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Differential-geometric formulation of the ideal MHD

Yuji Hattori
Institute of Fluid Science, Tohoku University
Katahira 2-1-1, Aobaku, Sendai, 980-77, Japan

Abstract

The differential-geometric formulation of ideal magnetohydrodynamics is developed recently. In this article we discuss its applications in three directions: (i) physical interpretation of Jacobi fields; (ii) some typical sectional curvaturtes; and (iii) derivation of su(N)-truncation.

1 Introduction

It is Arnold who revealed the differential-geometric structure of ideal hydrodynamics by his deep insight [2, 3]; that is, the total motion of fluid particles in ideal (ideal implies inviscid and incompressible throughout this article) fluid is a geodesic on the group of volume-preserving diffeomorphisms on a flow region; this group, sometimes denoted by SDiff(M), is denoted by $\mathcal{D}_v(M)$ where M is the flow region.

Strictly speaking, the theory by Arnold does not refer to the problem on analyticity. The above group of diffeomorphisms becomes a proper object in mathematics only when we decide a class of diffeomorphysms to be included in the group. However, when we consider a group of C^∞-diffeomorphisms on M, denoted by $\mathcal{D}^\infty(M)$, it cannot be even a complete space; if we consider its completion by H^s-norm, denoted by $\mathcal{D}^s(M)$, it becomes a topological group, but still not a Lie group. This problem has been solved in two directions. One is to extend the notion of Lie group; $\mathcal{D}^\infty(M)$ is an ILH-Lie group (Omori [16]). The other solution by Ebin and Marsden [4] is that sufficient smoothness of exponential mapping on $\mathcal{D}^s(M)$ can be proved. Therefore, we have mathematical foundation for the differential-geometric formulation of ideal hydrodynamics.

Recently we have shown that ideal magnetohydrodynamics (MHD) also generally admits a differential-geometric formulation [6]; that is, the equations of motion for ideal MHD flow correspond to the equations of geodesics on a semidirect product group. This has been proved for a limited case, i.e., two-dimensional case with periodic boundary, by Zeitlin and Kambe [19].

Although we have such beautiful property of ideal dynamics, there are a few attempts to utilize it in the analysis of fluid motion; actually, differential-geometric structures have been studied by calculating sectional curvatures of the group [2, 10, 11, 15]. We have tried to give a reasonable basis for exponential stretching of line elements using the differential-geometric formulation [8]. Nakamura discussed the application of Jacobi fields [14].

In this article, we are primarily concerned with physical applications of this mathematical formulation of ideal MHD. We proceed in three ways: (i) physical interpretation of Jacobi fields; (ii) calculation of sectional curvatures; and (iii) derivation of su(N)-truncations. We also discuss the relation between our formulation and non-canonical Hamiltonian formalism; ideal MHD is a non-canonical Hamiltonian system on a semidirect product space [12]. After our formulation, differential-geometric formulation using the same semidirect product with the product in Marsden et al. was given by Ono [17].

The paper is organized as follows; after summarizing the differential-geometric formulation in section 2, we discuss its relation to non-canonical Hamiltonian formalism in section 3; three directions of physical application given above are presented in section 4-6, respectively.

2 Differential-geometric formulation of ideal MHD

In this section, we briefly summarize the differential-geometric formulation [6] with some different notations.

Let G be a semidirect product of $\mathcal{D}_v(M)$ and $\mathcal{X}_0(M)$; in the preceding letter [6], they are denoted by SDiff(M) and Vect$_0$(M), respectively. The domain $M \in \mathbf{R}^3$ of MHD flow is assumed to be a flat torus with periodic boundary or a simply-connected finite region. The multiplication in G is defined as

$$(g,\gamma)\circ(h,\eta) = (g\circ h, \mathrm{Ad}_{h^{-1}}\gamma + \eta), \tag{1}$$

where $\mathrm{Ad}_{h^{-1}} = \tilde{L}_{h^{-1}}\tilde{R}_h$ is the usual adjoint action.

