ri \sim 1.093 \times 10^{-5} P^{-1}$ while the critical number tends to infinity for $P \rightarrow 0.17$. Beyond the curve of Fig.3, supercritical bifurcation occurs.

4.2. Cubic Degeneracy

Since the supercritical feature is obtained for relatively small Prandtl numbers, mercury would be the best example which may exhibit the cubic degeneracy for a high Richardson (or Rayleigh) number. Mercury has the Prandtl number $P = 0.025$ at the room temperature. The critical Richardson number, Reynolds number, and wavenumber which give the cubic degeneracy on (9) are

$$Ri_c = -5.93224610 \times 10^{-2}, \quad Re = 1.251430041986 \times 10^4,$$

$$\alpha = 0.9937969914465, \quad c_r = 0.21136070568. \tag{10}$$

Under the constant mass flux condition and a new normalization, $\phi_1(z = 0) = 1$, we evaluated all the coefficients involved in (9). The values are listed in Table I together with the ones for $P = 0.0001$.

Our concern here is bifurcation characteristics of (9) around the degenerate point. Let's set $a = b\, e^{i\theta}$. Equation (9) is thus written as

$$\begin{aligned} db/dt = (&\beta\lambda_{11r} + \gamma\lambda_{12r} + \delta\lambda_{13r} + \beta^2\lambda_{21r} + \beta\gamma\lambda_{22r} \\ &+\beta\delta\lambda_{23r} + \gamma^2\lambda_{24r} + \gamma\delta\lambda_{25r} + \delta^2\lambda_{26r})b \\ &+(\lambda_{14r} + \beta\lambda_{27r} + \gamma\lambda_{28r} + \delta\lambda_{29r})b^3 + \lambda_{210r}b^5 \\ &\equiv c_1 b + c_2 b^2 + c_3 b^5. \end{aligned} \tag{11}$$

At the degenerate point, λ_{11r} and λ_{14r} vanish. We denote the discriminant for the biquadratic equation $c_1 + c_2 b^2 + c_3 b^4 = 0$ as D. In order for b^2 to have two distinct positive roots, we require $-c_2/c_3 > 0$ and $c_1/c_3 > 0$. According to the numerical data in Table I, $\lambda_{210r} > 0$ for $P = 0.025$. We thus obtain the condition as

$$c_2 < 0, \quad c_1 > 0, \quad \text{and} \quad D > 0. \tag{12}$$

If we require b^2 to have one positive and one negative roots, the following should be satisfied:

$$c_1 < 0 \quad \text{and} \quad D > 0. \tag{13}$$

We pictured the conditions (12) and (13) in Fig.4 where $\delta = 0$.

Positiveness of λ_{210r} is not consistent with the assumption done by Eckhaus and Iooss who selected signs of coefficients so as to fit with Sen and Vashist's data. In purely hydrodynamic situations, where $P \rightarrow 0$ and $Ri \rightarrow 0$ hold, the stability characteristics should tend to the ones for isothermal plane Poiseuille flow. In that case, we have a negative value for λ_{210r} at the criticality. (Fujimura) Therefore, the sign of λ_{210r} is expected to change from positive to negative as the Prandtl number decreases. In fact, at $P = 0.0001$, for example, we find the negative λ_{210r} as was assumed by Eckhaus and Iooss. For the latter case, their analysis are valid. We pictured the modified conditions (12) and (13) in Fig.5 with $\delta = 0$.

4.3. Hyper Degeneracy

Careful numerical computation of the Landau constants clarifies that the cubic and quintic Landau constants lose their real parts simultaneously at the criticality at $P = 0.0028316448$. We pictured variations of the real parts of the Landau constants at $P = 0.0028316448$ as functions of $\alpha - \alpha_c$ in Fig.6. We took an amplitude expansion method provided by Herbert for the purpose instead of extending the method of multiple scales. Both the reduction methods give equivalent Stuart-Landau equations as far as the linear growth rate of the fundamental mode is small enough. (Fujimura) Now we find that Re λ_3 vanishes at $\alpha - \alpha_c = 0.002896$ which is sufficiently small whereas Re λ_4 has finite negative value. This situation is much more "hyper" than the Blasius boundary layer case where even the cubic order degeneracy and the quintic order one set in at different wavenumbers, separated by ~ 0.01. In order to unfold our problem completely, we need to involve at least ninth order nonlinear term in the Stuart-Landau equation. For that case, we have to classify positive solutions of bi-quartic equation. Since it is not easy matter within an elementary algebra, we have to solve the equation numerically. It will form our future works.

