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On Stein-Weiss Theorem and Mapping Properties of Potential Type Operators with Power and Power-Logarithmic Kernels

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Abstract

The conditions are given for the multidimensional potential type operators with power and power-logarithmic kernels to be bounded from the one weighted space of p-summable functions with power weight into another.

1. Introduction

Let R^n be the n-dimensional Euclidean space and I^α be the Riesz potential, or multidimensional fractional integral

$$(I^\alpha\varphi)(x)=c_{n,\alpha}\int_{R^n}\frac{\varphi(t)dt}{|x-t|^{n-\alpha}}\quad\left(\alpha>0,\ c_{n,\alpha}=\frac{\Gamma([n-\alpha]/2)}{2^\alpha\pi^{n/2}\Gamma(\alpha/2)}\right).\tag{1.1}$$

It is well known by the classical Hardy-Littlewood-Sobolev theorem (see eg. [1, § 25] and [2, Chapter 5, §1.2]) that if $1\leqq p\leqq\infty$, $1\leqq q\leqq\infty$ and $\alpha>0$, the Riesz potential I^α is a bounded operator from $L_p(R^n)$ into $L_q(R^n)$ if and only if

$$0<\alpha<n,\quad 1<p<\frac{n}{\alpha},\quad \frac{1}{q}=\frac{1}{p}-\frac{\alpha}{n}.\tag{1.2}$$

This result was generalized in many directions. The weighted analogue of it was first given by Stein and Weiss [3]. They proved that if $\alpha>0$, $1<p<\infty$, $1<q<\infty$ and

$$\alpha p-n<\mu<n(p-1),\quad \frac{1}{p}-\frac{\alpha}{n}\leqq\frac{1}{q}\leqq\frac{1}{p},\quad \frac{\nu+n}{q}=\frac{\mu+n}{p}-\alpha,\tag{1.3}$$

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then the the Riesz potential I^α is a bounded operator from $L_p(R^n; |x|^\mu)$ into $L_q(R^n; |x|^\nu)$, namely the estimate

$$\left(\int_{R^n} |x|^\nu |(I^\alpha\varphi)(x)|^q dx\right)^{1/q} \leqq k \left(\int_{R^n} |x|^\mu |\varphi(x)|^p dx\right)^{1/p} \tag{1.4}$$

holds, where the constant $k > 0$ does not depend on φ. Various generalizations and modifications of such a statement for the Riesz potential (1.1) and for other connected operators were given by many authors (see the monograph [1, § 29] for historical notices and survey of the results).

This paper is devoted to obtain such an estimates for the potential type operators with the power-logarithmic kernel

$$(I_\Omega^{\alpha,\beta}\varphi)(x) = \int_\Omega \log^\beta\left(\frac{\gamma}{|x-t|}\right)\frac{\varphi(t)dt}{|x-t|^{n-\alpha}} \quad (x \in \Omega) \tag{1.5}$$

for $0 < \alpha < n, \beta \geqq 0$ and $\gamma > \operatorname{mes}(\Omega)$ on a measurable set $\Omega \in R^n$. In Section 2 we prove the estimate (1.4) for the Riesz potential on Ω

$$(I_\Omega^\alpha\varphi)(x) = \int_\Omega \frac{\varphi(t)dt}{|x-t|^{n-\alpha}} \quad (x \in \Omega,\ 0 < \alpha < n). \tag{1.6}$$

The cases when Ω is a unit ball $B = \{t \in R^n : |t| \leqq 1\}$ in R^n and its exterior $B^c = \{t \in R^n : |t| \geqq 1\}$ are more important. We show that in these cases the condition $\alpha p - n < \mu < n(p-1)$ in (1.3) can be weakened till $\mu > \alpha p - n$ and $\mu < n(p-1)$ for B and B^c, respectively (see Theorem 1). Section 3 deals with an extension of the result by Stein and Weiss to the weighted space

$$L_p(R^n;\rho) = \left\{f : \|f\|_{L_p(\rho)} = \left\|\rho^{1/p}f\right\|_{L_p} = \left(\int_\Omega \rho(x)|f(x)|^p dx\right)^{1/p}\right\} \tag{1.7}$$

with the power weight

$$\rho(x) = (1+|x|)^\mu \prod_{k=1}^m |x-x_k|^{\mu_k} \tag{1.8}$$

concentrated at the finite points $x_1,\ x_2,\cdots,x_m$ of Ω with $0 \leqq |x_1| < |x_2| < \cdots < |x_m|$ and at infinity, where $\mu, \mu_1, \mu_2, \cdots, \mu_m \in R$. In Section 4 the results obtained are applied to prove the estimates for the potential type operator with power-logarithmic kernels (1.5), in particular, for the Riesz potential (1.6), in the weighted spaces $L_p(\Omega;\rho)$ with the power weight

$$\rho(x) = \begin{cases} \prod_{k=1}^m |x-x_k|^{\mu_k}, & \text{if } \operatorname{mes}(\Omega) < \infty, \\ (1+|x|)^\mu \prod_{k=1}^m |x-x_k|^{\mu_k}, & \text{if } \operatorname{mes}(\Omega) = \infty, \end{cases} \tag{1.9}$$

concentrated at the finite points $x_1,\ x_2,\cdots,x_m$ of Ω with $0 \leqq |x_1| < |x_2| < \cdots < |x_m|$ and at infinity (the latter when Ω is unbounded), where $\mu, \mu_1, \mu_2, \cdots, \mu_m \in R$.

2. Riesz Potential in the Case of a Simplest Power Weight

Let us consider the cases of the unit ball $B = \{t \in R^n : |t| \leqq 1\}$ in R^n and its exterior $B^c = \{t \in R^n : \ |t| \geqq 1\}$. Let $I_B^\alpha \varphi$ and $I_{B^c}^\alpha \varphi$ be the corresponding Riesz potentials:

$$(I_B^\alpha \varphi)(x) = c_{n,\alpha} \int_{|t|\leqq 1} \frac{\varphi(t)dt}{|x-t|^{n-\alpha}} \quad (|x| \leqq 1), \tag{2.1}$$

$$(I_{B^c}^\alpha \varphi)(x) = c_{n,\alpha} \int_{|t|\geqq 1} \frac{\varphi(t)dt}{|x-t|^{n-\alpha}} \quad (|x| \geqq 1), \tag{2.2}$$

where $0 < \alpha < n$ and $c_{n,\alpha}$ is given by (1.1).

Theorem 1. *Let real numbers α, p, q, μ and ν satisfy the conditions*

$$\alpha > 0,\ 1 < p < \infty,\ \ 1 < q < \infty,\ \ \frac{1}{p} - \frac{\alpha}{n} \leqq \frac{1}{q} \leqq \frac{1}{p},\ \ \frac{\nu + n}{q} = \frac{\mu + n}{p} - \alpha. \tag{2.3}$$

(i) *If $\mu > \alpha p - n$, then the Riesz potential (2.1) is a bounded operator from $L_p(B; |x|^\mu)$ into $L_q(B; |x|^\nu)$ and*

$$\left(\int_{|x|\leqq 1} |x|^\nu |(I_B^\alpha \varphi)(x)|^q dx\right)^{1/q} \leqq k_1 \left(\int_{|x|\leqq 1} |x|^\mu |\varphi(x)|^p dx\right)^{1/p} \tag{2.4}$$

holds with the constant $k_1 > 0$ not depending on φ.
(ii) *If $\mu < n(p-1)$, then the Riesz potential (2.2) is a bounded operator from $L_p(B^c; |x|^\mu)$ into $L_q(B^c; |x|^\nu)$ and*

$$\left(\int_{|x|\geqq 1} |x|^\nu |(I_{B^c}^\alpha \varphi)(x)|^q dx\right)^{1/q} \leqq k_2 \left(\int_{|x|\geqq 1} |x|^\mu |\varphi(x)|^p dx\right)^{1/p} \tag{2.5}$$

holds with the constant $k_2 > 0$ not depending on φ.