The Lie algebra of G is defined as a linear space $\mathcal{X}^R(G)$ of all right-invariant vector fields on G. Its bracket reads

$$\begin{aligned}[(u,\alpha)^R,(v,\beta)^R]|_{(h,\gamma)} = \quad & \tilde{R}_{(h,\gamma)}\Big((u\cdot\nabla)v-(v\cdot\nabla)u, \\ & (u\cdot\nabla)\beta-(\beta\cdot\nabla)u+(\alpha\cdot\nabla)v-(v\cdot\nabla)\alpha \Big),\end{aligned} \tag{2}$$

where $(u,\alpha)^R|_{(h,\gamma)} = \tilde{R}_{(h,\gamma)}(u,\alpha)$ with $(u,\alpha)\in T_{(e,0)}G$ is an element of $\mathcal{X}^R(G)$.

We construct a right-invariant metric on G by: (i) defining it at the identity; and (ii) extending it to every point with right action. That is,

$$<(u,\alpha),(v,\beta)>|_{(e,0)} = \int_M u\cdot v d^3x + \int_M \alpha(-\Delta^{-1})\beta d^3x \tag{3}$$

for $(u,\alpha),(v,\beta)\in T_{(e,0)}G$ and

$$<(u',\alpha'),(v',\beta')>|_{(h,\gamma)} =< \tilde{R}_{(h^{-1},-\mathrm{Ad}_h\gamma)}(u',\alpha'),\tilde{R}_{(h^{-1},-\mathrm{Ad}_h\gamma)}(v',\beta')>|_{(e,0)} \tag{4}$$

for $(u',\alpha'),(v',\beta')\in T_{(h,\gamma)}G$. For a given metric there is a Levi-Civita connection $\tilde{\nabla}$ derived by the following formula [9]

$$\begin{aligned}2<\tilde{\nabla}_X Y,Z>= \quad & X<Y,Z>+Y<Z,X>-Z<X,Y> \\ & +<Z,[X,Y]>+<Y,[Z,X]>-<X,[Y,Z]>.\end{aligned} \tag{5}$$

By this formula, we obtain the Levi-Civita connection on G for right-invariant vector fields

$$\begin{aligned}\tilde{\nabla}_{(u,\alpha)^R} \quad & (v,\beta)^R|_{(h,\gamma)} = \tilde{R}_{(h,\gamma)}\Big(P[(u\cdot\nabla)v-\frac{1}{2}(\alpha\times B_\beta+\beta\times B_\alpha)], \\ & \big[\frac{1}{2}\nabla\times(-u\times\beta+v\times\alpha)-\frac{1}{2}\nabla\times(\nabla\times(u\times B_\beta+v\times B_\alpha))\big]\Big)\end{aligned} \tag{6}$$

where B_α is a vector field on M satisfying

$$\nabla\times B_\alpha=\alpha,\quad \nabla\cdot B_\alpha=0. \tag{7}$$

With this connection we have the equation of geodesics

$$\tilde{\nabla}_X X=0,\quad X=\frac{d}{dt}\sigma(t). \tag{8}$$

In order to express this equation at the identity of G, or to obtain the equation in Eulerian picture, we apply $\tilde{R}_{\sigma(t)^{-1}}$ to eq.(8)

$$\tilde{R}_{\sigma(t)^{-1}}\tilde{\nabla}_X X=0. \tag{9}$$

The left-hand side of the above equation is calculated to be

$$\tilde{R}_{\sigma(t)^{-1}}\tilde{\nabla}_X X = \frac{\partial \tilde{R}_{\sigma(t)^{-1}}X}{\partial t} + \tilde{R}_{\sigma(t)^{-1}}\tilde{\nabla}_{X'_t}X'_t. \tag{10}$$

Here we have introduced the right-invariant vector field X'_t defined by

$$X'_{t_0}(\sigma(t_0)) = X(t_0), \quad X'_{t_0}|_{(g,\gamma)} = \tilde{R}_{(g\cdot\sigma(t_0)^{-1},\gamma)}X(t_0) \tag{11}$$

for fixed t_0.

