数理解析研究所講究録
第 890 巻 1994 年 1-18

FUNDAMENTAL THEOREMS IN LINEAR TRANSFORMS

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Abstract. Fundamental theorems with new viewpoints and methods for linear transforms in the framework of Hilbert spaces are introduced, and their miscellaneous applications are discussed.

1. INTRODUCTION

In 1976, I obtained the generalized isoperimetric inequality in my thesis (Saitoh [14]):

For a bounded regular region G in the complex $z = x + iy$ plane surrounded by a finite number of analytic Jordan curves and for any analytic functions $\varphi(z)$ and $\psi(z)$ on $\overline{G} = G \cup \partial G$,

$$\frac{1}{\pi} \iint_G |\varphi(z)\psi(z)|^2 dxdy \leqq \frac{1}{2\pi} \int_{\partial G} |\varphi(z)|^2 |dz| \frac{1}{2\pi} \int_{\partial G} |\psi(z)|^2 |dz|.$$

The crucial point in this paper is to determine completely the case that the equality holds in the inequality.

In order to prove this simple inequality, surprisingly enough, we must use the long historical results of

Riemann (1826-1866) - Klein (1849-1925)- Bergman - Szegö

- Nehari - Schiffer- Garabedian - Hejhal (1972, thesis),

in particular, a profound result of D. A. Hejhal which establishes the fundamental interrelationship between the Bergman and the Szegö reproducing kernels of G (Hejhal[8]). Furthermore, we must use the general theory of reproducing kernels by (Aronszajn [2]) in 1950. — These circumstances do still not exchange, since the paper (Saitoh [14]) has been published about 15 years ago.

The thesis will become a milestone on the development of the theory of reproducing kernels. In the thesis, we realized that miscellaneous applications of the general theory of reproducing kernels are possible in many concrete problems. See (Saitoh [19]) for the details. It seems that the general theory of reproducing kernels was, in a strict sense, not active in the theory of concrete reproducing kernels until the publication of the thesis. Indeed, after the publication of the thesis, we, for example, derived miscellaneous fundamental norm inequalities containing quadratic norm inequalities in matrices in many papers over 18. Furthermore, we got a general idea for linear transforms by using essentially the general theory of reproducing kernels which is the main theme of this exposition.

2. LINEAR TRANSFORMS IN HILBERT SPACES

In 1982 and 1983, we published the very simple theorems in (Saitoh [15, 16]). Certainly the results are very simple mathematically, but they seem to be very fundamental and applicable widely for general linear transforms. Moreover, the results will contain several new ideas for linear transforms, themselves.

We shall formulate a "linear transform "as follows:

$$F(t) \longrightarrow \boxed{\qquad\qquad} \longrightarrow f(p)$$

$$h(t,p)$$

$$f(p) = \int_T F(t)\overline{h(t,p)}dm(t), \quad p \in E. \tag{1}$$

Here, the input $F(t)$ (source) is a function on a set T, E is an arbitrary set, $dm(t)$ is a σ-finite positive measure on the dm measurable set T, and $h(t,p)$ is a function on $T \times E$ which determines the transform of the system.

This formulation will give a generalized form of a linear transform:

$$L(aF_1 + bF_2) = aL(F_1) + bL(F_2).$$

Indeed, following the Schwartz kernel theorem, we see that very general linear transforms are realized as integral transforms as in the above (1) by using generalized functions as the integral kernels $h(t,p)$.

We shall assume that $F(t)$ is a member of the Hilbert space $L_2(T, dm)$ satisfying

$$\int_T |F(t)|^2 dm(t) < \infty. \tag{2}$$

The space $L_2(T, dm)$ whose norm gives an energy integral will be the most fundamental space as the input function space. In other spaces we shall modify them in order to meet to our situation, or as a prototype case we shall consider primarily or, as the first stage, the linear transform (1) in our situation.

As a natural result of our basic assumption (2), we assume that

$$\text{for any fixed} \quad p \in E, \quad h(t,p) \in L_2(T, dm) \tag{3}$$

for the existence of the integral in (1).

We shall consider the two typical linear transforms:

We take $E = \{1, 2\}$ and let $\{\mathbf{e}_1, \mathbf{e}_2\}$ be some orthonormal vectors of $\mathbb{R}^2$. Then, we shall consider the linear transform from $\mathbb{R}^2$ to $\{x_1, x_2\}$ as follows:

$$\mathbf{x} \longrightarrow \begin{cases} x_1 = (\mathbf{x}, \mathbf{e}_1) \\ x_2 = (\mathbf{x}, \mathbf{e}_2). \end{cases} \tag{4}$$

For $F \in L_2(\mathbb{R}, dx)$ we shall consider the integral transform

$$u(x,t) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} F(\xi) e^{-\frac{(x-\xi)^2}{4t}} d\xi, \tag{5}$$

which gives the solution $u(x,t)$ of the heat equation

$$u_{xx}(x,t) = u_t(x,t) \quad \text{on} \quad \mathbb{R} \times \{t > 0\} \tag{6}$$

subject to the initial condition

$$u(x,0) = F(x) \quad \text{on} \quad \mathbb{R}. \tag{7}$$

3. IDENTIFICATION OF THE IMAGES OF LINEAR TRANSFORMS

We formulated linear transforms as the integral transforms (1) satisfying (2) and (3) in the framework of Hilbert spaces. In this general situation, we can identify the space of output functions $f(p)$ and we can characterize completely the output functions $f(p)$. For this fundamental idea, it seems that our mathematical community does still not realize this important fact, since the papers (Saitoh [15, 16]) have been published about ten years ago.

