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Signorini 型境界条件の超伝導現象への応用

— ON PENETRATION PHENOMENON OF MAGNETIC FIELD INTO SUPERCONDUCTING MATERIALS —

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0. Introduction

We shall deal with a behavior of penetration of magnetic field into a superconducting material in the presence of an applied magnetic field. As well known, a superconductor has the Meissner effect, i.e., if the magnetic field applying to a superconducting material is less than the critical field at a sufficiently low temperature, the magnetic field does not penetrate into the material.

This phenomenon discovered in 1933 by W.Meissner and R.Ochsenfeld, has been interpreted as follows : a superconductor has a property of perfect diamagnetism, so that surface current occurs on it's surface and the magnetic feild is excluded from it.

On the other hand, the superconductor can not preserve the Meissner effect under a magnetic field which is much greater than the critical field , so that a penetration of magnetic field into the superconductor arises and the superconductivity is lost partially or completely .

Our interest is to investigate the behavior of the penetration of magnetic field into the superconductor under the situation that a given magnetic field applying to the material is just around the critical field.

In order to understand this penetration phenomenon, we shall present a mathematical model using boundary conditions of Signorini type in section 1 . In section 2, a theorem of existence and uniqueness for the model equation will be shown. Our mathematical technique is to reduce the equation of the above model to certain variational inequality. After that, we shall use the standard method to prove the existence and uniqueness theorem (for example, Duvaut-Lions [3] and Glowinski-Lions-Trémolières [5]).

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1. Mathematical model for penetration of magnetic field into superconductor

Let Ω be a bounded, connected domain in R^3. Denote by Ω_0 a domain occupied with a superconducting material which is put in Ω and set $\Omega_1 = \Omega \backslash \overline{\Omega_0}$. According to the theory of superconductivity (Tinkham [6] , Du-Gunzburger-Peterson [2]), the following equations hold in Ω_0 :

(1.1) $$\text{rot } j_0 = -\frac{c}{4\pi\lambda^2} B_0 \quad \text{in } \Omega_0 \qquad \text{(London equation)}$$

and

(1.2) $$\text{rot } B_0 = \frac{4\pi}{c} j_0 \ , \ \text{div } B_0 = 0 \quad \text{in } \Omega_0 \qquad \text{(Maxwell equation)}$$

where j_0 is a current density vector in Ω_0, B_0 is a magnetic induction in Ω_0 and c is the velocity of light , and a parameter λ represents the penetration depth.
Hence using (1.1) and (1.2) , we have

(1.3) $$\text{rot rot } B_0 + \frac{1}{\lambda^2} B_0 = 0 \quad \text{in } \Omega_0 \ ,$$

(1.4) $$\text{div } B_0 = 0 \quad \text{in } \Omega_0 \ .$$

In Ω_1, the magnetic induction B_1 is generated by a stational current j_1 imposed on Γ_1 $(= \partial\Omega)$. Let us assume that B_1 satisfies a modified Maxwell's equation having a dumping effect in Ω_1 :

(1.5) $$\text{rot rot } B_1 + \epsilon B_1 = 0 \ , \qquad \text{div } B_1 = 0 \quad \text{in } \Omega_1$$

and on the boundary Γ_1 , B_1 satisfies

(1.6) $$\text{rot } B_1 \times n = \frac{4\pi}{c} j_1, \qquad B_1 \cdot n = 0 \quad \text{on } \Gamma_1.$$

Here n is the unit normal vector (directed toward the exterior of Ω_1) and $\epsilon(> 0)$ is small.
The first condition of (1.6) represents the direction of the given current j_1 and the second one also describes that the magnetic field does not exclude from Ω_1 .