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Table I. Coefficients involved in (9).

	$P = 0.025$	$P = 0.0001$
α	9.9379699×10^{-1}	$9.654567054 \times 10^{-1}$
Re	1.2514300×10^{4}	$9.5857326572 \times 10^{3}$
Ri	5.9322461×10^{-2}	$1.03361699 \times 10^{-1}$
λ_{11r}	0.0	0.0
λ_{12r}	7.8499×10^{1}	3.0211×10^{1}
λ_{13r}	-8.9281×10^{-2}	-3.7202×10^{-2}
λ_{14r}	2.53×10^{-5}	4.49×10^{-6}
λ_{21r}	-2.0204×10^{-1}	-1.8540×10^{-1}
λ_{22r}	-5.6772×10^{2}	-3.9494×10^{2}
λ_{23r}	4.4911×10^{-1}	7.5512×10^{-2}
λ_{24r}	-3.6452×10^{5}	-4.2413×10^{5}
λ_{25r}	1.0218×10^{3}	-2.2150×10^{2}
λ_{26r}	-2.5319×10^{-1}	5.7788×10^{-3}
λ_{27r}	5.1467×10^{2}	5.5574×10^{2}
λ_{28r}	3.4559×10^{5}	1.9284×10^{5}
λ_{29r}	-8.3300×10^{2}	-3.7679×10^{2}
λ_{210r}	9.0233×10^{4}	-5.6807×10^{3}