One reason why we do not have the idea of the identification of the images of linear transforms will be based on the definition itself of linear transforms. A linear transform is, in general, a linear mapping from a linear space into a linear space, and so, the image space of the linear mapping will be considered as a, a priori, given one. For this, our idea will show that the image spaces of linear transforms, in our situation, form the uniquely determined and intuitive ones which are, in general, different from the image spaces stated in the definitions of linear transforms.

Another reason will be based on the fact that the very fundamental theory of reproducing kernels by Aronszajn is still not popularized. The general theory seems to be a very fundamental one in mathematics, as in the theory of Hilbert spaces.

Recall the paper of (Schwartz [21]) for this fact which extended globally the theory of Aronszajn. — Our basic idea for linear transforms is very simple, mathematically, but it is, at first, found from the theory of Schwartz using the direct integrals of reproducing kernel Hilbert spaces. See (Saitoh [19]) for the details.

In order to identify the image space of the integral transform (1), we consider the Hermitian form

$$K(p,q) = \int_T h(t,q)\overline{h(t,p)}dm(t) \quad \text{on} \quad E \times E. \tag{8}$$

The kernel $K(p,q)$ is apparently a positive matrix on E in the sense of

$$\sum_{j=1}^{n}\sum_{j'=1}^{n} C_j\overline{C_{j'}}K(p_{j'},p_j) \geqq 0$$

for any finite points $\{p_j\}$ of E and for any complex numbers $\{C_j\}$. Then, following the fundamental theorem of Aronszajn-Moore, there exists a uniquely determined Hilbert space H_K comprising of functions $f(p)$ on E satisfying

$$\text{for any fixed} \quad q \in E, K(p,q) \quad \text{belongs to} \quad H_K \quad \text{as a function in} \quad p, \tag{9}$$

and

(10) for any $q \in E$ and for any $f \in H_K$

$$(f(p), K(p,q))_{H_K} = f(q).$$

Then, the point evaluation $f(p)(p \in E)$ is continuous on H_K and, conversely, a functional Hilbert space such that the point evaluation is continuous admits the reproducing kernel $K(p,q)$ satisfying (9) and (10). Then, we obtain

Theorem 1. The images $f(p)$ of the integral transform (1) for $F \in L_2(T,dm)$ form precisely the Hilbert space H_K admitting the reproducing kernel $K(p,q)$ in (8).

In Example (4), we can deduce naturally using Theorem 1 that

$$\|\{x_1, x_2\}\|_{H_K} = \sqrt{x_1^2 + x_2^2}$$

for the image.

In Example (5), we deduce naturally the very surprising result that the image $u(x,t)$ is extensible analytically onto the entire complex $z = x + iy$ plane and when we denote its analytic extension by $u(z,t)$, we have

$$\|u(z,t)\|_{H_K}^2 = \frac{1}{\sqrt{2\pi t}} \iint_C |u(z,t)|^2 e^{-\frac{y^2}{2t}} dx dy \tag{11}$$

(Saitoh [17]).

The images $u(x,t)$ of (5) for $F \in L_2(\mathbb{R}, dx)$ are characterized by (11); that is, $u(x,t)$ are entire functions in the form $u(z,t)$ with finite integrals (11).

In 1989, we deduced that (11) equals to

$$\sum_{j=0}^{\infty} \frac{(2t)^j}{j!} \int_{-\infty}^{\infty} (\partial_x^j u(x,t))^2 dx \tag{12}$$

by using the property that $u(x,t)$ is the solution of the heat equation (6) with (7) (Hayashi and Saitoh [7]). Hence, we see that the images $u(x,t)$ of (5) are also characterized by the property that $u(x,t) \in C^\infty$ with finite integrals (12).