Remark 1.1 From a physical point of view , the term ϵB_1 in (1.5) describes a dumping effect . In the case where $\epsilon = 0$ in (1.5) , (1.5) together with (1.6) describes Ampère's law ;

(1.7) $$\text{rot } B_1 = \frac{4\pi}{c} j_1 \text{ on } \Gamma_1, \ \ \text{rot } B_1 = 0 \text{ in } \Omega_1 \ \text{ and } \ \text{div } B_1 = 0 \text{ in } \Omega_1.$$

Finally as the transmisson condition prescribed on the surface Γ_0 of the superconductor, we shall impose the boundary conditions of Signorini type which could be regarded as a representation of the Meissner effect :

$$B_0 = B_1 \ (\equiv B) \quad \text{on } \Gamma_0 \ , \tag{1.8}$$

$$|\ rot\ B_1 \times n - rot\ B_0 \times n\ | \leq g \quad \text{on } \Gamma_0 \ , \tag{1.9}$$

$$|\ rot\ B_1 \times n - rot\ B_0 \times n\ | \ < \ g \implies B_T = 0, \tag{1.10}$$

$$|\ rot\ B_1 \times n - rot\ B_0 \times n\ | \ = \ g \implies B_T = 0 \quad \text{or} \tag{1.11}$$

$$\text{there exists } \mu \ > \ 0 \text{ such that } B_T = \ -\frac{1}{\mu}\left(rot\ B_1 \times n - rot\ B_0 \times n \right) ,$$

where g represents an intensity of critical current which is a inherent constant to this material, $B_T = n \times (B \times n)$ and $|x|$ denotes Euclidean norm in R^3 , i.e., $|x| = \sqrt{x_1^2 + x_2^2 + x_3^2}$ for $x = (x_1, x_2, x_3)$.

Remark 1.2 Boundary conditions (1.9) - (1.11) can be rewritten by the following two equivalent forms (1.12) and (1.13) ;

$$\begin{cases} |\ rot\ B_1 \times n - rot\ B_0 \times n\ | \leq g \quad \text{on } \Gamma_0 \ , \\ g\ |B_T| + (rot\ B_1 \times n - rot\ B_0 \times n) \cdot B_T = 0 \quad \text{on } \Gamma_0 \end{cases} \tag{1.12}$$

and

$$-\Big(rot\ B_1 \times n - rot\ B_0 \times n\Big) \in g\ \partial\ (\ |B_T|\ ). \tag{1.13}$$

Here $\partial(|x|)$ in (1.13) means the multi-valued subdifferential at x $(\in R^3)$, i.e.,

$$\partial(|x|) = \begin{cases} x\ /\ |x| & \text{if } x \neq 0, \\ \{\ y \in R^3;\ |y| \leq 1\ \} & \text{if } x = 0\ . \end{cases} \tag{1.14}$$

We note particularly that these conditions (1.8) - (1.11) are one of the characteristics of our mathematical modelling (cf. Fujita-Kawarada [4], treating an example appearing in flow problems).

Problem I Given a positive constant g and a current $j_1 \in \left(H^{-1/2}(\Gamma_1)\right)^3$, find $\{B_0, B_1\} \in \left(H^1(\Omega_0)\right)^3 \times \left(H^1(\Omega_1)\right)^3$ satisfying (1.3)-(1.4), (1.5)-(1.6) and (1.8)-(1.11).

Remark 1.3 The physical correspondences of the conditions (1.8) - (1.11) are as follows. Set

$$\Gamma_{00} = \{x \in \Gamma_0 \; ; \; |\, (\, rot\ B_1 \times n - rot\ B_0 \times n\ )(x)\, | < g\ \},$$

$$\Gamma_{01} = \{x \in \Gamma_0 \; ; \; |\, (\, rot\ B_1 \times n - rot\ B_0 \times n\ )(x)\, | = g\ \}.$$

(i) $\Gamma_{00} = \Gamma_0$ implies that the magnetic field does not penetrate into Ω_0 and the superconductivity preserves. i.e., $B_0 = 0$ in Ω_0 (Meissner effect).

In fact, from (1.10), we have

$$B_T = n \times (B_0 \times n) = 0 \quad \text{on } \Gamma_0. \tag{1.15}$$

Note that $(rot\ B \times n) \cdot n = 0$. Using (1.15) together with (1.3) and Green's formula, we then deduce that

$$0 = (rot\ B_0, rot\ B_0) + \int_{\Gamma_0} (rot\ B_0 \times n) \cdot B_0\ d\sigma + \frac{1}{\lambda^2}\ (B_0, B_0)$$

$$= (rot\ B_0, rot\ B_0) + \int_{\Gamma_0} (rot\ B_0 \times n) \cdot \Big(n \times (B_0 \times n)\Big)\ d\sigma + \frac{1}{\lambda^2}\ (B_0, B_0)$$

$$= (rot\ B,\ rot\ B_0) + \frac{1}{\lambda^2}\ (B_0, B_0),$$

from which $B_0 = 0$ in Ω_0.