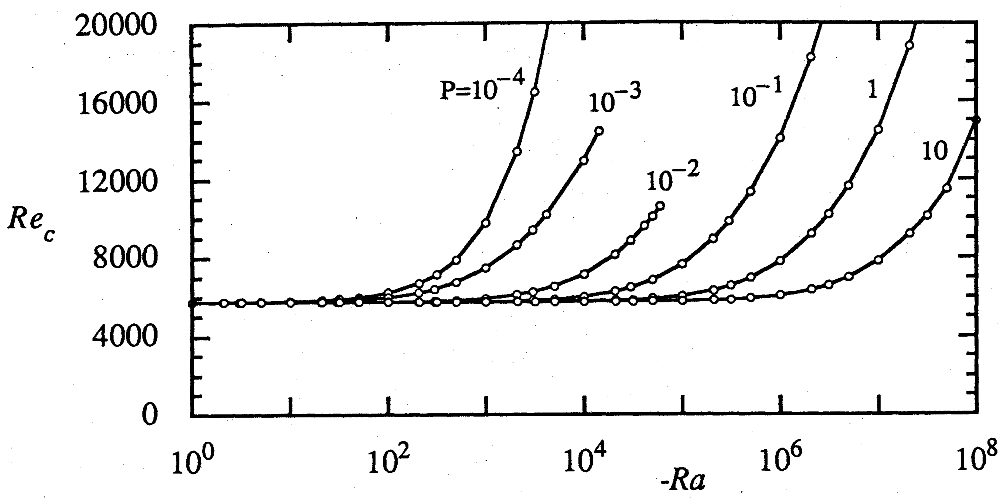


Fig. 1

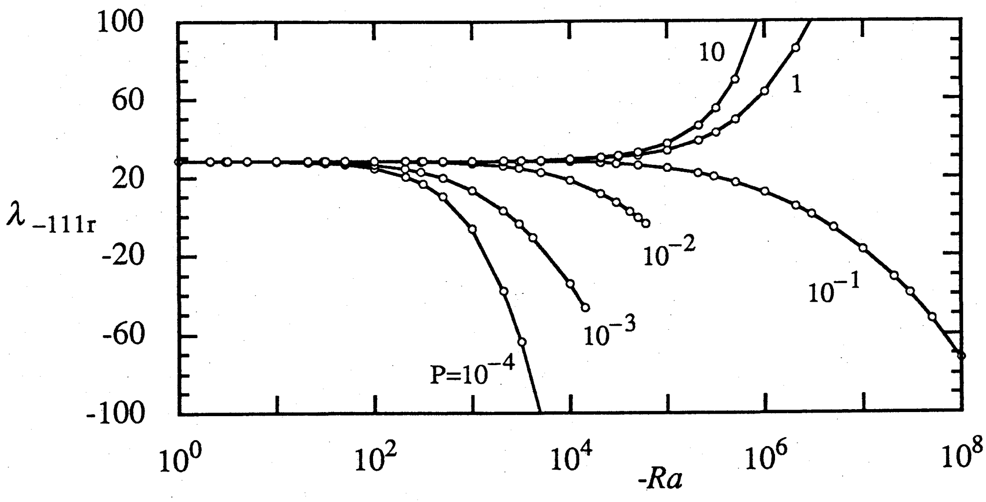

100
60
20
-20
-60
-100
λ_{-111r}
10^0
10^2
10^4
10^6
10^8
-Ra
10
1
10^{-1}
10^{-2}
10^{-3}
$P=10^{-4}$