4. RELATIONSHIP BETWEEN MAGNITUDES OF INPUT AND OUTPUT FUNCTIONS — A GENERALIZED PYTHAGORAS THEOREM

Our second Theorem is,

Theorem 2. In the integral transform (1), we have the inequality

$$\|f\|_{H_K}^2 \leqq \int_T |F(t)|^2 dm(t).$$

Furthermore, there exist the functions F^* with the minimum norms satisfying (1), and we have the isometrical identity

$$\|f\|_{H_K}^2 = \int_T |F^*(t)|^2 dm(t).$$

In Example (4), we have, surprisingly enough, the Pythagoras theorem

$$\|\mathbf{x}\|^2 = x_1^2 + x_2^2.$$

In Example (5), we have the isometrical identity

$$\int_T |F(x)|^2 dx = \frac{1}{\sqrt{2\pi t}} \iint_C |u(z,t)|^2 e^{-\frac{y^2}{2t}} dx dy, \tag{13}$$

whose integrals are independent of $t > 0$. — At this moment, we will be able to say that by the general principle (Theorems 1 and 2) for linear transforms we can prove the Pythagoras (B. C. 572-492) theorem apart from the idea of "orthogonality", and we can understand Theorem 2 as a generalized theorem of Pythagoras in our general situation of linear transforms.

By using the general principle, we derived miscellaneous Pythagoras type theorems in many papers over 30. We shall refer to one typical example (Saitoh [18]).

For the solution $u(x,t)$ of the most simple wave equation

$$u_{tt}(x,t) = c^2 u_{xx}(x,t) \quad (c > 0 : \text{constant})$$

subject to the initial conditions

$$u_t(x,t)|_{t=0} = F(x), \quad u(x,0) = 0 \quad \text{on} \quad \mathbb{R}$$

for $F \in L_2(\mathbb{R}, dx)$, we obtain the isometrical identity

$$\frac{1}{2}\int_{-\infty}^{\infty} |F(x)|^2 dx = \frac{2\pi c}{t} \int_{-\infty}^{\infty} \left| l.i.m._{N\to\infty} \int_{-N}^{N} u(x,t) \exp(\frac{ix\xi}{2\pi ct}) dx \right|^2 \frac{d\xi}{\left(\frac{\sin\frac{\xi}{2}}{\frac{\xi}{2}}\right)^2}, \tag{14}$$

whose integrals are independent of $t > 0$.

Recall here the conservative law of energy

$$\frac{1}{2}\int_{-\infty}^{\infty} |F(x)|^2 dx = \frac{1}{2}\int_{-\infty}^{\infty} (u_t(x,t)^2 + c^2 u_x(x,t)^2) dx. \tag{15}$$

To compare the two integrals (14) and (15) will be very interesting, because (15) contains the derived functions $u_t(x,t)$ and $u_x(x,t)$, meanwhile (14) contains the values $u(x,t)$ only.

In the viewpoint of the conservative law of energy in (14) and (15), could we give some physical interpretation of the isometrical identities in (13) and (12) whose integrals are independent of $t > 0$ in the heat equation ?

5. INVERSION FORMURAS FOR LINEAR TRANSFORMS

In our **Theorem 3**, we establish the inversion formula

$$f \longrightarrow F^* \tag{16}$$

of the integral transform (1) in the sense of Theorem 2.

The basic idea to derive the inversion formula (16) is, first, to represent $f \in H_K$ in the space H_K in the form

$$f(q) = (f(p), K(p,q))_{H_K},$$

secondly, to consider as follows:

$$\begin{aligned} f(q) &= (f(p), \int_T h(t,q)\overline{h(t,p)}dm(t))_{H_K} \\ &= \int_T (f(p), \overline{h(t,p)})_{H_K} \overline{h(t,q)}dm(t) \\ &= \int_T F^*(t)\overline{h(t,q)}dm(t) \end{aligned}$$

and, finally, to deduce that

$$F^*(t) = (f(p), \overline{h(t,p)})_{H_K}. \tag{17}$$

In these arguments, however, the integral kernel $h(t,p)$ does, in general, not belong to H_K as a function of p and so, (17) is, in general, not valid.

For this reason, we shall realize the norm in H_K in terms of a σ-finite positive measure $d\mu$ in the form

$$\|f\|^2_{H_K} = \int_E |f(p)|^2 d\mu(p).$$

Then, for some suitable exhaustion $\{E_N\}$ of E, we obtain, in general, the inversion formula

$$F^*(t) = s - \lim_{N\to\infty} \int_{E_N} f(p)h(t,p)d\mu(p) \tag{18}$$

in the sense of the strong convergence in $L_2(T, dm)$ (Saitoh [19]).

Note that F^* is a member of the visible component of $L_2(T, dm)$ which is the orthocomplement of the null space (the invisible component)

$$\{F_0 \in L_2(T, dm); \int_T F_0(t)\overline{h(t,p)}dm(t) = 0 \quad \text{on} \quad E\}$$

of $L_2(T, dm)$. Therefore, our inversion formula (16) will be considered as a very natural one.

By our Theorem 3, for example, in Example (5) we can establish the inversion formulas

$$u(z,t) \longrightarrow F(x)$$

and

$$u(x,t) \longrightarrow F(x)$$

for any fixed $t > 0$ (Saitoh [17], and Byun and Saitoh [5]).

