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# Asymptotic Behavior for Nonlinear Systems of Phase Transitions

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## 1. Introduction

We consider the following nonlinear system:

$$\frac{\partial\rho(u)}{\partial t}+\frac{\partial w}{\partial t}-\triangle u=f(t,x) \qquad \text{in } Q:=(0,+\infty,)\times\Omega, \tag{1.1}$$

$$\nu\frac{\partial w}{\partial t}+\beta(w)+g(w)\ni u \quad \text{in } Q \tag{1.2}$$

with lateral boundary condition:

$$\frac{\partial u}{\partial n}+\alpha_N(x)u=h_N(t,x) \quad \text{on } \Sigma:=(0,+\infty)\times\Gamma, \tag{1.3}$$

and initial conditions:

$$u(0,\cdot)=u_0, \quad w(0,\cdot)=w_0 \quad \text{in } \Omega. \tag{1.4}$$

Here $\Omega$ is a bounded domain in $R^N$ $(N\geq 1)$ with smooth boundary $\Gamma:=\partial\Omega$; $\rho$ is a monotone increasing and bi-Lipschitz continuous function on $R$; $\nu$ is a positive constant; $\beta$ is a maximal monotone graph in $R\times R$; $g$ is a smooth function defined on $R$; $\alpha_N$ is a non-negative, bounded and measurable function on $\Gamma$ such that $\alpha_N>0$ on a subset of $\Gamma$ with positive measure; $f, h_N, u_0$ and $w_0$ are given data.

For simplicity problem (1.1)-(1.4) is denoted by (CP). This is a simplified model for a class of solid-liquid phase change problems, and in this context $u$ represents a function related to temperature and $w$ a non-conserved order parameter (the state variable characterizing phase). For instance, we have the following examples:

(1) Stefan problem with phase relaxation, in which $\beta$ is the subdifferential of the indicator function of the interval $[0,1]$ and $g\equiv 0$. This case was discussed as a melting problem with supercooling and superheating effect in [12,5].

(2) Phase-field model with constraint, in which $\beta$ is the same as in (1), $\rho(u)=u$, $g(w)=w^3-cw$ with a positive constant $c$, and a diffusion term $-\kappa\triangle w$ is added to the left

side of (1.2). This is a phase-field model with constraint $0 \leq w \leq 1$ and was discussed in [6,9,11]. We may consider system (1.1)-(1.4) as an approximation of this problem with small $\kappa > 0$.

Furthermore we refer to [2,1] for papers dealing with similar problems.

In this paper, we discuss the large-time behavior of the solution $\{u, w\}$. In fact, under the condition that $f(t,x) \to f^\infty(x)$ and $h_N(t,x) \to h_N^\infty(x)$ in an appropriate sense as $t \to +\infty$, it will be shown that as $t \to +\infty$, $u(t,\cdot)$ and $w(t,\cdot)$ converge to a solution $\{u^\infty, w^\infty\}$ of the corresponding steady-state problem

$$\begin{cases} -\triangle u^\infty = f^\infty(x) \quad \text{in } \Omega, \\ \dfrac{\partial u^\infty}{\partial n} + \alpha_N(x) u^\infty = h_N^\infty(x) \quad \text{on } \Gamma, \\ \beta(w^\infty) + g(w^\infty) \ni u^\infty \quad \text{in } \Omega. \end{cases}$$

## 2. Existence and uniqueness result for (CP)

Problem (CP) is discussed under the following assumptions (A1)-(A6):

(A1) $\rho : R \longrightarrow R$ is an increasing and bi-Lipschitz continuous function.

(A2) $\beta$ is a maximal monotone graph in $R \times R$ such that for some numbers $\sigma_*, \sigma^*$ with $-\infty < \sigma_* < \sigma^* < +\infty$

$$\overline{D(\beta)} = [\sigma_*, \sigma^*];$$

note in this case that $R(\beta) = R$, so that there is a non-negative proper l.s.c. convex function $\hat{\beta}$ on $R$ whose subdifferential $\partial\hat{\beta}$ coincides with $\beta$ in $R$, and in the context of solid-liquid system we can consider that $w = \sigma_*$ (resp. $\sigma^*$) indicates the pure solid (resp. liquid) phase and any intermediate value $w$ indicates a state of mixture.