(ii) Let x be in Γ_{01}. Then the penetration of magnetic field into superconduction arises at x because of Lorentz's force.

In fact from (1.11) , it follows that

$$rot\ B_1 \times n - rot\ B_0 \times n = -\mu\ B_T \tag{1.16}$$

$$= -\mu\ \Big(n \times (B \times n)\Big) = -\mu\ \Big(B - (B \cdot n)n\Big)$$

Hence

$$\Big(\ rot\ B_1 \times n - rot\ B_0 \times n\ \Big) \cdot B_T = -\mu\ |B_T|^2\ \leq 0 \tag{1.17}$$

and

$$B \cdot \Big((rot\ B_1 - rot\ B_0) \times n\Big) = -\mu\ \Big(|B|^2 - (B \cdot n)^2\Big) = -\mu\ |B_T|^2,$$

from which

$$n \cdot \Big((rot\ B_1 - rot\ B_0) \times B \Big) = \mu\ |B_T|^2. \tag{1.18}$$

Therefore we could regard that the left hand side of (1.18) stands for the normal component of Lorenz' force operated to the superconducting material . And we could also deduce that the only tangential component of B on Γ_0 contributes to the penetration of magnetic field into the superconductor from (1.16) - (1.18).

2. Existence and uniqueness theorem for penetration of magnetic field

In this section, we shall prove a theorem of existence and uniqeness for our mathematical model . Here we shall use a variational inequality formulation which is equivalent to the original problem I.

Now let us introduce some notations .

For any vector valued function u defined in Ω , set $u_0 = u|_{\Omega_0}$ and $u_1 = u|_{\Omega_1}$. We recall that $\Omega_0 \subset \Omega$, $\Omega_1 = \Omega \backslash \overline{\Omega_0}$ and $\Gamma_0 = \partial\Omega_0$, $\Gamma_1 = \partial\Omega$. We then define a Hilbert space V :

$$V \equiv \{u \mid u \in \Big(L^2(\Omega)\Big)^3, \text{ rot } u \in \Big(L^2(\Omega)\Big)^3, \text{ div } u = 0 \text{ in } \Omega, \quad u \cdot n = 0 \text{ on } \Gamma_1 \ \} \tag{2.1}$$

equipped with an inner product

$$\Big(u\ ,\ v\Big) \equiv \Big(rot\ u, rot\ v\Big)_\Omega + \Big(u,v\Big)_\Omega \qquad \text{for } u,v \in V.$$

We remark that this norm

$$\|u\| = \sqrt{\Big(u,u\Big)}$$

is equivalent to the usual norm of $\Big(H^1(\Omega)\Big)^3$ (cf. Duvaut-Lions [3]).

Let $a(u,v)$ be a bilinear form on $V \times V$ given by

$$a(u,v) = \Big(rot\ u, rot\ v\Big)_\Omega + \frac{1}{\lambda^2}\Big(u,v\Big)_{\Omega_0} + \epsilon\Big(u,v\Big)_{\Omega_1} \qquad \text{for } u,v \in V, \tag{2.2}$$

and we define a functional $j\ :\ V \to \mathrm{R}$ such that

$$j(u) = \int_{\Gamma_0} g\ |u_T|\ d\sigma \qquad \text{for } u \in V. \tag{2.3}$$

where $u_T = n \times (u \times n)$. Then we set up a variational inequality as follows.

Problem II Find $u \in V$ such that

$$a(u, v-u) + j(v) - j(u) \ge \int_{\Gamma_1} \frac{4\pi}{c} j_1 \cdot (v-u)\ d\sigma \tag{2.4}$$

for any $v \in V$.

Hereafter let us put $B_0 = u_0$ *and* $B_1 = u_1$.

Lemma 2.1. *Problem I and Problem II are equivalent.*

Proof. Let u be a solution of Problem I.