From eq.(6) and (10) the equation of geodesics becomes

$$\frac{\partial u}{\partial t} + P[(u\cdot\nabla)u - \alpha \times B_\alpha] = 0, \tag{12}$$

$$\frac{\partial \alpha}{\partial t} - P[\nabla \times (\nabla \times (u \times B_\alpha))] = 0 \tag{13}$$

for $(u, \alpha) = \tilde{R}_{\sigma(t)^{-1}}X(t) \in T_{(e,0)}G$. If we write the projection explicitly as $P[w] = w - \nabla p$ with a function p on M and integrate the equation (12) into that for B_α, the above equations turn out to be the 3d-iMHD equations

$$\frac{\partial u}{\partial t} + (u\cdot\nabla)u = -\nabla p + j \times B, \tag{14}$$

$$\frac{\partial B}{\partial t} = \nabla \times (u \times B). \tag{15}$$

3 Relation to non-canonical Hamiltonian formalism

The non-canonical Hamiltonian structure of MHD equations without diffusivity was first recognized by Morrison and Greene [13]. It turns out to be a Hamiltonian system on a Lie algebra of semidirect product group [12]. The algebra is

$$\mathcal{X}_0(M) \times \mathcal{F}(M) \times \Lambda^1(M) \tag{16}$$

where $\mathcal{F}(M)$ is a space of functions on M and $\Lambda^1(M)$ is a space of 1-form fields on M; the former corresponds to the density fields and the latter to the magnetic fields. If we impose incompressibility, the algebra is reduced to

$$\mathcal{X}_0(M) \times \Lambda^1(M). \tag{17}$$

The bracket of this algebra is

$$[(u, B_u), (v, B_v)] = ((u \cdot \nabla)v - (v \cdot \nabla)u, \mathrm{ad}^*_u B_v - \mathrm{ad}^*_v B_u). \tag{18}$$

The above algebra is seen to have close connection with our algebra $\mathcal{X}_0(M) \times \mathcal{X}_0(M)$ as follows. For appropriate domains and boundary conditions, the following correspondence between current fields and magnetic fields is one-to-one

$$\nabla\times : B^\sharp \mapsto j, \quad (P[(\nabla\times)^{-1}])^\flat : j \mapsto B.$$

Here we denoted by $B^\sharp \in \mathcal{X}_0(M)$ the dual of $B \in \Lambda^1(M)$ and the dual of $w \in \mathcal{X}_0(M)$ is denoted by $w^\flat \in \Lambda^1(M)$. With this correspondence, we can extend the bracket (2) of $\mathcal{X}^R(G)$ to that of $\mathcal{X}^R(\mathcal{D}_v(M) \ltimes \Lambda^1(M))$ The result is

$$[(u, 0)^R, (0, B)^R] = (0, (P[u \times (\nabla \times B^\sharp)])^\flat)^R \tag{19}$$

for $u \in \mathcal{X}_0(M), B \in \Lambda^1(M)$. The general expression of bracket is recovered with (19) and $[(0, B_1)^R, (0, B_2)^R] = 0$ by its bilinearity.

Now the relation between two brackets is clear from the following identity

$$P[(\mathcal{L}_v B)^\sharp] = -P[v \times (\nabla \times B^\sharp)]$$

(note that it holds $\mathcal{L}_v B = \mathrm{ad}^*_v B$). That is, the bracket (19) is the restriction of the bracket (18) to $\mathcal{X}^R(\mathcal{D}_v(M) \ltimes (\mathcal{X}_0(M))^\flat)$. It is easy to check that the restricted bracket is actually a bracket.