Proof. We shall follow Stein and Weiss [3]. First we consider the case $q = p$. We have to prove the estimates

$$\left(\int_{|x|\leqq 1} |x|^\nu |(I_B^\alpha \varphi)(x)|^p dx\right)^{1/p} \leqq k_1 \left(\int_{|x|\leqq 1} |x|^\mu |\varphi(x)|^p dx\right)^{1/p} \quad (k_1 > 0) \tag{2.6}$$

and

$$\left(\int_{|x|\geqq 1} |x|^\nu |(I_{B^c}^\alpha \varphi)(x)|^p dx\right)^{1/p} \leqq k_2 \left(\int_{|x|\geqq 1} |x|^\mu |\varphi(x)|^p dx\right)^{1/p} \quad (k_2 > 0). \tag{2.7}$$

We first prove (2.6). According to (2.1) we have to show that

$$\left(\int_{|x|\leqq 1} |(J\phi)(x)|^p dx\right)^{1/p} \leqq c_1 \left(\int_{|x|\leqq 1} |\phi(x)|^p dx\right)^{1/p} \quad (c_1 > 0), \tag{2.8}$$

where

$$(J\phi)(x) = \int_{|x|\leqq 1} |x|^{\nu/p} |x-t|^{\alpha-n} |t|^{-\mu/p} \phi(t) dt \quad \left(\phi(t) = |t|^{\mu/p}\varphi(t)\right). \tag{2.9}$$

We represent (2.9) in the form

$$(J\phi)(x) = \int_{B_1} K_1(x,t)\phi(t)dt + \int_{B_2} K_2(x,t)\phi(t)dt + \int_{B_3} K_3(x,t)\phi(t)dt$$
$$= (J_1\phi)(x) + (J_2\phi)(x) + (J_3\phi)(x), \tag{2.10}$$

where

$$K_1(x,t) = \begin{cases} |x|^{\nu/p}|x-t|^{\alpha-n}|t|^{-\mu/p}, & (x,t) \in B_1 = \{(x,t) : |x| \leqq 1, |t| \leqq |x|/2\}, \\ 0, & (x,t) \notin B_1, \end{cases}$$

$$K_2(x,t) = \begin{cases} |x|^{\nu/p}|x-t|^{\alpha-n}|t|^{-\mu/p}, & (x,t) \in B_2 = \{(x,t) : |t| \leqq 1, |t| \geqq 2|x|\}, \\ 0, & (x,t) \notin B_2, \end{cases}$$

$$K_3(x,t) = \begin{cases} |x|^{\nu/p}|x-t|^{\alpha-n}|t|^{-\mu/p}, & \\ \qquad (x,t) \in B_3 = \{(x,t) : |x| \leqq 1, |t| \leqq 1, |x|/2 < |t| < 2|x|\}, & \\ 0, & (x,t) \notin B_3. \end{cases} \tag{2.11}$$

We prove the estimate (2.8) for $J_1\phi$. Since $|t| \leqq |x|/2$, $|x-t| \geqq |x|/2$ and therefore

$$|K_1(x,t)| \leqq 2^{n-\alpha}|x|^{\alpha-n+\nu/p}|t|^{-\mu/p}, \ (x,t) \in B_1,$$

and hence

$$|(J_1\phi)(x)| \leqq 2^{n-\alpha}|x|^{\alpha-n+\nu/p} \int_{B_1} |t|^{-\mu/p}|\phi(t)|dt. \tag{2.12}$$

Let $S = \{\sigma \in R^n : \ |\sigma| = 1\}$ be the unit sphere in R^n with the surface element $d\sigma$ and the surface area $|\sigma_n| = 2\pi^{n/2}/\Gamma(n/2)$. We now estimate the integral

$$\int_{|x|\leqq 1} |(J_1\phi)(x)|^p dx.$$

Making the substitution $x = R\sigma$ for $R = |x|$ and $\sigma = x/|x| \in S$, and using (2.12) and the definition of B_1, we have

$$\int_{|x| \leqq 1} |(J_1\phi)(x)|^p dx = \int_S \left(\int_0^1 |(J_1\phi)(x)|^p R^{n-1} dR\right) d\sigma$$
$$\leqq 2^{(n-\alpha)p} \int_S \left(\int_0^1 \left| R^{\alpha-n+\nu/p} \int_{B_1} |t|^{-\mu/p}|\phi(t)|dt\right|^p R^{n-1}dR\right) d\sigma$$
$$\leqq 2^{(n-\alpha)p}|\sigma_n| \int_0^1 |(\tilde{J}_1\phi)(R)|^p R^{n-1} dR, \tag{2.13}$$

where

$$(\tilde{J}_1\phi)(R) = \int_{\tilde{B}_1} R^{\alpha-n+\nu/p}|t|^{-\mu/p}|\phi(t)|dt, \ \ \tilde{B}_1 = \left\{(R,t) : \ |t| \leqq \frac{R}{2}\right\}. \tag{2.14}$$

Changing the variable $t = r\theta$ with $r = |t|, \theta = t/|t| \in S$, we can rewrite (2.14) as

$$(\tilde{J}_1\phi)(R) = \int_S \left\{ R^{\alpha-n+\nu/p} \int_0^{R/2} r^{n-1-\mu/p} |\phi(r\theta)| dr \right\} d\theta = \int_S (J_\theta \phi)(R) d\theta. \tag{2.15}$$

Here $J_\theta \phi$ is given by

$$(J_\theta\phi)(R) = R^{\alpha-n+\nu/p} \int_0^{R/2} r^{n-1-\mu/p} |\phi(r\theta)| dr$$

and, after the substitution $r = Rt$ and noting $\nu = \mu - \alpha p$ implied by (2.3), it can be rewritten as

$$(J_\theta\phi)(R) = \int_0^{1/2} t^{n-1-\mu/p} |\phi(Rt\theta)| dt. \tag{2.16}$$

Let $h(R)$ be a function given on $[0,1]$ and such that

$$\int_0^1 |h(R)|^{p'} R^{n-1} dR = 1, \ \left(\int_0^1 |(J_\theta\phi)(R)|^p R^{n-1} dR \right)^{1/p} = \int_0^1 |(J_\theta\phi)(R)| R^{n-1} h(R) dR.$$

Using (2.16) and applying Fubini's theorem and Hölder's inequality, we have

$$\begin{aligned}
\left(\int_0^1 |(J_\theta\phi)(R)|^p R^{n-1} dR \right)^{1/p} &= \int_0^1 |(J_\theta\phi)(R)| R^{n-1} h(R) dR \\
&= \int_0^{1/2} t^{n-1-\mu/p} dt \int_0^1 |\phi(Rt\theta)| R^{n-1} h(R) dR \\
&\leqq \int_0^{1/2} t^{n-1-\mu/p} \left(\int_0^1 |\phi(Rt\theta)|^p R^{n-1} dR \right)^{1/p} \left(\int_0^1 |h(R)|^{p'} R^{n-1} dR \right)^{1/p'} dt \\
&= \int_0^{1/2} t^{n-1-\mu/p} \left(\int_0^1 |\phi(Rt\theta)|^p R^{n-1} dR \right)^{1/p} dt.
\end{aligned}$$