Taking the scalar product of (1.3) and (1.5) with $v-u$ and using Green's formula , we have

$$0=\left(rot\ rot\ u_0+\frac{1}{\lambda^2}\ u_0,v-u_0\right)_{\Omega_0}+\left(rot\ rot\ u_1+\epsilon u_1,v-u_1\right)_{\Omega_1}$$

$$=\left(rot\ u_0,rot\ (v-u_0)\right)_{\Omega_0}+\int_{\Gamma_0}(rot\ u_0\times n)\cdot(v-u_0)\ d\sigma$$

$$+\frac{1}{\lambda^2}\ (u_0,v-u_0)_{\Omega_0}+\ \epsilon\ \left(u_1,v-u_1\right)_{\Omega_1}+\left(rot\ u_1,rot\ (v-u_1)\right)_{\Omega_1}$$

$$-\int_{\Gamma_0}(rot\ u_1\times n)\cdot(v-u_1)\ d\sigma-\int_{\Gamma_1}(rot\ u_1\times n)\cdot(v-u_1)\ d\sigma$$

$$=a(u,v-u)-\int_{\Gamma_0}(rot\ u_1\times n-rot\ u_0\times n)\cdot v\ d\sigma$$

$$+\int_{\Gamma_0}(rot\ u_1\times n-rot\ u_0\times n)\cdot u_0\ d\sigma-\int_{\Gamma_1}(rot\ u_1\times n)\cdot(v-u_1)\ d\sigma$$

by (1.8) (*i.e.*, $u_0=u_1$ on Γ_0)

$$\leq a(u,v-u)+\int_{\Gamma_0}g\ |v_T|\ d\sigma-\int_{\Gamma_0}g\ |u_T|\ d\sigma$$

$$-\int_{\Gamma_1}\frac{4\pi}{c}j_1\cdot(v-u)\ d\sigma\quad \text{by } (1.9)-(1.11) \text{ and } (1.6)$$

$$=a(u,v-u)+j(v)-j(u)-\int_{\Gamma_1}\frac{4\pi}{c}j_1\cdot(v-u)\ d\sigma.$$

Hence it follows

$$a(u,v-u)+j(v)-j(u)\geq\int_{\Gamma_1}\frac{4\pi}{c}j_1\cdot(v-u)\ d\sigma.$$

This means that u is a solution of Problem II.

Next let u be a solution of Problem II. By taking $v-u=\pm\varphi$ ($\varphi\in V\cap\left(\mathcal{D}(\Omega_i)\right)^3, i=0,1$) in (2.4), we find $u=\{u_0,u_1\}$ such that

$$(2.5)\qquad rot\ rot\ u_0+\frac{1}{\lambda^2}\ u_0=0\quad \text{in}\ \ \Omega_0$$

$$(2.6) \qquad rot\ rot\ u_1 + \epsilon\, u_1 = 0 \qquad \text{in } \Omega_1.$$

It is obvious that div $u = 0$ in Ω and $u \cdot n = 0$ on Γ_1 because $u \in V$.
Moreover from (2.5), (2.6) and Green's formula, we have

$$0 = a(u, v-u) - \int_{\Gamma_0} (rot\ u_1 \times n - rot\ u_0 \times n) \cdot (v-u)\, d\sigma$$

$$- \int_{\Gamma_1} (rot\ u_1 \times n) \cdot (v-u)\, d\sigma\ ,$$

which , in combination with (2.4) , shows that

$$(2.7) \qquad j(v) - j(u) + \int_{\Gamma_0} (rot\ u_1 \times n - rot\ u_0 \times n) \cdot (v-u)\, d\sigma$$

$$+ \int_{\Gamma_1} (rot\ u_1 \times n - \frac{4\pi}{c} j_1) \cdot (v-u)\, d\sigma \geq 0\ .$$

In fact,

$$0 = a(u, v-u) - \int_{\Gamma_0} (rot\ u_1 \times n - rot\ u_0 \times n) \cdot (v-u)\, d\sigma$$

$$- \int_{\Gamma_1} (rot\ u_1 \times n) \cdot (v-u)\, d\sigma.$$

$$\geq -j(v) + j(u) + \int_{\Gamma_1} \frac{4\pi}{c} j_1 \cdot (v-u)\, d\sigma$$

$$- \int_{\Gamma_0} (rot\ u_1 \times n - rot\ u_0 \times n) \cdot (v-u)\, d\sigma - \int_{\Gamma_1} (rot\ u_1 \times n) \cdot (v-u)\, d\sigma,$$

from which we obtain (2.7). Therefore (2.7) shows that

$$(2.8) \qquad rot\ u_1 \times n = \frac{4\pi}{c} j_1 \quad \text{on } \Gamma_1,$$

$$(2.9) \qquad j(v) - j(u) + \int_{\Gamma_0} (rot\ u_1 \times n - rot\ u_0 \times n) \cdot (v-u)\, d\sigma \geq 0.$$