4 Physical interpretation of Jacobi fields

One of the applications of the above formulation can be performed by studying Jacobi fields. A variation of geodesics on G is defined as

$$\tau_t^s : [-\epsilon, \epsilon] \times [-\epsilon, \epsilon] \to G$$

where each $\tau_t^{s_0}$ (s_0: fixed) is a geodesic. With this definition, a Jacobi field is defined as

$$W_t = (\tilde{\nabla}_s \tau_t^s)_{s=0}.$$

The equation for Jacobi fields is derived from the above definition and the equation of geodesics to be

$$\tilde{\nabla}_t^2 W_t + R(W_t, Y_t)Y_t = 0 \tag{20}$$

where $Y_t = \tilde{\nabla}_t \tau_t^0$.

Nakamura [14] found that a special choice of variation leads to a simplified equation of (20); a right translation of a geodesic also becomes a geodesic. Thus we can take $\tau_t^s = \tilde{R}_{a^s}\sigma$ as a variation, where a^s is a curve on G with $a^0 = e$ and σ is a geodesic. Then the following equation determines the evolution of Jacobi field W_t

$$\tilde{\nabla}_t W_t = (\tilde{\nabla}_s \tilde{\nabla}_t \tau_t^s)_{s=0}. \tag{21}$$

For the case of $\mathcal{D}_v(M)$ this reduces to be

$$\frac{\partial w}{\partial t} + (u \cdot \nabla)w = (w \cdot \nabla)u \tag{22}$$

where $w = \tilde{R}_{\sigma_t^{-1}} W_t, u = \tilde{R}_{\sigma_t^{-1}} \tilde{\nabla}_t \sigma_t$. This is an equation for infinitesimal line elements kinematically convected and stretched by the flow field u.

We can proceed in the similar way for ideal MHD. If we express the equation 21 at the identity, we obtain

$$\frac{\partial w}{\partial t} + (u \cdot \nabla)w = (w \cdot \nabla)u \tag{23}$$

$$\frac{\partial \zeta}{\partial t} + \nabla \times (-u \times \zeta + w \times \alpha) = 0 \tag{24}$$

where $(w, \zeta) = \tilde{R}_{\sigma_t^{-1}} W_t, (u, j) = \tilde{R}_{\sigma_t^{-1}} \tilde{\nabla}_t \sigma_t$ and $\sigma_t = (g, \alpha)$. The former (23) is again the equation for line elements. The latter equation is interpreted as the equation for deviation of translated charge; the second term in (24) represents the effect of initial translated charge and the third term represents the effect of different labeling of fluid particles.

5 Sectional curvatures

The sectional curvature of the section spanned by tangent vectors X, Y is defined as

$$K(X, Y) = \frac{R}{< X, X >< Y, Y > - < X, Y >^2} \tag{25}$$

where $R =< R(X,Y)Y,X >$. It is not enlightening to present the explicit expression of sectional curvature of G generally. Its tensor form R is decomposed as

$$\begin{aligned} R &= < R(u,v)v,u > +2 < R(u,v)\beta,\alpha > \\ &\quad +2 < R(u,\beta)v,\alpha > + < R(u,\beta)\beta,u > \\ &\quad + < R(\alpha,v)v,\alpha > + < R(\alpha,\beta)\beta,\alpha > \end{aligned} \tag{26}$$

for $X = (u,\alpha), Y = (v,\beta)$. Here we write simply as $< R((u,0),(v,0))(0,\beta),(0,\alpha) >=< R(u,v)\beta,\alpha >$ and so on.

We focus on three typical types of sectional curvature here: (i) pure hydrodynamic section; (ii) pure magnetic section; and (iii) section spanned by pure hydrodynamic vector and pure magnetic vector.