Fig. 2

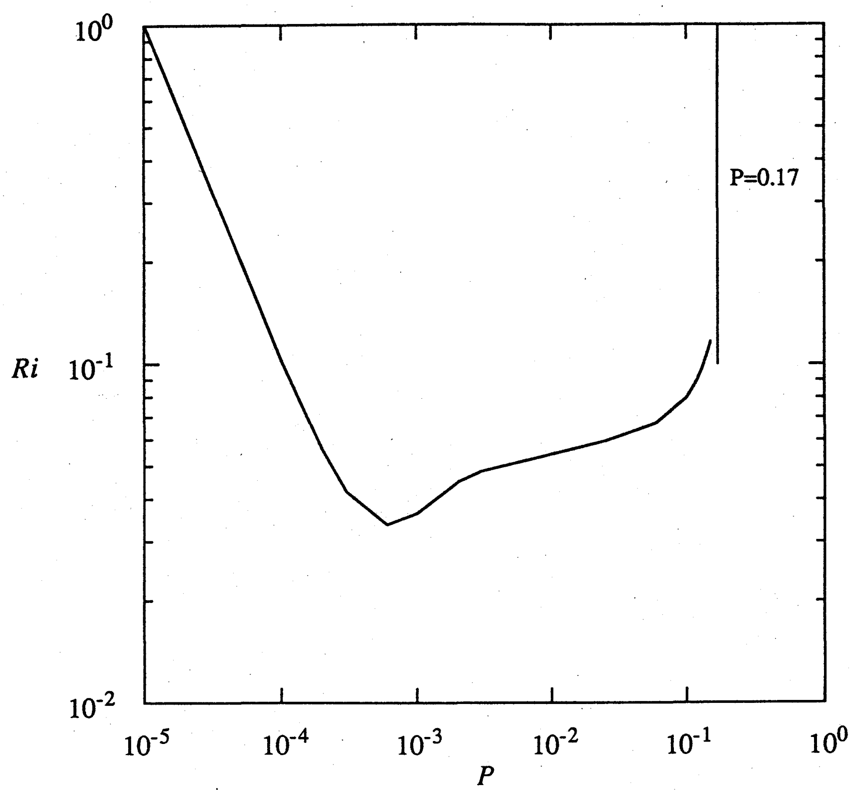
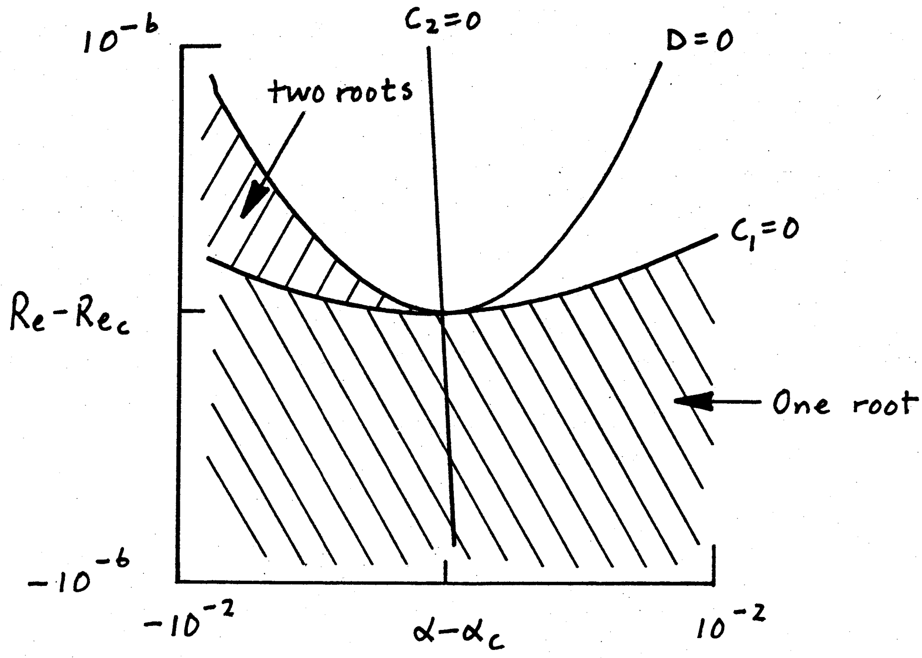
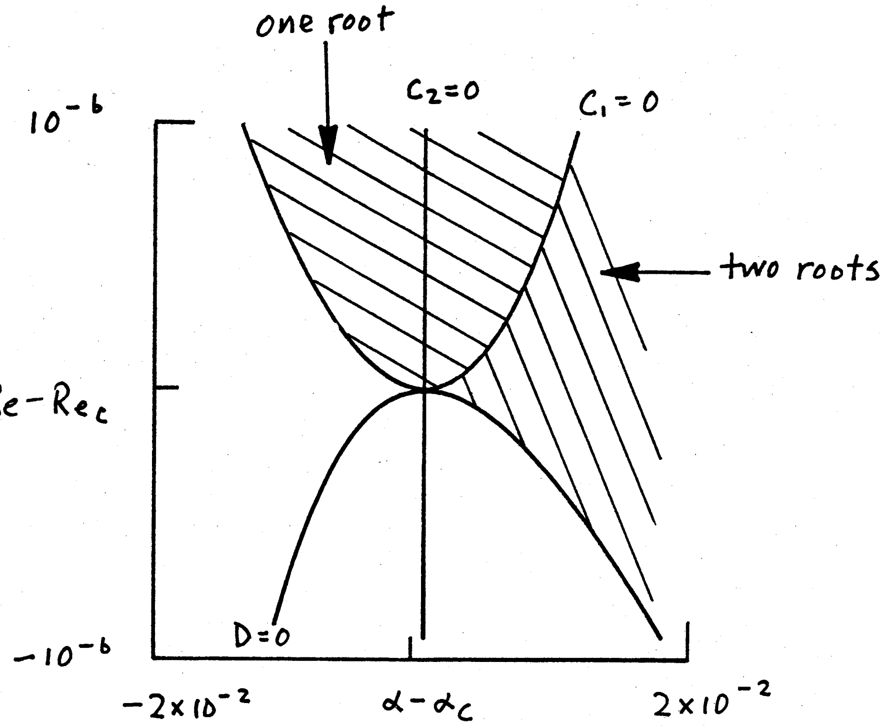

10^0
10^{-1}
10^{-2}
Ri
10^{-5}
10^{-4}
10^{-3}
10^{-2}
10^{-1}
10^0
P
P=0.17

Fig. 3



10^{-6}
$C_2=0$
$D=0$
two roots
$C_1=0$
$Re-Re_c$
One root
-10^{-6}
-10^{-2}
$\alpha-\alpha_c$
10^{-2}

Fig. 4



one root
$C_2=0$
$C_1=0$
10^{-6}
two roots
$Re-Re_c$
$D=0$
-10^{-6}
-2×10^{-2}
$\alpha-\alpha_c$
2×10^{-2}