Making the change $Rt = r$ in the inner integral, we obtain

$$\begin{aligned}
\int_0^1 |(J_\theta\phi)(R)|^p R^{n-1} dR &\leqq \int_0^{1/2} t^{n-1-\mu/p-n/p} \left(\int_0^R |\phi(r\theta)|^p r^{n-1} dr \right)^{1/p} dt \\
&\leqq c \left(\int_0^1 |\phi(r\theta)|^p r^{n-1} dr \right)^{1/p}
\end{aligned} \tag{2.17}$$

in view of the convergence of the integral

$$c = \int_0^{1/2} t^{n-1-\mu/p-n/p} dt = \frac{p}{np-\mu-n} 2^{(\mu+n)/p-n}.$$

Then by (2.15) and Hölder's inequality we find

$$\begin{aligned}
|(\tilde{J}_1\phi)(R)|^p &\leqq \left| \int_S |(J_\theta\phi)(R)| d\theta \right|^p \\
&\leqq \left[\left(\int_S |(J_\theta\phi)(R)|^p d\theta \right)^{1/p} |\sigma_n|^{1/p'} \right]^p = |\sigma_n|^{p-1} \int_S |(J_\theta\phi)(R)|^p d\theta.
\end{aligned} \tag{2.18}$$

Then according to (2.17), (2.18) and Fubini's theorem we have

$$\begin{aligned}\int_0^1 |(\tilde{J}_1\phi)(R)|^p R^{n-1}dR &\leqq |\sigma_n|^{p-1}\int_0^1 R^{n-1}dR\int_S |(J_\theta\phi)(R)|^p d\theta \\ &\leqq |\sigma_n|^{p-1}\int_S d\theta\int_0^1 |(J_\theta\phi)(R)|^p R^{n-1}dR \\ &\leqq c|\sigma_n|^{p-1}\int_S d\theta\int_0^1 |\phi(r\theta)|^p r^{n-1}dr = c|\sigma_n|^{p-1}\int_{|x|\leqq 1}|\phi(x)|^p dx.\end{aligned}$$

Substituting this into (2.13) we arrive at the estimate (2.8) for $J_1\phi$:

$$\left(\int_{|x|\leqq 1}|(J_1\phi)(x)|^p dx\right)^{1/p} \leqq c_1\left(\int_{|x|\leqq 1}|\phi(x)|^p dx\right)^{1/p} \quad (c_1 = 2^{\alpha-n}|\sigma_n|c^{1/p}). \tag{2.19}$$

The arguments similar to the above lead to the estimate (2.8) for $J_2\phi$ defined in (2.10):

$$\left(\int_{|x|\leqq 1}|(J_2\phi)(x)|^p dx\right)^{1/p} \leqq c_2\left(\int_{|x|\leqq 1}|\phi(x)|^p dx\right)^{1/p} \quad (c_2 > 0). \tag{2.20}$$

The estimate for $J_3\phi$ of (2.10) follows from the corresponding result on R^n given by Stein and Weiss [3, pp. 509-510]:

$$\begin{aligned}\left(\int_{|x|\leqq 1}|(J_3\phi)(x)|^p dx\right)^{1/p} &\leqq \left(\int_{R^n}|(J_3\phi_B)(x)|^p dx\right)^{1/p} \leqq c_3\left(\int_{R^n}|\phi_B(x)|^p dx\right)^{1/p} \\ &= c_3\left(\int_{|x|\leqq 1}|\phi(x)|^p dx\right)^{1/p} \quad (c_3 > 0),\end{aligned} \tag{2.21}$$

where $\phi_B(x)$ is the function concentrated on the unit ball B:

$$\phi_B(x) = \phi(x) \quad (x \in B), \qquad \phi_B(x) = 0 \quad (x \notin B). \tag{2.22}$$

Applying Minkowski's inequality to (2.10) and taking (2.19) - (2.21) into account we arrive at the estimate (2.8), and so (2.6) is proved.

The inequality (2.7) can be proved similarly. This completes the proof of the theorem in the case $p = q$.

Let now consider the case $1 < p < q < \infty$. The relations (2.4) and (2.5) are known to be equivalent to the following ones [3]:

$$\left|\int_{|t|\leqq 1}\int_{|x|\leqq 1}\frac{f(t)g(x)}{|x|^{-\nu/q}|x-t|^{n-\alpha}|t|^{\mu/p}}\,dt\,dx\right| \leqq c_4\,\|f\|_p\,\|g\|_{q'} \quad (c_4 > 0), \tag{2.23}$$

and

$$\left|\int_{|t|\geqq 1}\int_{|x|\geqq 1}\frac{f(t)g(x)}{|x|^{-\nu/q}|x-t|^{n-\alpha}|t|^{\mu/p}}\,dt\,dx\right| \leqq c_5\,\|f\|_p\,\|g\|_{q'} \quad (c_5 > 0), \tag{2.24}$$

where $\|f\|_p = \|f\|_{L_p(\Omega)}$ with $\Omega = B$ or $\Omega = B^c$.

We prove (2.23). We represent the integral in the left hand side of (2.23) as

$$I = \int_{|t|\leqq 1}\int_{|x|\leqq 1} \frac{f(t)g(x)}{|x|^{-\nu/q}|x-t|^{n-\alpha}|t|^{\mu/p}}\,dt\,dx = I_1 + I_2 + I_3, \tag{2.25}$$

where

$$I_i = \int\int_{D_i} \frac{f(t)g(x)}{|x|^{-\nu/q}|x-t|^{n-\alpha}|t|^{\mu/p}}\,dt\,dx \quad (i = 1,2,3), \tag{2.26}$$

with

$$\begin{aligned} D_1 &= \left\{(x,t) : |t| \leqq 1,\ |x| \leqq 1,\ \frac{1}{2}|t| \leqq |x| \leqq 2|t|\right\}, \\ D_2 &= \left\{(x,t) : |t| \leqq 1,\ |x| < \frac{1}{2}|t|\right\}, \\ D_3 &= \{(x,t) : |t| \leqq 1,\ |x| > 2|t|\}. \end{aligned} \tag{2.27}$$

When $(x,t) \in D_1$, we find that

$$|x-t|^{\mu/p-\nu/q} \leqq 3^{\mu/p-\nu/q}|x|^{\mu/p-\nu/q} \leqq 3^{\mu/p-\nu/q}2^{|\nu|/q}|x|^{\mu/p}|t|^{-\nu/q}$$

by noting $\mu/p - \nu/q \geqq 0$ from the condition, and hence after applying Hölder's inequality we have

$$\begin{aligned} |I_1| &\leqq c_6 \int_{|t|\leqq 1} |f(t)| \left(\int_{|x|\leqq 1} \frac{|g(x)|dx}{|x-t|^{n-\alpha+\mu/p-\nu/q}}\right) dt \\ &\leqq c_6 \|f\|_p \left[\int_{|t|\leqq 1} \left(\int_{|x|\leqq 1} \frac{|g(x)|dx}{|x-t|^{n-\alpha+\mu/p-\nu/q}}\right)^{p'} dt\right]^{1/p'} \quad (c_6 = 3^{\mu/p-\nu/q}2^{|\nu|/q}). \end{aligned}$$