Our inversion formula will give a new viewpoint and a new method for integral equations of Fredholm of the first kind which are fundamental in the theory of integral equations. The characteristics of our inversion formula are as follows:

(i) Our inversion formula is given in terms of the reproducing kernel Hilbert space H_K which is intuitively determined as the image space of the integral transform (1).
(ii) Our inversion formula gives the visible component F^* of F with the minimum $L_2(T, dm)$ norm.
(iii) The inverse F^* is, in general, given in the sense of the strong convergence in $L_2(T, dm)$.
(iv) Our integral equation (1) is, in general, an ill-posed problem, but our solution F^* is given as a solution of a well-posed problem in the sense of Hadamard (1902, 1923).

At this moment, we can see a reason why we meet to ill-posed problems; that is, because we do consider the problems not in the natural image spaces H_K, but in some artificial spaces.

6. DETERMING OF THE SYSTEM BY INPUT AND OUTPUT FUNCTIONS

In our **Theorem 4,** we can construct the integral kernel $h(t,p)$ conversely, in terms of the isometrical mapping $\widetilde{L}$ from a reproducing kernel Hilbert space H_K onto $L_2(T, dm)$ and the reproducing kernel $K(p,q)$ in the form

$$h(t,p) = \widetilde{L}K(\cdot,p). \tag{19}$$

7. GENERAL APPLICATIONS

Our basic assumption for the integral transform (1) is (3). When this assumption is not valid, we will be able to apply the following techniques to meet our assumption (3).

(a) We restrict the sets E or T, or we exchange the set E.
(b) We multiply a positive continuous function ρ in the form $L_2(T, \rho dm)$.
For example, in the Fourier transform

$$\int_{-\infty}^{\infty} F(t)e^{-itx}dt, \tag{20}$$

we consider the integral transform with the weighted function such that

$$\int_{-\infty}^{\infty} F(t)e^{-itx}\frac{dt}{1+t^2}.$$

(c) We integrate the integral kernel $h(t,p)$.
For example, in the Fourier transform (20), we consider the integral transform

$$\int_{-\infty}^{\infty} F(t)\left(\int_0^{\hat{x}} e^{-itx}dx\right)dt = \int_{-\infty}^{\infty} F(t)\left(\frac{e^{-it\hat{x}}-1}{-it}\right)dt.$$

By these techniques we can apply our general method even for integral transforms with integral kernels of generalized functions. Furthermore, for the integral transforms with the integral kernels of

miscellaneous Green's functions,
Cauchy's kernel,

and

Poisson's kernel

and also for even the cases of Fourier transform and Laplace transform, we could derive new results. See Saitoh [19] for the details, for example.

Recall the Whittaker-Kotelnikov-Shannon sampling theorem:

In the integral transform

$$f(t) = \frac{1}{\sqrt{2\pi}}\int_{-\pi}^{\pi} F(\omega)e^{i\omega t}d\omega$$

for functions $F(\omega)$ satisfying

$$\int_{-\pi}^{\pi} |F(\omega)|^2 d\omega < \infty,$$

we have the expression formula

$$f(t) = \sum_{n=-\infty}^{\infty} f(n)\frac{\sin\pi(n-t)}{\pi(n-t)} \quad \text{on} \quad (-\infty,\infty).$$

All the signals $f(t)$ are expressible in terms of the discrete data $f(n)$ (n: integers) only, and so many scientiests are interested in this theorem and this theorem is applied in miscellaneous fields. Furthermore, very interesting relations between fundamental theorems and formulas of signal analysis, of analytic number theory and of applied analysis are found recently (Klusch [9]).

In our general situation (1), the essence of the sampling theorem is given clearly and simply as follows:

For a sequence of points $\{p_n\}$ of E, if $\{h(t,p_n)\}_n$ is a complete orthonormal system in $L_2(T,dm)$, then for any $f \in H_K$, we have the sampling theorem

$$f(p) = \sum_n f(p_n)K(p,p_n) \quad \text{on} \quad E.$$

Meanwhile, the theory of wavelets is developing enormously in both mathematical sciences and pure mathematics which was created by (Morlet [10, 11]) about ten years ago. The theory is applicable in signal analysis, numerical analysis and many other fields as in Fourier transforms. Since the theory is that of integral transforms in the framework of Hilbert spaces, our general theory for integral transforms will be applicable to the wavelet theory, globally, in particular, our method will give a good understanding, as a unified one, for the wavelet transform, frames, multiresolution analysis and the sampling theory in the theory of wavelets (Saitoh [19]).

8. ANALYTIC EXTENSION FORMULAS

The equality of the two integrals (11) and (12) means that a C^∞ function $g(x)$ with a finite integral

$$\sum_{j=0}^{\infty}\frac{(2t)^j}{j!}\int_{-\infty}^{\infty}|\partial_x^j g(x)|^2dx < \infty,$$

is extensible analytically onto $\boldsymbol{C}$ and when we denote its analytic extension by $g(z)$, we have the identity

$$\sum_{j=0}^{\infty}\frac{(2t)^j}{j!}\int_{-\infty}^{\infty}|\partial_x^j g(x)|^2dx = \frac{1}{\sqrt{2\pi t}}\iint_{\boldsymbol{C}}|g(z)|^2\exp\{-\frac{y^2}{2t}\}dxdy. \tag{21}$$

In this way, we derived miscellaneous analytic extension formulas in many papers over 15 with H. Aikawa and N. Hayashi, and the analytic extension formulas are applied to the investigation of analyticity of solutions of nonlinear partial differential equations. See, for example (Hayashi and Saitoh [7]).