5.1 Pure hydrodynamic section

Using the connection (6), we obtain the sectional curvature of the pure hydrodynamic section spanned by $(u,0),(v,0)$ as

$$\begin{aligned} R_H &= < R(u,v)v,u > \\ &= < Q[(v\cdot\nabla)v], Q[(u\cdot\nabla)u] > -|Q[(u\cdot\nabla)v]|^2 \end{aligned} \tag{27}$$

in tensor form. Here we denote by Q the projection operator from $\mathcal{X}(M)$ to the space of all vector fields in gradient form; i.e. $Q = I - P$. Of course, this is identical with the case of $\mathcal{D}_v(M)$ [11]. From the above expression it holds

Prop. 1 ([11]) *if u satisfies $Q[(u\cdot\nabla)u] = 0$ (e.g. $u = (u_1(z), u_2(z), 0)$), then $R_H \le 0$.*

5.2 Pure magnetic section

The sectional curvature of the pure magnetic section spanned by $(0,\alpha),(0,\beta)$ is calculated to be

$$\begin{aligned} R_M &= < R(\alpha,\beta)\beta,\alpha > \\ &= - < P[(B_\beta\cdot\nabla)B_\beta], P[(B_\alpha\cdot\nabla)B_\alpha] > \\ &\quad +\frac{1}{4}|P[(B_\alpha\cdot\nabla)B_\beta + (B_\beta\cdot\nabla)B_\alpha]|^2 \end{aligned} \tag{28}$$

in tensor form. From this expression we obtain

Prop. 2 *if* B_α *satisfies* $P[(B_\alpha \cdot \nabla)B_\alpha] = 0$, *then* $R_M \geq 0$.

The condition $P[(B_\alpha \cdot \nabla)B_\alpha] = 0$ implies that B_α is a "steady velocity field" in ideal HD, or equivalently, $u = B_\alpha$ is a steady solution of the Euler equation

$$\frac{\partial u}{\partial t} + (u \cdot \nabla)u = -\nabla p.$$

This condition is satisfied for the field which has Beltrami property, i.e. $\nabla \times B_\alpha \parallel B_\alpha$ everywhere; it is called a force-free field in MHD since the Lorentz force vanishes; $\alpha \times B_\alpha = 0$.

5.3 Section spanned by pure hydrodynamic vector and pure magnetic vector

For the section spanned by $(u, 0)$ and $(0, \beta)$, the sectional curvature is calculated to be

$$\begin{aligned} R_{HM} &= < R(u,\beta)\beta, u > \\ &= < P[(B_\beta \cdot \nabla)B_\beta], P[(u \cdot \nabla)u] > \\ &\quad + \frac{1}{4}|\nabla \times (u \times B_\beta) + P[u \times \beta]|^2 - |P[u \times \beta]|^2 \end{aligned} \tag{29}$$

in tensor form. It holds

Prop. 3 *if* u *or* B_β *is a "steady velocity field" and* $u \parallel \beta$ *everywhere, then* $R_{HM} \geq 0$.
If u *or* B_β *is a "steady velocity field" and* $u \parallel B_\beta$ *everywhere, then* $R_{HM} \leq 0$.

6 Derivation of su(N)-truncations

Another application of the differential-geometric formulation is a derivation of su(N)-truncation proposed by Zeitlin [18].

When we perform numerical calculations of hydrodynamic or magnetohydrodynamic phenomena, the equation should be truncated to a finite number of degrees of freedom. Let us consider the case of the two-dimensional Euler equation with periodic boundary condition; the spectral method with Fourier expansion is often used in this case. Then the equations for Fourier modes become

$$\frac{d}{dt}y^{\mathbf{k}} + \sum_{\mathbf{l}\in C, \mathbf{m}=\mathbf{k}-\mathbf{l}} \{\frac{|\mathbf{m}|^2}{|\mathbf{k}|^2}(\mathbf{l} \times \mathbf{m})\} y^{\mathbf{l}} y^{\mathbf{m}} = 0 \tag{30}$$

where $y^{\mathbf{k}}$ is a Fourier amplitude of vorticity for wavenumber vector $\mathbf{k}$ and C is a region of wavenumber vectors whose Fourier amplitude is not truncated. Equations (30) are often called an Inviscid Truncated System (ITS).