From here by using the Hardy-Littlewood-Sobolev result given in Section 1 with α being replaced by $\alpha - \mu/p + \nu/q$, p by q' and q by p', we obtain

$$|I_1| \leqq c_7 \|f\|_p \|g\|_{q'}. \tag{2.28}$$

If $(x,t) \in D_2$,

$$|x-t| \geqq \frac{1}{2}|t|, \quad |x-t|^{\alpha-n} \leqq 2^{n-\alpha}|t|^{\alpha-n},$$

and after applying Hölder's inequality we have

$$\begin{aligned} |I_2| &\leqq c_8 \int_{|t|\leqq 1} |f(t)||t|^{\alpha-n-\mu/p} \left(\int_{|x|<|t|} |g(x)||x|^{\nu/q}dx\right) dt \\ &\leqq c_8 \|f\|_p \left(\int_{|t|\leqq 1} \left[|t|^{\alpha-\mu/p+\nu/q}(Kg)(t)\right]^{p'} dt\right)^{1/p'} \\ &= c_8 \|f\|_p \left(\int_{|t|\leqq 1} [(Kg)(t)]^{q'}[(Kg)(t)]^{p'-q'}|t|^{(\alpha-\mu/p+\nu/q)p'}dt\right)^{1/p'} \quad (c_8 = 2^{n-\alpha}), \end{aligned} \tag{2.29}$$

where

$$(Kg)(t) = |t|^{-n-\nu/q} \int_{|x|<|t|} |g(x)||x|^{\nu/q} dx \tag{2.30}$$

is the operator with the homogeneous kernel $K(x,t) = |t|^{-n-\nu/q}|x|^{\nu/q}$ of degree $-n$.

From the condition $1 < p < q < \infty$ we have $1 < q' < p' < \infty$ and $p' - q' > 0$. Then by Hölder's inequality and the condition $\mu > \alpha p - n$ of the theorem being equivalent to $\nu + n > 0$, we obtain

$$(Kg)(t) \leqq |t|^{-n-\nu/q} \|g\|_{q'} \left(\int_{|x|<|t|} |x|^{\nu} dx \right)^{1/q} = \left(\frac{|\sigma_n|}{\nu + n} \right)^{1/q} \|g\|_{q'} |t|^{-n/q'}.$$

Using the assumption (2.3) we obtain

$$[(Kg)(t)]^{p'-q'} |t|^{(\alpha-\mu/p+\nu/q)p'} \leqq c_9 \|g\|_{q'}^{p'-q'} \quad \left(c_9 = \left(\frac{|\sigma_n|}{\nu+n} \right)^{(p'-q')/q} \right).$$

Substituting this estimate into (2.29), we have

$$|I_2| \leqq c_{10} \|f\|_p \|g\|_{q'}^{1-q'/p'} \left(\int_{|t|\leqq 1} [(Kg)(t)]^{q'} dt \right)^{1/p'} \quad \left(c_{10} = 2^{n-\alpha} \left(\frac{|\sigma_n|}{\nu+n} \right)^{(1-q'/p')/q} \right).$$

In view of Lemma 2.1 of [3] which gives conditions for an integral operator with a homogeneous kernel of order $-n$ to be bounded in $L_{q'}$-space, we have the estimate

$$\left(\int_{|t|\leqq 1} [(Kg)(t)]^{q'} dt \right)^{1/q'} \leqq c_{11} \|g\|_{q'}$$

to the integral (2.30). From here we arrive at the estimate

$$|I_2| \leqq c_{12} \|f\|_p \|g\|_{q'}^{1-q'/p'} \|g\|_{q'}^{q'/p'} = c_{12} \|f\|_p \|g\|_{q'}. \tag{2.31}$$

The estimate for I_3

$$|I_3| \leqq c_{13} \|f\|_p \|g\|_{q'} \tag{2.32}$$

can be proved similarly.

Substituting the estimates (2.28), (2.31) and (2.32) into (2.25) we obtain the relation (2.23).

The inequality (2.24) may be deduced similarly and the theorem is completely proved.

Remark 1. Theorem 1 is evidently true for any ball $B_b = \{t \in R^n : |t| \leqq b\}$ in R^n and its exterior $B_b^c = \{t \in R^n : |t| \geqq b\}$ with $0 < b < \infty$.

Theorem 1 and Remark 1 imply the corresponding statements for the Riesz potential (1.6).

Theorem 2. *Let Ω be a measurable set in R^n and $d \in R^n$ be a finite point. Let the conditions* (2.3) *be satisfied. Assume that either* (i) *Ω is a bounded set in R^n with $d \in \Omega$*

and $\mu > \alpha p - n$, *or* (ii) Ω *is an unbounded set in* R^n *with* $d \notin \Omega$ *and* $\mu < n(p-1)$. *Then the Riesz potential* I_Ω^α *is bounded from* $L_p(\Omega; |x-d|^\mu)$ *into* $L_q(\Omega; |x-d|^\nu)$:

$$\left(\int_\Omega |x-d|^\nu |(I_\Omega^\alpha \varphi)(x)|^q dx\right)^{1/q} \leqq k_1 \left(\int_\Omega |x-d|^\mu |\varphi(x)|^p dx\right)^{1/p} \tag{2.33}$$

for the case (i) *and from* $L_p(\Omega; (|x|-|d|)^\mu)$ *into* $L_q(\Omega; (|x|-|d|)^\nu)$:

$$\left(\int_\Omega (|x|-|d|)^\nu |(I_\Omega^\alpha \varphi)(x)|^q dx\right)^{1/q} \leqq k_2 \left(\int_\Omega (|x|-|d|)^\mu |\varphi(x)|^p dx\right)^{1/p} \tag{2.34}$$

for the case (ii). *Here constants* $k_1 > 0$ *and* $k_2 > 0$ *do not depend on* φ.

Proof. We prove (2.33). The case $\Omega = R^n$ is reduced to Theorem 1 by simple replacement $t-d$ by t. If $\Omega \neq R^n$, then we use the function φ_Ω defined as in (2.22) to obtain (2.33) and (2.34). In fact, for example, let Ω be a bounded set in R^n. Then there is a ball B_b such that $\Omega \subseteq B_b$. Let φ_Ω be the function

$$(\varphi_\Omega)(x) = (\varphi)(x) \quad (x \in \Omega); \qquad (\varphi_\Omega)(x) = 0 \quad (x \in B_b \setminus \Omega).$$

Then applying Theorem 1(i) and Remark 1 we obtain

$$\begin{aligned}\left(\int_\Omega |x-d|^\nu |(I_\Omega^\alpha \varphi)(x)|^q dx\right)^{1/q} &= \left(\int_\Omega |x-d|^\nu |(I_{B_b}^\alpha \varphi_\Omega)(x)|^q dx\right)^{1/q} \\ &\leqq \left(\int_{B_b} |x-d|^\nu |(I_{B_b}^\alpha \varphi_\Omega)(x)|^q dx\right)^{1/q} \\ &\leqq k_1 \left(\int_{B_b} |x-d|^\mu |\varphi_\Omega(x)|^p dx\right)^{1/p} \\ &= k_1 \left(\int_\Omega |x-d|^\mu |\varphi(x)|^p dx\right)^{1/p},\end{aligned} \tag{2.35}$$

which proves (2.33). The relation (2.34) can be proved similarly.

3. Riesz Potential in the Case of a General Power Weight

We consider mapping properties of the Riesz potential I^α given in (1.1) from $L_p(R^n;\rho)$ into $L_q(R^n;r)$ with the power weights ρ and r of the form (1.8). In what follows, we shall denote by $c_1, c_2, c_3, \cdots$ the different positive constants which do not depend on the function $\varphi \in L_p(R^n;\rho)$.