One typical result of another type is obtained from the integral transform

$$v(x,t) = \frac{1}{t}\int_0^t F(\xi)\frac{x\exp[\frac{-x^2}{4(t-\xi)}]}{2\sqrt{\pi}(t-\xi)^{\frac{3}{2}}}\xi d\xi$$

in connection with the heat equation

$$u_t(x,t) = u_{xx}(x,t) \quad \text{for} \quad v(x,t) = tu(x,t)$$

satisfying the conditions

$$u(0,t) = tF(t) \quad \text{on} \quad t \geqq 0$$

and

$$u(x,0) = 0 \quad \text{on} \quad x \geqq 0.$$

Then, we obtain:

Let $\Delta(\frac{\pi}{4})$ denote the sector $\{|argz| < \frac{\pi}{4}\}$. For any analytic function $f(z)$ on $\Delta(\frac{\pi}{4})$ with a finite integral

$$\iint_{\Delta(\frac{\pi}{4})} |f(z)|^2 dxdy < \infty,$$

we have the identity

$$\iint_{\Delta(\frac{\pi}{4})} |f(z)|^2 dxdy = \sum_{j=0}^{\infty} \frac{2^j}{(2j+1)!} \int_0^{\infty} x^{2j+1} |\partial_x^j f(x)|^2 dx. \tag{22}$$

Conversely, for any smooth function $f(x)$ with a finite integral in (22) on $(0, \infty)$, there exits an analytic extension $f(z)$ onto $\Delta(\frac{\pi}{4})$ satisfying (22) (Aikawa, Hayashi and Saitoh [1]).

9. BEST APPROXIMATION FORMULAS

As we saw, when we consider linear transforms in the framework of Hilbert spaces, we get naturally the idea of reproducing kernel Hilbert spaces. As a natural extension of our theorems, we have the fundamental theorems for approximations of functions in the framework of Hilbert spaces (Byun and Saitoh [5]).

For a function F on a set X, we shall look for a function which is the nearest one to F among some family of functions $\{f\}$. In order to formulate the "nearest" precisely, we shall consider F as a member of some Hilbert space $H(X)$ comprising of functions on X. Meanwhile, as the family $\{f\}$ of approximation functions, we shall consider some reproducing kernel Hilbert space H_K comprising of functions f on, in general, a set E containing the set X. Here the reproducing kernel Hilbert space H_K as a family of approximation functions will be considered as a natural one, since the point evaluation $f(p)$ is continuous on H_K.

We shall also set the natural assumptions for the relation between the two Hilbert spaces $H(X)$ and H_K:

(23) for the restriction $f|_X$ of the members f of H_K to the set X, $f|_X$ belongs to the Hilbert space $H(X)$,

and

(24) the linear operator $Lf = f|_X$ is continuous from H_K into $H(X)$.

In this natural situation, we can discuss the best approximation problem

$$\inf_{f \in H_K} \|Lf - F\|_{H(X)}$$

for a member F of $H(X)$.

For the sake of the nice properties of the restriction operator L and its adjoint L^*, we can obtain "algorithms" to decide whether the best approximations f^* of F in the sense of

$$\inf_{f \in H_K} \|Lf - F\|_{H(X)} = \|Lf^* - F\|_{H(X)}$$

exist. Further, when there exist the best approximations f^*, we can give "algorithms" obtaining constructively them. Moreover, we can give the representations of f^* in terms of the given function F and the reproducing kernel $K(p,q)$. Meanwhile, when the best approximations f^* do not exist, we can construct the sequence $\{f_n\}$ of H_K satisfying

$$\inf_{f \in H_K} \|Lf - F\|_{H(X)} = \lim_{n\to\infty} \|Lf_n - F\|_{H(X)}.$$

As one example (Byun and Saitoh [6]), for an $L_2(\mathbb{R}, dx)$ function $h(x)$, we shall approximate it by the family of functions $u_F(x,t)$ for any fixed $t > 0$ which are the solutions of the heat equation (6) with (7) for $F \in L_2(\mathbb{R}, dx)$.

Then, we can see that there exists a member F of $L_2(\mathbb{R}, dx)$ such that

$$u_F(x,t) = h(x) \quad \text{on} \quad \mathbb{R}$$

if and only if

$$\iint_C \left| \int_{-\infty}^{\infty} h(\xi) \exp\{-\frac{\xi^2}{8t} + \frac{\xi z}{4t}\} d\xi \right|^2 \exp\left\{ \frac{-3x^2 + y^2}{12t} \right\} dxdy < \infty.$$

If this condition is not valid for h, then we can construct the sequence $\{F_n\}$ satisfying

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} |u_{F_n}(x,t) - h(x)|^2 dx = 0;$$

that is, for any $L_2(\mathbb{R}, dx)$ function $h(x)$, we can construct the initial functions $\{F_n\}$ whose heat distributions $u_{F_n}(x,t)$ of t time later converge to $h(x)$.