Since the Liouville property holds for ITS, we can discuss the spectrum of ITS statistically; that is, taking account of two invariants, energy and enstrophy,

$$E = \sum_{\mathbf{k}} \frac{|y^{\mathbf{k}}|^2}{|\mathbf{k}|^2}, \quad \Omega = \sum_{\mathbf{k}} |y^{\mathbf{k}}|^2,$$

we have the following canonical ensemble average

$$< |y_{\mathbf{k}}|^2 >= \frac{|\mathbf{k}|^2}{\alpha + \beta|\mathbf{k}|^2}$$

from the corresponding partition function.

However, ITS does not inherit the Hamiltonian structure of the original equation. As its result, ITS does not have invariants other than two invariants above, though the two-dimensional Euler equation conserves the integral of any function of vorticity $\int f(\omega)d^2x$.

The su(N)-truncation is another truncation of the Euler equation which solves the above problem; it is derived by the differential-geometric formulation with a Lie group SU(N) instead of $\mathcal{D}_v(M)$. Choosing a basis of $T_eG = \mathrm{su}(N)$ as

$$\{e_{(j_1,j_2)} = \frac{i}{2}\zeta^{\frac{1}{2}j_1j_2}A^{j_1}B^{j_2}\} \quad \mathbf{j} = (j_1, j_2) \in C$$

where $\zeta = e^{4\pi i/N}$, $C = [-n, n] \times [-n, n] \subset \mathbf{Z}^2$ and

$$A = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & \zeta & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \zeta^{N-1} \end{pmatrix}$$

$$B = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ 1 & 0 & \cdots & 0 \end{pmatrix}$$

we have the following structure constants

$$c_{\mathbf{kl}}^{\mathbf{m}} = \{\sin \frac{2\pi}{N}(\mathbf{k} \times \mathbf{l})\}\, \delta_{\mathbf{k}+\mathbf{l}|_{\mathrm{mod}\,N},\mathbf{m}}.$$

If we define the metric tensor as

$$a_{\mathbf{kl}} = |\mathbf{k}|^2 \delta_{\mathbf{k+l},\mathbf{0}},$$

the equation of geodesics at T_e^*G turns out to be

$$\frac{d}{dt}y^{\mathbf{k}} + \sum_{\mathbf{l}\in C, \mathbf{m}=\mathbf{k}-\mathbf{l}|_{\mathrm{mod}\,N}} \{\frac{|\mathbf{m}|^2}{|\mathbf{k}|^2} \sin\frac{2\pi}{N}(\mathbf{l}\times\mathbf{m})\} y^{\mathbf{l}} y^{\mathbf{m}} = 0. \tag{31}$$

These equations are quite similar to equaions (30); the differences are: (i) the presence of operation *mod* in (31) and (ii) the coefficients of nonlinear terms; note that $\frac{N}{2\pi}\sin\frac{2\pi}{N}(\mathbf{l}\times\mathbf{m}) \to \mathbf{l}\times\mathbf{m}$, as $N\to\infty$.

The remarkable feature of su(N)-truncation (31) is that it has phase-dependent and higher-order invariants

$$I_m = \sum_{\mathbf{k}_1+\cdots+\mathbf{k}_m|_{\mathrm{mod}\,N}=\mathbf{0}} C_{\mathbf{k}_1\cdots\mathbf{k}_m} \omega_{\mathbf{k}_1}\cdots\omega_{\mathbf{k}_m}, \tag{32}$$

$$C_{\mathbf{k}_1\cdots\mathbf{k}_m} = \exp\Big\{\frac{2\pi i}{N}[\mathbf{k}_1\times\mathbf{k}_2 + (\mathbf{k}_1+\mathbf{k}_2)\times\mathbf{k}_3 + \cdots + (\mathbf{k}_1+\cdots+\mathbf{k}_{m-2})\times\mathbf{k}_{m-1}]\Big\}. \tag{33}$$

These invariants correspond to $\int \omega^m d^2x$. Therefore, the su(N)-truncation has an advantage in discussing the effect of these phase-dependent invariants [5].