Substituting $\pm\lambda v, \lambda > 0$ for v in (2.9),

$$\lambda j(v) - j(u) + \int_{\Gamma_0} (rot\ u_1 \times n - rot\ u_0 \times n) \cdot (\pm\lambda v - u)\, d\sigma \geq 0,\ \forall \lambda \geq 0$$

i.e.,

(2.10)
$$\lambda \int_{\Gamma_0} \left(g\ |v_T| \pm (rot\ u_1 \times n - rot\ u_0 \times n) \cdot v \right)\ d\sigma$$
$$- \int_{\Gamma_0} \left(g\ |u_T| + (rot\ u_1 \times n - rot\ u_0 \times n) \cdot u \right)\ d\sigma \geq 0.$$

As $\lambda \to \infty$ in (2.10), then

$$\left| \int_{\Gamma_0} (rot\ u_1 \times n - rot\ u_0 \times n) \cdot v\ d\sigma \right| \leq \int_{\Gamma_0} g\ |v_T|\ d\sigma$$
$$\leq \int_{\Gamma_0} g|v|\ d\sigma\ ,$$

which means that

$$v \mapsto \int_{\Gamma_0} \frac{1}{g}\ (rot\ u_1 \times n - rot\ u_0 \times n) \cdot g\ v\ d\ \sigma$$

is continuous on $\left(H^{1/2}(\Gamma_0)\right)^3$, having the topology induced by $\left(L^1(\Gamma_0)\right)^3$, and has norm ≤ 1 for the norm

$$\int_{\Gamma_0} g|v|d\sigma \quad \text{on} \quad \left(L^1(\Gamma_0)\right)^3.$$

Using the fact that $\left(H^{1/2}(\Gamma_0)\right)^3$ is dense in $\left(L^1(\Gamma_0)\right)^3$ and the Hahn-Banach theorem , we have

$$\frac{1}{g}\ (rot\ u_1 \times n - rot\ u_0 \times n) \in \left(L^\infty(\Gamma_0)\right)^3,$$

with norm ≤ 1 . Hence

(2.11)
$$|rot\ u_1 \times n - rot\ u_0 \times n| \leq g \quad \text{a.e.} \quad \text{on } \Gamma_0\ .$$

Also as $\lambda \to 0$,

(2.12)
$$\int_{\Gamma_0} \left(g\ |u_T| + (rot\ u_1 \times n - rot\ u_0 \times n) \cdot u \right)\ d\sigma \leq 0\ .$$

By use of (2.11) , we get

$$g\ |u_T| + (rot\ u_1 \times n - rot\ u_0 \times n) \cdot u \geq 0 \quad \text{a.e.} \quad \text{on } \Gamma_0\ ,$$

which , in combination with (2.12), shows that

$$g\ |u_T| + (rot\ u_1 \times n - rot\ u_0 \times n) \cdot u = 0 \quad \text{a.e.} \quad \text{on } \Gamma_0$$

i.e.,

$$g\ |u_T| + (rot\ u_1 \times n - rot\ u_0 \times n) \cdot u_T = 0 \quad \text{a.e.} \quad \text{on } \Gamma_0\ .$$

Therefore u is a solution of Problem I. ■

We now state the existence and uniqueness theorem for our model.

Theorem 2.1. *There exists one and only one $u \in V$ satisfying (2.4).*

proof. First note that the norm $\|\cdot\|$ and the norm associated with the inner product given by ((u,v)) = a(u,v) are equivalent . Then a(u,v) is a symmetric bilinear form , continuous on V and coercive on V. Also using the property of trace operator defined on $\left(H^1(\Omega_0)\right)^3$, j is an l.s.c. proper convex functional. Hence under these conditions , (2.4) has a unique solution u $\in V$ (for example see Baiochi-Capel [1], Glowinski-Lions-Trémolières [5]). ■

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