10. APPLICATIONS TO RANDOM FIELDS ESTIMATIONS

We assume that the random field is of the form

$$u(x) = s(x) + n(x),$$

where $s(x)$ is the useful signal and $n(x)$ is noise. Without loss of generality, we can assume that the mean values of $u(x)$ and $n(x)$ are zero. We assume that the covariance functions

$$R(x,y) = \overline{u(x)u(y)}$$

and

$$f(x,y) = \overline{u(x)s(y)}$$

are known. We shall consider the general form of a linear estimation $\hat{u}$ of u in the form

$$\hat{u}(x) = \int_T u(t)h(x,t)dm(t)$$

for an $L_2(T, dm)$ space and for a function $h(x,t)$ belonging to $L_2(T, dm)$ for any fixed $x \in E$. For a desired information As for a linear operator A of s, we wish to determine the function $h(x,t)$ satisfying

$$\inf \overline{(\hat{u} - As)^2}$$

which gives the minimum of variance by the least square method. Many topics in filtering and estimation theory in signal, image processing, underwater acoustics, geophysics, optical filtering etc., which are initiated by N. Wiener (1894-1964), will be given in this framework. Then, we see that the linear transform $h(x,t)$ is given by the integral equation

$$\int_T R(x',t)h(x,t)dm(t) = f(x',x)$$

(Ramm [12]). Therefore, our random fields estimation problems will be reduced to find the inversion formula

$$f(x',x) \longrightarrow h(x,t)$$

in our framework. So, our general method for integral transforms will be applied to these problems. For this situation and another topics and methods for the inversion formulas, see (Ramm [12]) for the details.

11. APPLICATIONS TO SCATTERING AND INVERSE PROBLEMS

Scattering and inverse problems will be considered as the problems determining unobservable quantities by observable quantities. These problems are miscellaneous and are, in general, difficult. In many cases, the problems are reduced to some integral equations of Fredholm of the first kind and then, our method will be applicable to the equations. Meanwhile, in many cases, the problems will be reduced to determine the inverse F^* from the data $f(p)$ on some subset of E in our integral transform (1). See, for example, Ramm [13].

In each case, we shall state a typical example.

We shall consider the Poisson equation

$$\Delta u = -\rho(\mathbf{r}) \quad \text{on} \quad \mathbb{R}^3 \tag{25}$$

for real-valued $L_2(\mathbb{R}^3, d\mathbf{r})$ source functions ρ whose supports are contained in a sphere $r < a(|\mathbf{r}| = r)$. By using our method to the integral transform

$$u(\mathbf{r}) = \frac{1}{4\pi}\int_{r'<a}\frac{1}{|\mathbf{r}-\mathbf{r}'|}\rho(\mathbf{r}')d\mathbf{r}',$$

we can get the characteristic property and natural representation of the potentials u on the outside of the sphere $\{r < a\}$. Furthermore, we can obtain the surprisingly simple representations of ρ^* in terms of u on any sphere $(a', \theta', \varphi')(a < a')$, which have the minimum $L_2(\mathbb{R}^3, d\mathbf{r})$ norms among ρ satisfying (25) on $r > a$, in the form:

$$\begin{aligned}\rho^*(r,\theta,\varphi) = &\frac{1}{4\pi}\sum_{n=0}^{\infty}\frac{(2n+1)^2(2n+3)}{a^{2n+3}}r^n a'^{n+1}\\ &\times\sum_{m=0}^{n}\frac{\varepsilon_m(n-m)!}{(n+m)!}P_n^m(\cos\theta)\int_0^{\pi}\int_0^{2\pi}u(a',\theta',\varphi')\\ &\times P_n^m(\cos\theta')\cos m(\varphi'-\varphi)\sin\theta' d\theta' d\varphi'.\end{aligned}$$

Here, ε_m is the Neumann factor $\varepsilon_m = 2 - \delta_{m0}$.

Next, we shall consider an analytical real inversion formula of the Laplace transform

$$f(p) = \int_0^{\infty} e^{-pt}F(t)dt, \quad p > 0;$$

$$\int_0^{\infty}|F(t)|^2 dt < \infty.$$

For the polynomial of degree $2N+2$

$$\begin{aligned}P_N(\xi) = &\sum_{0\leqq\nu\leqq n\leqq N}\frac{(-1)^{\nu+1}(2n)!}{(n+1)!\nu!(n-\nu)!(n+\nu)!}\xi^{n+\nu}\\ &\cdot\left\{\frac{2n+1}{n+\nu+1}\xi^2 - (\frac{2n+1}{n+\nu+1}+3n+1)\xi + n(n+\nu+1)\right\},\end{aligned}$$

we set

$$F_N(t) = \int_0^{\infty} f(p)e^{-pt}P_N(pt)dp.$$

Then, we have

$$\lim_{N\to\infty}\int_0^{\infty}|F(t)-F_N(t)|^2 dt = 0.$$

Furthermore, the estimation of the error of $F_N(t)$ is also given (Byun and Saitoh [4]).