Theorem 3. *We assume*

$$0 < \alpha < n, \quad 1 < p < \infty, \quad 1 < q < \infty, \quad \frac{1}{p} - \frac{\alpha}{n} < \frac{1}{q} \leqq \frac{1}{p}; \tag{3.1}$$

$$\alpha p - n < \mu_k < n(p-1) \; (k = 0, 1, \cdots, m), \quad \mu \equiv \mu_0 - \sum_{k=1}^m \mu_k; \tag{3.2}$$

$$\frac{\nu_k + n}{q} = \frac{\mu_k + n}{p} - \alpha \; (k = 0, 1, \cdots, m), \quad \nu \equiv \nu_0 - \sum_{k=1}^m \nu_k \tag{3.3}$$

and

$$\rho(x) = (1+|x|)^{\mu} \prod_{k=1}^{m} |x - x_k|^{\mu_k}, \quad r(x) = (1+|x|)^{\nu} \prod_{k=1}^{m} |x - x_k|^{\nu_k} \tag{3.4}$$

with $0 \leqq |x_1| < |x_2| < \cdots < |x_m| < \infty$. *Then the Riesz potential* (1.1) *is bounded from* $L_p(R^n;\rho)$ *into* $L_q(R^n;r)$:

$$\left(\int_{R^n} (1+|x|)^{\nu} \prod_{k=1}^{m} |x-x_k|^{\nu_k} |(I^{\alpha}\varphi)(x)|^q dx\right)^{1/q}$$
$$\leqq k_3 \left(\int_{R^n} (1+|x|)^{\mu} \prod_{k=1}^{m} |x-x_k|^{\mu_k} |\varphi(x)|^p dx\right)^{1/p}, \tag{3.5}$$

where the constant $k_3 > 0$ *does not depend on* φ.

Proof. Let $d_0 = 0$, d_1, $d_2, \cdots, d_m$ be positive numbers such that

$$d_0 \leqq |x_1| < d_1 < |x_2| < d_2 < \cdots < |x_{m-1}| < d_{m-1} < |x_m| < d_m < \infty, \ d_m > 2|x_m|. \tag{3.6}$$

These numbers split R^n into $m+1$ sets

$$B_k = \{t \in R^n : \ d_{k-1} \leqq |t| < d_k\}, \ k = 1, 2, \cdots, m; \ B_{m+1} = \{t \in R^n : \ |t| \geqq d_m\}. \tag{3.7}$$

Each of such sets contains the simplest power weight concentrated in one point of the weights in (3.4). That is, the spherical layers B_1, $B_2, \cdots, B_m$ contain the weights $|x - x_1|^{\mu_1}$, $|x-x_2|^{\mu_2}, \cdots, |x-x_m|^{\mu_m}$ concentrated at x_1, $x_2, \cdots, x_m$, respectively, and B_{m+1} contains the weight $|x|^{\mu_0}$ concentrated at infinity.

We represent the integral in the left hand side of (3.5) as the sum of the integrals over the sets (3.7):

$$I = \left[\int_{R^n} (1+|x|)^{\nu} \prod_{k=1}^{m} |x-x_k|^{\nu_k} |(I^{\alpha}\varphi)(x)|^q dx\right]^{1/q} = \sum_{i=1}^{m+1} \sum_{j=1}^{m+1} I_{ij}, \tag{3.8}$$

where

$$I_{ij} = \left[\int_{B_i} (1+|x|)^{\nu} \prod_{k=1}^{m} |x-x_k|^{\nu_k} \left|\int_{B_j} \frac{\varphi(t)dt}{(x-t)^{n-\alpha}}\right|^q dx\right]^{1/q} \quad (i,j = 1, 2, \cdots, m+1). \tag{3.9}$$

It is enough to prove the relation (3.5) for any I_{ij}:

$$|I_{ij}| \leqq c_1 \left(\int_{R^n} (1+|x|)^{\mu} \prod_{k=1}^{m} |x-x_k|^{\mu_k} |\varphi(x)|^p dx\right)^{1/p} \tag{3.10}$$
$$(c_1 = c_1(i,j) \ i,j = 1, 2, \cdots, m+1).$$

First we treat the case $i = j = 1, 2, \cdots, m$.. For fixed $i = 1, 2, \cdots, m$, let $x \in B_i$, then

$$|x_k| - d_i \leqq |x - x_k| \leqq |x_k| + d_i \quad (1 \leqq i < k \leqq m),$$
$$d_{i-1} - |x_k| \leqq |x - x_k| \leqq |x_k| + d_i \quad (m \geqq i > k \geqq 1), \tag{3.11}$$

and we have

$$(1+|x|)^{\nu}\prod_{k=1}^{m}|x-x_k|^{\nu_k}\leqq c_2|x-x_i|^{\nu_i}\quad (x\in B_i),\quad c_2=c_2(i)\quad (i=1,2,\cdots,m), \tag{3.12}$$

which leads us to the case of the simple power weight $|x-x_i|^{\nu_i}$ at $x_i\in B_i$. Substituting this estimate into (3.9), using Theorem 2 (with $\Omega=B_i$ and $\rho(x)=|x-x_i|^{\mu_i}$) and taking (3.11) into account we arrive at the estimate (3.10):

$$\begin{aligned}|I_{ii}| &\leqq c_2^{1/q}\left[\int_{B_i}|x-x_i|^{\nu_i}\left(\int_{B_i}\frac{|\varphi(t)|dt}{|x-t|^{n-\alpha}}\right)^q dx\right]^{1/q}\\ &\leqq c_2^{1/q}k_1\left(\int_{B_i}|x-x_i|^{\mu_i}|\varphi(x)|^p dx\right)^{1/p}\\ &\leqq c_3\left(\int_{B_i}(1+|x|)^{\mu}\prod_{k=1}^{m}|x-x_k|^{\mu_k}|\varphi(x)|^p dx\right)^{1/p}\\ &\leqq c_3\left(\int_{R^n}(1+|x|)^{\mu}\prod_{k=1}^{m}|x-x_k|^{\mu_k}|\varphi(x)|^p dx\right)^{1/p}\quad (c_3=c_3(i),\ i=1,2,\cdots,m).\end{aligned} \tag{3.13}$$

Let now $i=j=m+1$. Then

$$|I_{m+1,m+1}|\leqq\left[\int_{B_{m+1}}(1+|x|)^{\nu}\prod_{k=1}^{m}|x-x_k|^{\nu_k}\left(\int_{B_{m+1}}\frac{|\varphi(t)|dt}{|x-t|^{n-\alpha}}\right)^q dx\right]^{1/q}. \tag{3.14}$$

For $x\in B_{m+1}, |x|\geqq d_m$ and therefore

$$|x|\leqq 1+|x|\leqq\left(\frac{1+d_m}{d_m}\right)|x|\quad (x\in B_{m+1}),$$

$$\frac{|x|}{2}\leqq|x-x_k|\leqq 2|x|\quad (k=1,2,\cdots,m;x\in B_{m+1}). \tag{3.15}$$

Hence the analogue of (3.12) is valid:

$$(1+|x|)^{\nu}\prod_{k=1}^{m}|x-x_k|^{\nu_k}\leqq c_4|x|^{\nu+\sum_{k=1}^m\nu_k}=c_4|x|^{\nu_0}\quad (x\in B_{m+1},\ c_4=c_4(m+1)>0), \tag{3.16}$$

which leads us to the case of the simple weight $|x|^{\nu_0}$ at infinity. Substituting (3.16) into (3.14), using Theorem 2 with $\Omega=B_{m+1}$ and $\rho(x)=|x|^{\mu_0}$ and taking (3.15) into account we obtain similarly to (3.13) the estimate:

$$|I_{m+1,m+1}|\leqq c_5\left(\int_{R^n}(1+|x|)^{\mu}\prod_{k=1}^{m}|x-x_k|^{\mu_k}|\varphi(x)|^p dx\right)^{1/p}\quad (c_5=c_5(m+1)). \tag{3.17}$$