(A3) $g : R \to R$ is a Lipschitz continuous function with compact support in $R$; in this case note that there is a non-negative primitive $\hat{g}$ of $g$.

(A4) $f \in L^2_{loc}(R_+; L^2(\Omega))$.

(A5) $h_N \in W^{1,2}_{loc}(R_+; L^2(\Gamma))$ with $\sup_{t \geq 0} |h_N|_{W^{1,2}(t,t+1;L^2(\Gamma))} < +\infty$.

(A6) $u_0 \in L^2(\Omega)$ and $w_0 \in L^2(\Omega)$ with $\hat{\beta}(w_0) \in L^1(\Omega)$.

We introduce some function spaces and a convex function in order to discuss (CP) in the framework of abstract evolution equations of the form

$$U'(t) + \partial\varphi^t(U(t)) + G(U(t)) \ni \tilde{f}(t).$$

Let $V := H^1(\Omega)$ with norm

$$|z|_V := \{|\nabla z|^2_{L^2(\Omega)} + \int_\Gamma \alpha_N |z|^2 d\Gamma\}^{\frac{1}{2}},$$

and denote by $V^*$ the dual space of $V$ and by $\langle \cdot, \cdot \rangle$ the duality pairing between $V^*$ and $V$. Then, identifying $L^2(\Omega)$ with its dual space by means of the usual inner product

$$(v, z) := \int_\Omega vz dx,$$

we see that

$$V \subset L^2(\Omega) \subset V^*$$

with compact injections.

Let $F$ be the duality mapping from $V$ onto $V^*$ which is given by the formula

$$\langle Fv, z \rangle = \int_\Omega \nabla v \cdot \nabla z dx + \int_\Gamma \alpha_N vz d\Gamma \quad \text{for any } v, z \in V.$$

It is easy to see that $V^*$ becomes a Hilbert space with inner product $(\cdot, \cdot)_*$ given by

$$(v, z)_* := \langle v, F^{-1} z \rangle \; (= \langle z, F^{-1} v \rangle) \quad \text{for any } v, z \in V^*.$$

Now, consider the product space

$$X := V^* \times L^2(\Omega),$$

which becomes a Hilbert space with inner product $(\cdot, \cdot)_X$ given by

$$([e_1, w_1], [e_2, w_2])_X := (e_1, e_2)_* + \nu(w_1, w_2) \quad \text{for any } [e_i, w_i] \in X \; (i = 1, 2).$$

Next, given the boundary data $h_N$, choose $h : R_+ \longrightarrow H^1(\Omega)$ such that for each $t \geq 0$

$$\int_\Omega \nabla h(t) \cdot \nabla z dx + \int_\Gamma \alpha_N h(t) z d\Gamma = \int_\Gamma h_N(t) z d\Gamma \quad \text{for all } z \in V;$$

note from (A5) that $\sup_{t \geq 0} |h|_{W^{1,2}(t,t+1;H^1(\Omega))} < +\infty$.

Also, using $h$ and $\hat{\beta}$, for each $t \geq 0$, define a proper l.s.c. convex function $\varphi^t$ on $X$ by

$$\varphi^t(U) = \begin{cases} \int_\Omega \rho^*(e - w) dx + \int_\Omega \hat{\beta}(w) dx - (h(t), e) \\ \qquad \text{if } U = [e, w] \in L^2(\Omega) \times L^2(\Omega) \text{ with } \hat{\beta}(w) \in L^1(\Omega), \\ +\infty \quad \text{otherwise,} \end{cases}$$

where $\rho^*$ is a non-negative primitive of $\rho^{-1}$. We denote by $\partial \varphi^t$ the subdifferential of $\varphi^t$ in $X$ and its characterization is given by the following theorem.