We can derive an su(N)-truncation of ideal MHD in the similar way using the semidirect product group $G = \mathrm{SU}(N) \rtimes \mathrm{su}(N)$; its multiplication law is

$$(g,\alpha)\circ(h,\beta) = (gh, \mathrm{Ad}_{h^{-1}}\alpha + \beta).$$

For the following basis of T_eG

$$\{(e_{\mathbf{j}}, 0), (0, e_{\mathbf{k}'})\},$$

we have the structure constants

$$c_{\mathbf{kl}}^{\mathbf{m}} = \{\sin\frac{2\pi}{N}(\mathbf{k}\times\mathbf{l})\}\delta_{\mathbf{k+l}|_{\mathrm{mod}\,N},\mathbf{m}},$$

$$c_{\mathbf{k}'\mathbf{l}}^{\mathbf{m}'} = -c_{\mathbf{lk}'}^{\mathbf{m}'} = \{\sin\frac{2\pi}{N}(\mathbf{k}'\times\mathbf{l})\}\delta_{\mathbf{k}'+\mathbf{l}|_{\mathrm{mod}\,N},\mathbf{m}'},$$

$$c_{\mathbf{k}'\mathbf{l}}^{\mathbf{m}} = c_{\mathbf{kl}'}^{\mathbf{m}} = c_{\mathbf{kl}}^{\mathbf{m}'} = c_{\mathbf{k}'\mathbf{l}'}^{\mathbf{m}} = c_{\mathbf{k}'\mathbf{l}'}^{\mathbf{m}'} = 0.$$

Choosing a metric tensor as

$$a_{\mathbf{kl}} = |\mathbf{k}|^2\delta_{\mathbf{k}+\mathbf{l},\mathbf{0}}, \qquad a_{\mathbf{k}'\mathbf{l}} = 0, \tag{34}$$

$$a_{\mathbf{k}'\mathbf{l}'} = \frac{1}{|\mathbf{k}'|^2}\delta_{\mathbf{k}'+\mathbf{l}',\mathbf{0}}, \tag{35}$$

we have the following equation of geodesics

$$\frac{d}{dt}y^{\mathbf{k}} + \sum_{\mathbf{l}\in C,\mathbf{m}=\mathbf{k}-\mathbf{l}|_{\mathrm{mod}N}} \frac{|\mathbf{m}|^2}{|\mathbf{k}|^2}\{\sin\frac{2\pi}{N}(\mathbf{l}\times\mathbf{m})\}y^{\mathbf{l}}y^{\mathbf{m}} - \sum_{\mathbf{l}'\in C,\mathbf{m}'=\mathbf{k}-\mathbf{l}'|_{\mathrm{mod}N}} \frac{\{\sin\frac{2\pi}{N}(\mathbf{l}'\times\mathbf{m}')\}}{|\mathbf{l}'|^2|\mathbf{k}|^2}z^{\mathbf{l}'}z^{\mathbf{m}'} = 0 \tag{36}$$

$$\frac{d}{dt}z^{\mathbf{k}'} + \sum_{\mathbf{l}\in C,\mathbf{m}'=\mathbf{k}'-\mathbf{l}|_{\mathrm{mod}N}} \frac{|\mathbf{k}'|^2}{|\mathbf{m}'|^2}\{\sin\frac{2\pi}{N}(\mathbf{l}\times\mathbf{m}')\}y^{\mathbf{l}}z^{\mathbf{m}'} = 0. \tag{37}$$

The equations (36) and (37) are easily seen to correspond to (14) and (15) respectively. This su(N)-truncation of ideal MHD again has phase-dependent invariants correspond to invariants of two-dimensional ideal MHD fluid $\int a^m d^2x$ and $\int \omega a^m d^2x$. Its properties will be reported in the forthcoming paper [7].

7 Summary

We have so far discussed the differential-geometric formulation of ideal MHD and its applications. The formulation itself gives a new insight to the dynamics of magnetic fluids. However, whether it can be a poweful tool in the analysis of MHD flow is not evident at present; it depends on the future study.

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