Further, we consider the case $i,j=1,2,\cdots,m,\ i\neq j$. Let $i<j$. In view of (3.12) we have

$$|I_{ij}|\leqq c_6\left[\int_{B_i}|x-x_i|^{\nu_i}\left(\int_{B_j}\frac{|\varphi(t)|dt}{|x-t|^{n-\alpha}}\right)^q dx\right]^{1/q}\quad (c_6=c_6(i,j),\ 1\leqq i<j\leqq m). \tag{3.18}$$

Here $d_{i-1} \leqq |x| \leqq d_i \leqq d_{j-1} \leqq |t| \leqq d_j$ and we find

$$|x-t| \geqq |t| - d_i,\ |x-t| \geqq d_{j-1} - |x| \geqq d_i - |x| \quad (x \in B_i, t \in B_j). \tag{3.19}$$

If $1 < p = q$, using (3.19) and Hölder's inequality, we have

$$\begin{aligned}
|I_{ij}| &\leqq c_6 \left[\int_{B_i} |x-x_i|^{\nu_i} \left(\int_{B_j} \frac{|\varphi(t)|}{|x-t|^{(n-\alpha)/p}|x-t|^{(n-\alpha)/p'}}\, dt\right)^p dx\right]^{1/p} \\
&\leqq c_6 \left(\int_{B_i} \frac{|x-x_i|^{\nu_i}}{(d_{j-1}-|x|)^{n-\alpha}} dx\right)^{1/p} \int_{B_j} (|t-x_j|^{\mu_j/p}|\varphi(t)|) \frac{|t-x_j|^{-\mu_j/p}}{(|t|-d_i)^{(n-\alpha)/p'}} dt \\
&\leqq c_7 \left(\int_{B_j} |t-x_j|^{\mu_j}|\varphi(t)|^p dt\right)^{1/p} \left(\int_{B_j} |t-x_j|^{-\mu_j p'/p}(|t|-d_i)^{\alpha-n} dt\right)^{1/p'} \\
&= c_8 \left(\int_{B_j} |t-x_j|^{\mu_j}|\varphi(t)|^p dt\right)^{1/p} \quad (c_8 = c_8(i,j),\ 1 \leqq i < j \leqq m).
\end{aligned} \tag{3.20}$$

We note that the integrals in (3.20) are convergent for $\nu_i > -n, \alpha > 0, \mu_j p'/p < n$ which are equivalent to $\mu_i > \alpha p - n,\ \alpha > 0,\ \mu_i < n(p-1)$ by (3.3) and the latter is valid from the assumption (3.2).

If $p < q$ then by the assumption (3.1) we can choose ε such that $0 < \varepsilon < \alpha - n/p + n/q$. We set

$$\beta = \frac{n}{q} - \varepsilon,\quad \gamma = n - \alpha - \frac{n}{q} + \varepsilon,\quad \beta + \gamma = n - \alpha. \tag{3.21}$$

Using (3.19) and Hölder's inequality, we obtain

$$\begin{aligned}
|I_{ij}| &\leqq c_6 \left(\int_{B_i} \frac{|x-x_i|^{\nu_i} dx}{(d_{j-1}-|x|)^{q\beta}}\right)^{1/q} \int_{B_j} (|t-x_j|^{\mu_j/p}|\varphi(t)|)|t-x_j|^{-\mu_j/p}(|t|-d_i)^{-\gamma} dt \\
&\leqq c_9 \left(\int_{B_j} |t-x_j|^{\mu_j}|\varphi(t)|^p dt\right)^{1/p} \left(\int_{B_j} |t-x_j|^{-\mu_j p'/p}(|t|-d_i)^{-\gamma p'} dt\right)^{1/p'} \\
&= c_{10} \left(\int_{B_j} |t-x_j|^{\mu_j}|\varphi(t)|^p dt\right)^{1/p} \quad (c_{10} = c_{10}(i,j),\ 1 \leqq i < j \leqq m).
\end{aligned} \tag{3.22}$$

The integrals in (3.22) are convergent due to the conditions (3.1) - (3.3) and the choice of ε, β and γ. From (3.20) and (3.22) we arrive at the estimate (3.10):

$$|I_{ij}| \leqq c_{11} \left(\int_{R^n} (1+|x|)^{\mu} \prod_{k=1}^{m} |x-x_k|^{\mu_k}|\varphi(x)|^p dx\right)^{1/p} \tag{3.23}$$

$$(c_{11} = c_{11}(i,j),\ (1 \leqq i < j \leqq m)),$$

by (3.11).

In the case $1 \leqq j < i \leqq m$ the relation

$$|I_{ij}| \leqq c_{12} \left(\int_{R^n} (1+|x|)^{\mu} \prod_{k=1}^{m} |x-x_k|^{\mu_k}|\varphi(x)|^p dx\right)^{1/p} \tag{3.24}$$

$$(c_{12} = c_{12}(i,j),\ 1 \leqq i < j \leqq m)),$$

is proved similarly to (3.23).

Let finally $1 \leqq i \leqq m$ and $j = m+1$. On the basis of (3.11) and similarly to (3.18) we obtain

$$|I_{i,m+1}| \leqq c_{13}\left[\int_{B_i}|x-x_i|^{\nu_i}\left(\int_{B_{m+1}}\frac{|\varphi(t)|dt}{|x-t|^{n-\alpha}}\right)^q dx\right]^{1/q} \quad (c_{13}=c_{13}(i,m+1)). \tag{3.25}$$

We choose $d_{m+1} \in R$ such that $d_m < d_{m+1} < \infty$ and set

$$B_{m+1,1}=\{t\in R^n:\ d_m \leqq |t| \leqq d_{m+1}\},\ B_{m+1,2}=\{t\in R^n:\ |t|>d_{m+1}\}. \tag{3.26}$$

Then from (3.25) we have

$$|I_{i,m+1}| \leqq c_{14}\left\{\left[\int_{B_i}|x-x_i|^{\nu_i}\left(\int_{B_{m+1,1}}\frac{|\varphi(t)|dt}{|x-t|^{n-\alpha}}\right)^q dx\right]^{1/q}\right.$$
$$\left.+\left[\int_{B_i}|x-x_i|^{\nu_i}\left(\int_{B_{m+1,2}}\frac{|\varphi(t)|dt}{|x-t|^{n-\alpha}}\right)^q dx\right]^{1/q}\right\}, \quad (c_{14}=c_{14}(i,m+1)).\tag{3.27}$$