**Theorem 2.1.** *(cf. [5,9]) Let $t \geq 0$, $[e^*, w^*] \in X$ and $[e, w] \in D(\partial \varphi^t)$. Then $[e^*, w^*] \in \partial \varphi^t([e, w])$ if and only if conditions (a) and (b) below are satisfied:*

*(a)* $e^* = F(\rho^{-1}(e-w)-h(t))$, *that is,* $\rho^{-1}(e-w)-h(t) \in V$ *and*

$$\langle e^*, z\rangle = \int_\Omega \nabla(\rho^{-1}(e-w)-h(t))\cdot\nabla z dx + \int_\Gamma \alpha_N(\rho^{-1}(e-w)-h(t))zd\Gamma$$

*for all* $z \in V$;

*(b)* *there exists a function* $\xi \in L^2(\Omega)$ *such that* $\xi \in \beta(w)$ *a.e. on* $\Omega$ *and*

$$\nu w^* = \xi - \rho^{-1}(e-w) \quad \text{in } L^2(\Omega).$$

*Moreover, for* $U_i^* = [e_i^*, w_i^*] \in \partial\varphi^t(U_i)$ *with* $U_i = [e_i, w_i] \in D(\partial\varphi^t)$ $(i = 1, 2)$,

$$(U_1^* - U_2^*, U_1 - U_2)_X = |(e_1 - w_1) - (e_2 - w_2)|_{L^2(\Omega)} + (\xi_1 - \xi_2, w_1 - w_2),$$

*where* $\xi_i \in L^2(\Omega)$ *is as any function* $\xi$ *in (b) for each* $i = 1, 2$.

A weak formulation for (CP) is given as follows.

**Definition 2.1.** A couple $\{u, w\}$ of functions $u : R_+ \longrightarrow V^*$ and $w : R_+ \longrightarrow L^2(\Omega)$ is called a (weak) solution of (CP) on $R_+$, if the following conditions (w1)-(w3) are fulfilled for any finite $T > 0$:

(w1) $\rho(u) \in C([0,T];V^*) \cap W^{1,2}_{loc}((0,T];V^*) \cap L^2(0,T;L^2(\Omega)), u \in L^2_{loc}((0,T];H^1(\Omega))$, $w \in C([0,T];L^2(\Omega)) \cap W^{1,2}_{loc}((0,T];L^2(\Omega))$, and $\hat\beta(w) \in L^1(0,T;L^1(\Omega))$.

(w2) $\rho(u)(0) = \rho(u_0)$ and

$$\langle u'(t) + w'(t), z\rangle + \int_\Omega \nabla(u(t)-h(t))\cdot\nabla z dx + \int_\Gamma \alpha_N(u(t)-h(t))zd\Gamma = (f(t), z)$$

for all $z \in V$ and a.e. $t \in [0,T]$, where the prime $'$ denotes the derivative in time.

(w3) there exists $\xi \in L^2_{loc}((0,T];L^2(\Omega))$ such that $\xi \in \beta(w)$ a.e. on $Q_T := (0,T)\times\Omega$ and

$$\nu(w'(t), z) + (\xi(t) + g(w(t)), z) = (u(t), z)$$

for all $z \in L^2(\Omega)$ and a.e. $t \in [0,T]$.

According to Theorem 2.1, (CP) can be reformulated as an evolution equation in $X$ in the following form:

$$\begin{cases} U'(t) + \partial\varphi^t(U(t)) + G(U(t)) \ni \tilde f(t), & \text{in } X,\ t \geq 0, \\ U(0) = [\rho(u_0) + w_0, w_0], \end{cases}$$

where $U(t) = [\rho(u(t)) + w(t), w(t)], G(U(t)) = [0, \frac{1}{\nu}g(w(t))]$ and $\tilde f(t) = [f(t), 0]$.