Compare our formula with (Boas and Widder [3], 1940), and with (Ramm [13], p.221, 1992):

For the Laplace transform

$$\int_0^b e^{-pt}F(t)dt = f(p),$$

we have

$$F(t) = \frac{2tb^{-1}}{\pi}\frac{d}{du}\int_0^u \frac{G(v)}{(u-v)^{\frac{1}{2}}}dv\Big|_{u=t^2b^{-2}};$$

$$G(v) = v^{-\frac{1}{2}}\frac{2}{\pi}\int_0^\infty dy\cos(y\cosh^{-1}v^{-1})\cosh\pi y$$

$$\times\int_0^\infty dz\cos(zy)(\cosh z)^{-\frac{1}{2}}\int_0^\infty dpf(p)J_0\left(p\frac{b}{(\cosh z)^{\frac{1}{2}}}\right).$$

In this very complicated formula, unfortunately, the characteristic properties of the both functions F and f making hold the inversion formula are not given.

12. NONHARMONIC TRANSFORMS

In our general transform (1), suppose that $\varphi(t,p)$ is near to the integral kernel $h(t,p)$ in the following sense:

For any $F \in L_2(T, dm)$,

$$\left\|\int_T F(t)\overline{(h(t,p)-\varphi(t,p))}dm(t)\right\|_{H_K}^2 \leqq \omega^2\int_T |F(t)|^2 dm(t)$$

where $0 < \omega < 1$ and ω is independent of $F \in L_2(T, dm)$.

Then, we can see that for any $f \in H_K$, there exists a function F_φ^* belonging to the visible component of $L_2(T, dm)$ in (1) such that

$$f(p) = \int_T F_\varphi^*(t)\overline{\varphi(t,p)}dm(t) \quad \text{on} \quad E \tag{26}$$

and

$$(1-\omega)^2\int_T |F_\varphi^*(t)|^2 dm(t) \leqq \|f\|_{H_K}^2$$
$$\leqq (1+\omega)^2\int_T |F_\varphi^*(t)|^2 dm(t).$$

The integral kernel $\varphi(t,p)$ will be considered as a perturbation of the integral kernel $h(t,p)$. When we look for the inversion formula of (26) following our general method, we must calculate the kernel form

$$K_\varphi(p,q) = \int_T \varphi(t,q)\overline{\varphi(t,p)}dm(t) \quad \text{on} \quad E\times E.$$

We will, however, in general, not able to calculate this kernel.

Suppose that the image $f(p)$ of (26) belongs to the known space H_K. Then, we can construct the inverse F_φ^* by using our inversion formula in H_K repeately and by constructing some approximation of F_φ^* by our inverses.

In particular, for the reproducing kernel $K(p,q) \in H_K(q \in E)$ we construct (or we get, by some other method or directly) the function $\hat{\varphi}(t,p)$ satisfying

$$K(p,q) = \int_T \hat{\varphi}(t,q)\overline{\varphi(t,p)}dm(t) \quad \text{on} \quad E \times E,$$

where $\hat{\varphi}(t,q)$ belongs to the visible component of $L_2(T, dm)$ in (26) for any fixed $q \in E$. Then, we have an idea of "nonharmonic integral transform" and we can formulate the inversion formula of (26) in terms of the kernel $\hat{\varphi}(t,q)$ and the space H_K, globally (Saitoh [19], chapter 7).

13. NONLINEAR TRANSFORMS

Our generalized isoperimetric inequality will mean that for an analytic function $\varphi(z)$ on $\overline{G}$ satisfying

$$\int_{\partial G} |\varphi(z)|^2 |dz| < \infty,$$

the image of the most simple nonlinear transform

$$\varphi(z) \longrightarrow \varphi(z)^2$$

belongs to the space of analytic functions satisfying

$$\iint_G |\varphi(z)|^4 dxdy < \infty$$

and we have the norm inequality

$$\frac{1}{\pi}\iint_G |\varphi(z)^2|^2 dxdy \leqq \left\{\frac{1}{2\pi}\int_{\partial G} |\varphi(z)|^2 |dz|\right\}^2.$$

We established in Theorem 1 the method of the identification of the images of linear transforms, and also we will be able to look for some Hilbert spaces containing the image spaces of nonlinear transforms. In these cases, however, the spaces will be too large for the image spaces, as in the generalized isoperimetric inequality. See Saitoh [14, 19] for the details. So, the inversion formulas for nonlinear transforms will be, in general, very involved essentially. In many nonlinear transforms of reproducing kernel Hilbert spaces, however, there exist some norm inequalities as in the generalized isoperimetric inequality. See Saitoh [19] for some examples. We shall state one example in the strongly nonlinear transform

$$f(x) \longrightarrow e^{f(x)}$$

which is obtained recently in (Saitoh [20]):

For an absolutely continuous real-valued function f on $(a,b)(a>0)$ satisfying

$$f(a)=0,$$
$$\int_a^b f'(x)^2 x dx < \infty,$$

we obtain the inequality

$$1 + a\int_a^b |(e^{f(x)})'|^2 dx \leqq e^{\int_a^b f'(x)^2 x dx}.$$

Here, we should note that the equality holds for many functions $f(x)$.