Making similar arguments to the above, we obtain that, if $1<p=q$,

$$|I_{i,m+1}| \leqq c_{15}\left[\left(\int_{B_i}\frac{|x-x_i|^{\nu_i}dx}{(d_m-|x|)^{n-\alpha}}\right)^{1/p}\int_{B_{m+1,1}}|\varphi(t)|(|t|-d_i)^{(\alpha-n)/p'}dt\right.$$
$$\left.+\left(\int_{B_i}|x-x_i|^{\nu_i}dx\right)^{1/p}\int_{B_{m+1,2}}|\varphi(t)|(|t|-d_i)^{\alpha-n}dt\right]$$
$$\leqq c_{15}\left[\left(\int_{B_i}\frac{|x-x_i|^{\nu_i}dx}{(d_m-|x|)^{n-\alpha}}\right)^{1/p}\left(\int_{B_{m+1,1}}|t|^{-\mu_0p'/p}(|t|-d_i)^{\alpha-n}dt\right)^{1/p'}\right.$$
$$\times\left(\int_{B_{m+1,1}}|t|^{\mu_0}|\varphi(t)|^p dt\right)^{1/p}$$
$$+\left(\int_{B_i}|x-x_i|^{\nu_i}dx\right)^{1/q}\left(\int_{B_{m+1,2}}|t|^{-\mu_0p'/p}(|t|-d_i)^{(\alpha-n)p'}dt\right)^{1/p'}$$
$$\left.\times\left(\int_{B_{m+1,2}}|t|^{\mu_0}|\varphi(t)|^p dt\right)^{1/p}\right]$$
$$\leqq c_{16}\left(\int_{B_{m+1,1}}|t|^{\mu_0}|\varphi(t)|^p dt\right)^{1/p}+c_{17}\left(\int_{B_{m+1,2}}|t|^{\mu_0}|\varphi(t)|^p dt\right)^{1/p}$$
$$\leqq c_{17}\left(\int_{B_{m+1}}|t|^{\mu_0}|\varphi(t)|^p dt\right)^{1/p} \quad (c_{17}=c_{17}(i,m+1)). \tag{3.28}$$

the integrals in (3.28) being convergent since $\nu_i+n>0$, $\mu_0p'/p<n$ and $(n-\alpha+\mu_0/p)p'>n$ according to the conditions (3.1) - (3.3) of the theorem.

If $p < q$, then by the assumption (3.1) we choose and $0 < \varepsilon < \alpha - n/p + n/q$ and set β and γ as in (3.21) then

$$|I_{i,m+1}| \leqq c_{15} \left[\left(\int_{B_i} \frac{|x - x_i|^{\nu_i} dx}{(d_m - |x|)^{q\beta}} \right)^{1/q} \left(\int_{B_{m+1,1}} |t|^{-\mu_0 p'/p} (|t| - d_i)^{-\gamma p'} dt \right)^{1/p'} \right.$$

$$\times \left(\int_{B_{m+1,1}} |t|^{\mu_0} |\varphi(t)|^p dt \right)^{1/p}$$

$$+ \left(\int_{B_i} |x - x_i|^{\nu_i} dx \right)^{1/q} \left(\int_{B_{m+1,2}} |t|^{-\mu_0 p'/p} (|t| - d_i)^{(\alpha - n)p'} dt \right)^{1/p'}$$

$$\left. \times \left(\int_{B_{m+1,2}} |t|^{\mu_0} |\varphi(t)|^p dt \right)^{1/p} \right]$$

$$\leqq c_{18} \left(\int_{B_{m+1}} |t|^{\mu_0} |\varphi(t)|^p dt \right)^{1/p} \quad (c_{18} = c_{18}(i, m+1)). \tag{3.29}$$

From (3.28) and (3.29) we arrive at the estimate (3.10)

$$|I_{i,m+1}| \leqq c_{19} \left(\int_{R^n} (1 + |x|)^{\mu} \prod_{k=1}^{m} |x - x_k|^{\mu_k} |\varphi(x)|^p dx \right)^{1/p} \tag{3.30}$$

$$(c_{19} = c_{19}(i, m+1), \ 1 \leqq i \leqq m),$$

by (3.15).

In the case $i = m + 1$ and $1 \leqq j \leqq m$, the relation

$$|I_{m+1,j}| \leqq c_{20} \left(\int_{R^n} (1 + |x|)^{\mu} \prod_{k=1}^{m} |x - x_k|^{\mu_k} |\varphi(x)|^p dx \right)^{1/p} \tag{3.31}$$

$$(c_{20} = c_{20}(m+1, j), \ 1 \leqq j \leqq m),$$

is proved similarly to (3.30).

Applying Minkowski's inequality to (3.8) and using the relations (3.13), (3.17), (3.23), (3.24), (3.30) and (3.31), we arrive at the required estimate (3.5).

Using Theorem 2(ii), the following assertion is proved similarly to Theorem 3.

Theorem 4. *Let the conditions* (3.1) - (3.3) *be satisfied and*

$$\rho(x) = (1 + |x|)^{\mu} \prod_{k=1}^{m} \Big| |x| - R_k \Big|^{\mu_k}, \quad r(x) = (1 + |x|)^{\nu} \prod_{k=1}^{m} \Big| |x| - R_k \Big|^{\nu_k}, \tag{3.32}$$

for $0 \leqq R_1 < R_2 < \cdots < R_m < \infty$. *Then the Riesz potential* (1.1) *is bounded from* $L_p(R^n; \rho)$ *into* $L_q(R^n; r)$:

$$\left(\int_{R^n} (1 + |x|)^{\nu} \prod_{k=1}^{m} \Big| |x| - R_k \Big|^{\nu_k} |(I^{\alpha} \varphi)(x)|^q dx \right)^{1/q}$$

$$\leqq k_4 \left(\int_{R^n} (1+|x|)^{\mu} \prod_{k=1}^{m} \Big||x| - R_k\Big|^{\mu_k} |\varphi(x)|^p dx \right)^{1/p} \tag{3.33}$$

with the constant $k_4 > 0$ being independent of φ.

Remark 2. Unlike Theorems 1 and 2, the limiting case $1/p - \alpha/n = 1/q$ excludes in Theorems 3 and 4 due to the impossibility to apply the method above of the estimations of I_{ij} for $j = i+1, 1 \leqq i \leqq m$ and $i = j+1, 1 \leqq j \leqq m$.

4. Potential Type Operators with Power and Power Logarithmic Kernels

Let $I_\Omega^\alpha \varphi$ be the Riesz potential given in (1.1). The following statements (Theorems 5 and 6) are the direct corollaries of Theorem 3.

Theorem 5. *Let Ω be a measurable bounded set in R^n and assume* (3.1),

$$\alpha p - n < \mu_k < n(p-1) \quad (k = 1, 2, \cdots, m); \tag{4.1}$$

$$\frac{\nu_k + n}{q} = \frac{\mu_k + n}{p} - \alpha \quad (k = 1, 2, \cdots, m) \tag{4.2}$$

and

$$\rho(x) = \prod_{k=1}^{m} |x - x_k|^{\mu_k}, \quad r(x) = \prod_{k=1}^{m} |x - x_k|^{\nu_k} \tag{4.3}$$

with $x_1, x_2, \cdots, x_m \in \Omega, |x_1| < |x_2| < \cdots < |x_m|$. Then the Riesz potential I_Ω^α is a bounded operator from $L_p(\Omega; \rho)$ into $L_q(\Omega; r)$:

$$\left(\int_\Omega \prod_{k=1}^{m} |x - x_k|^{\nu_k} \, |(I_\Omega^\alpha \varphi)(x)|^q \, dx \right)^{1/q} \leqq k_3 \left(\int_\Omega \prod_{k=1}^{m} |x - x_k|^{\mu_k} |\varphi(x)|^p dx \right)^{1/p}, \tag{4.4}$$

with the constant $k_3 > 0$ being independent of φ.

Theorem 6. *Let Ω be a measurable unbounded set in R^n and assume* (3.1) - (3.3). *Then the Riesz potential I_Ω^α is bounded from $L_p(\Omega; \rho)$ into $L_q(\Omega; r)$:*

$$\left(\int_\Omega (1+|x|)^{\nu} \prod_{k=1}^{m} |x - x_k|^{\nu_k} \, |(I_\Omega^\alpha \varphi)(x)|^q \, dx \right)^{1/q}$$

$$\leqq k_3 \left(\int_\Omega (1+|x|)^{\mu} \prod_{k=1}^{m} |x - x_k|^{\mu_k} |\varphi(x)|^p dx \right)^{1/p}, \tag{4.5}$$

with the constant $k_3 > 0$ being independent of φ, where constants x_k, ν_k, μ_k $(k = 1, 2, \cdots, m), \nu, \mu$ are taken as in Theorem 5 and weight functions ρ and r as in (3.4).