Fig. 5

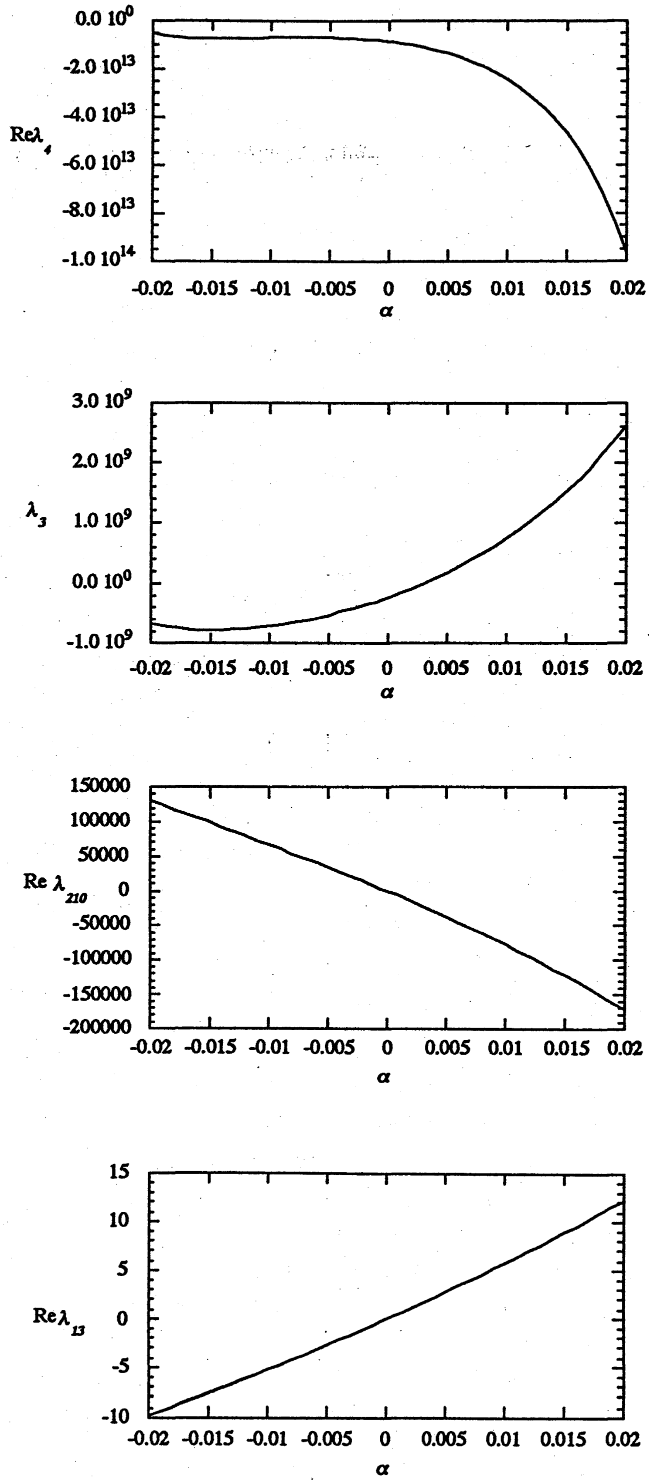

0.0 10^0
-2.0 10^13
-4.0 10^13
Reλ4
-6.0 10^13
-8.0 10^13
-1.0 10^14
-0.02 -0.015 -0.01 -0.005 0 0.005 0.01 0.015 0.02
α
3.0 10^9
2.0 10^9
λ3
1.0 10^9
0.0 10^0
-1.0 10^9
-0.02 -0.015 -0.01 -0.005 0 0.005 0.01 0.015 0.02
α
150000
100000
50000
Re λ210
0
-50000
-100000
-150000
-200000
-0.02 -0.015 -0.01 -0.005 0 0.005 0.01 0.015 0.02
α
15
10
5
Reλ13
0
-5
-10
-0.02 -0.015 -0.01 -0.005 0 0.005 0.01 0.015 0.02
α

Fig. 6