14. EPILOGUE

The author believes we could show the importance of our Theorems 1-4. These theorems were applied in many papers over 30, and our fundamental theorems and their applications were partially published in the research book (Saitoh [19]) in 1988, happily. The author hopes, however, that our theorems should be more popular in mathematical sciences and should be applied to many fields, more widely.

At the last, the author would like to say that the core of the originality in this expositive paper is not in mathematical results but to show clearly the importance of our Theorems 1-4 for linear transforms.

ACKNOWLEDGMENT

I would like to thank Takehisa Abe for many valuable comments and suggestions on this paper.

References

1. H. Aikawa, N. Hayashi and S. Saitoh, *The Bergman space on a sector and the heat equation*, Complex Variables **15** (1990), 27-36.
2. N. Aronszajn, *Theory of reproducing kernels*, Trans. Amer. Math. Soc. **68** (1950), 337-404.
3. R. P. Boas and D. V. Widder, *An inverse formula for the Laplace integral*, Duke Math. J. **6** (1940), 1-26.
4. D. -W. Byun and S. Saitoh, *A real inversion formula for the Laplace transform*, Zeit. Analysis und ihre Anwendungen **12** (1993), 597-603.

5. D. -W. Byun and S. Saitoh, *Approximation by the solutions of the heat equation*, J. Approximation Theory **78** (1994), 226-238.
6. D. -W. Byun and S. Saitoh, *Best approximations in reproducing kernel Hilbert spaces*, Proceedings of the Second International Colloquium on Numerical Analysis (1994), 55-61. VSP-Holland.
7. N. Hayashi and S. Saitoh, *Analyticity and smoothing effect for the Schrödinger equation*, Ann. Inst. Henri Poincaré **52** (1990), 163-173.
8. D. A. Hejhal, *Theta Functions, Kernel Functions and Abel Integrals*, Memoirs of Amer. Math. Soc. **129** (1972).
9. D. Klusch, *The sampling theorem, Dirichlet series and Hankel transforms*, J. of Computational and Applied Math. **44** (1992), 261-273.
10. J. Morlet, G. Arens, I. Fourgeau and D. Giard, *Wave propagation and sampling theory*, Geophysics **47** (1982), 203-236.
11. J. Morlet, *Sampling theory and wave propagation*, in "NATO ASI Series, Vol.1, Issues in Acoustic Signal Image Processing and Recognition, C. H. Chen ed.," Springer -Verlag, Berlin, 1983, pp. 233-261.
12. A. G. Ramm, "Random Fields Estimation Theory," Longman Scientific / Wiley, New York, 1990.
13. A. G. Ramm, "Multidimensional Inverse Scattering Problems," Longman Scientific & Technical, 1992.
14. S. Saitoh , *The Bergman norm and the Szegö norm*, Trans. Amer. Math. Soc. **249** (1979), 261-279.
15. ———, *Integral transforms in Hilbert spaces*, Proc. Japan Acad. Ser. A. Math. Sci. **58** (1982), 361-364.
16. ———, *Hilbert spaces induced by Hilbert space valued functions*, Proc. Amer. Math. Soc. **89** (1983), 74-78.
17. ———, *The Weirstrass transform and an isometry in the heat equation*, Applicable Analysis **16** (1983), 1-6.
18. ———, *Some isometrical identities in the wave equation*, International J. Math. and Math. Sci. **7** (1984), 117-130.
19. ———, "Theory of Reproducing Kernels and its Applications," Pitman Res. Notes in Math. Series 189, Longman Scientific & Technical, England, 1988.
20. ———, *An integral inequality of exponential type for real-valued functions*, in "World Scientific Series in Applicable Analysis, Inequalities and Applications 3," World Scientific, 1994, pp. 537-541.
21. L. Schwartz, *Sous-espaces hilbertiens d'espaces vectoriels topologiques et noyaux associès (noyaux reproduisants)*, J. Analyse Math. **13** (1964), 115-256.

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1991 *Mathematics Subject Classification.* Primary 30A10, 30B40, 30C40, 35B05, 35R30, 44A05, 41A50.

Key words and phrases. Hilbert space, reproducing kernel, linear transform, integral transform, inversion formula, Pythagoras' theorem, Fredholm integral equation of the first kind, heat equation, wave equation, analytic extension, ill and well-posed problems, sampling theorem, best approximation, random estimation, scattering problem, inverse problem, norm inequality, nonharmonic transform, nonlinear transform, wavelets.