We note that the constant k_3 in Theorems 4 and 5 is the same as in Theorem 2.

Let now Ω be a measurable bounded set in R^n. We discuss the potential type operator with power-logarithmic kernel given in (1.5):

$$(I_\Omega^{\alpha,\beta}\varphi)(x) = \int_\Omega \log^\beta \left(\frac{\gamma}{|x-t|}\right) \frac{\varphi(t)dt}{|x-t|^{n-\alpha}} \quad (x \in \Omega) \tag{4.6}$$

for $0 < \alpha < n, \beta \geqq 0, \gamma > \mathrm{mes}\,(\Omega)$. Since $r^{\alpha-n}\log^\beta(\gamma/r) \leqq cr^{\alpha-n-\varepsilon}$ for sufficiently small $c > 0$ and $\varepsilon > 0$, the estimate

$$\left|(I_\Omega^{\alpha,\beta}\varphi)(x)\right| \leqq c\left|(I_\Omega^{\alpha-\varepsilon}\varphi)(x)\right| \tag{4.7}$$

holds. Then Theorem 5 lead us to the following assertion giving mapping properties of the operator $I_\Omega^{\alpha,\beta}$.

Theorem 7. *Let Ω be a measurable bounded set in R^n and assume*

$$0<\alpha<n, \quad \beta \geqq 0, \quad 1<p<\infty, \quad 1<q<\infty, \quad \frac{1}{p}-\frac{\alpha}{n}<\frac{1}{q}\leqq\frac{1}{p}; \tag{4.8}$$

$$\alpha p - n < \mu_k < n(p-1), \quad \delta_k > \nu_k = \frac{(\mu_k + n - \alpha p)q}{p} - 1 \quad (k = 1, \cdots, m) \tag{4.9}$$

and

$$\rho(x) = \prod_{k=1}^m |x-x_k|^{\mu_k}, \quad r(x) = \prod_{k=1}^m |x-x_k|^{\delta_k} \tag{4.10}$$

with $x_1, x_2, \cdots, x_m \in \Omega, |x_1| < |x_2| < \cdots < |x_m|$. Then the operator $I_\Omega^{\alpha,\beta}$ is bounded from $L_p(\Omega;\rho)$ into $L_q(\Omega;r)$:

$$\left(\int_\Omega \prod_{k=1}^m |x-x_k|^{\delta_k} \left|(I_\Omega^{\alpha,\beta}\varphi)(x)\right|^q dx\right)^{1/q} \leqq k_5 \left(\int_\Omega \prod_{k=1}^m |x-x_k|^{\mu_k} |\varphi(x)|^p dx\right)^{1/p}, \tag{4.11}$$

with the constant $k_5 > 0$ being independent of φ.

Proof. We choose ε such that

$$0 < \varepsilon < \min\left(\alpha, \frac{\delta_1-\nu_1}{q}, \cdots, \frac{\delta_m-\nu_m}{q}\right). \tag{4.12}$$

Using the boundedness of Ω, the relation (4.7) and Theorem 4 with α and ν_k being replaced by $\alpha-\varepsilon$ and $\delta_k = \nu_k + \varepsilon q$, respectively, we have

$$\begin{aligned}
&\left(\int_\Omega \prod_{k=1}^m |x-x_k|^{\delta_k} \left|(I_\Omega^{\alpha,\beta}\varphi)(x)\right|^q dx\right)^{1/q} \\
&\leqq c_1 \left(\int_\Omega \prod_{k=1}^m |x-x_k|^{\nu_k+\varepsilon q+(\delta_k-\nu_k-\varepsilon q)} \left|(I_\Omega^{\alpha-\varepsilon}\varphi)(x)\right|^q dx\right)^{1/q} \\
&\leqq c_2 \left(\int_\Omega \prod_{k=1}^m |x-x_k|^{\nu_k+\varepsilon q} \left|(I_\Omega^{\alpha-\varepsilon}\varphi)(x)\right|^q dx\right)^{1/q} \\
&\leqq k_5 \left(\int_\Omega \prod_{k=1}^m |x-x_k|^{\mu_k} |\varphi(x)|^p dx\right)^{1/p} \quad (k_5 > 0).
\end{aligned} \tag{4.13}$$

The theorem is proved.

Corollary. *Let $-\infty < a < b < \infty$ and let*

$$0 < \alpha < 1, \quad \beta \geqq 0, \quad 1 < p < \infty, \quad 1 < q < \infty, \quad \frac{1}{p} - \alpha < \frac{1}{q} \leqq \frac{1}{p}; \tag{4.14}$$

$$\alpha p - 1 < \mu_k < p - 1, \quad \delta_k > \nu_k = \frac{(\mu_k + 1 - \alpha p)q}{p} - 1 \quad (k = 1, \cdots, m) \tag{4.15}$$

and

$$\rho(x) = \prod_{k=1}^{m} |x - x_k|^{\mu_k}, \quad r(x) = \prod_{k=1}^{m} |x - x_k|^{\delta_k} \tag{4.16}$$

with $a \leqq |x_1| < |x_2| < \cdots < |x_m| \leqq b$. Then the operator

$$(I^{\alpha,\beta}\varphi)(x) = \int_a^b \log^\beta \left(\frac{\gamma}{|x - t|} \right) \frac{\varphi(t)dt}{|x - t|^{1-\alpha}} \quad (a < x < b) \tag{4.17}$$

with $\gamma > b - a$ is bounded from $L_p([a, b]; \rho)$ into $L_q([a, b]; r)$:

$$\left(\int_a^b \prod_{k=1}^{m} |x - x_k|^{\delta_k} \left| (I^{\alpha,\beta}\varphi)(x) \right|^q dx \right)^{1/q} \leqq k_6 \left(\int_a^b \prod_{k=1}^{m} |x - x_k|^{\mu_k} |\varphi(x)|^p dx \right)^{1/p}, \tag{4.18}$$

the constant $k_6 > 0$ being independent of φ.

Remark 3. Using Theorem 4, similar statements to Theorems 5 - 7 may be proved for the power weight $\rho(x)$ of the form

$$\rho(x) = \begin{cases} \prod_{k=1}^{m} \Big| |x| - R_k \Big|^{\mu_k}, & \text{if } \operatorname{mes}(\Omega) < \infty, \\ (1 + |x|)^\mu \prod_{k=1}^{m} \Big| |x| - R_k \Big|^{\mu_k}, & \text{if } \operatorname{mes}(\Omega) = \infty, \end{cases} \tag{4.19}$$

where $0 \leqq R_1 < R_2 < \cdots < R_m < \infty$, $\mu_k \in R$, $k = 1, 2, \cdots, m$, and at least one point of the spheres $S_{R_k} = \{t \in R^n : |t| = R_k\}$, $k = 1, 2, \cdots, m$, belongs to Ω. This means that the weights $\Big||x| - R_1\Big|^{\mu_1}, \cdots, \Big||x| - R_m\Big|^{\mu_m}$ are concentrated on the spheres $S_{R_1}, \cdots, S_{R_m}$ and the weight $|x|^{\mu_0}$ at infinity (the latter when Ω is unbounded).

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