As to the solvability of (CP) we have:

**Theorem 2.2.** *(cf. [6,9]) Assume that (A1)-(A6) hold. Then, for any $T > 0$, (CP) admits one and only one solution $\{u, w\}$ on $[0, T]$ such that*

$$\begin{cases} t^{\frac{1}{2}}\rho(u)' \in L^2(0,T;V^*), & t^{\frac{1}{2}}u \in L^2(0,T;H^1(\Omega)), \\ t\rho(u)' \in L^2(0,T;L^2(\Omega)), & tu \in L^\infty(0,T;H^1(\Omega)), \end{cases}$$

$$t^{\frac{1}{2}}w' \in L^2(0,T;L^2(\Omega)), \quad t\hat{\beta}(w) \in L^\infty(0,T;L^1(\Omega)),$$
$$t^{\frac{1}{2}}\xi \in L^2(0,T;L^2(\Omega))$$

*where $\xi$ is the function in condition (w3).*

## 3. Large-time behavior of the solution

Further suppose that there are $h_N^\infty \in L^2(\Gamma)$ and $f^\infty \in L^2(\Omega)$ such that

$$h_N - h_N^\infty \in L^2(R_+;L^2(\Gamma)), \quad f - f^\infty \in L^2(R_+;L^2(\Omega)), \tag{3.1}$$

and consider the steady-state problem (3.2)-(3.3):

$$-\triangle u^\infty = f^\infty(x) \text{ in } \Omega, \quad \frac{\partial u^\infty}{\partial n} + \alpha_N(x)u^\infty = h_N^\infty(x) \quad \text{on } \Gamma, \tag{3.2}$$

$$\beta(w^\infty) + g(w^\infty) \ni u^\infty \quad \text{in } \Omega. \tag{3.3}$$

We should note that problem (3.2) does not include $w^\infty$, and it has a unique solution $u^\infty \in H^1(\Omega)$ in the variational sense, i.e.,

$$\int_\Omega \nabla(u^\infty - h^\infty)\cdot\nabla z dx + \int_\Gamma \alpha_N(u^\infty - h^\infty)zd\Gamma = (f^\infty, z) \quad \text{for all } z \in V, \tag{3.4}$$

where $h^\infty \in H^1(\Omega)$ such that

$$\int_\Omega \nabla h^\infty \cdot \nabla z dx + \int_\Gamma \alpha_N h^\infty z d\Gamma = \int_\Gamma h_N^\infty z d\Gamma \quad \text{for all } z \in V.$$

We see from (3.1) that $h - h^\infty \in L^2(R_+;H^1(\Omega))$.

In the sequel we mean by ($P^\infty$) the algebraic relation (3.3) with the solution $u^\infty \in H^1(\Omega)$ of (3.4), and $w^\infty = w^\infty(x)$ is called a solution of ($P^\infty$).

As the following example shows, the steady-state problem ($P^\infty$) has in general infinitely many solutions.

**Example 3.1.** Consider the case when

$$f^\infty(x) \equiv 0, \quad h_N^\infty(x) \equiv l_0, \quad \alpha_N(x) \equiv 1, \quad \beta = \partial I_{[-1,1]} \text{ and } g(w) = w^3 - w$$

where $l_0$ is a constant. Then, clearly $u^\infty \equiv l_0$ and we have the following three possibilities:

(i) when $l_0 > \frac{2}{3\sqrt{3}}$ (resp. $l_0 < -\frac{2}{3\sqrt{3}}$), the algebraic relation

$$\beta(r) + g(r) \ni u^\infty (= l_0) \tag{3.5}$$

has exactly one solution $r = 1$ (resp. $-1$).

(ii) when $l_0 = \frac{2}{3\sqrt{3}}$ (resp. $-\frac{2}{3\sqrt{3}}$), (3.5) has exactly two solutions $r = -\frac{1}{\sqrt{3}}$ (resp. $\frac{1}{\sqrt{3}}$), 1 (resp. $-1$).

(iii) when $|l_0| < \frac{2}{3\sqrt{3}}$, (3.5) has exactly three solutions $r = \xi_-, \xi_0, \xi_+$ with $-1 \leq \xi_- < \xi_0 < \xi_+ \leq 1$.

Physically (i) means that if the temperature is kept high (resp. low) enough, then the limit state (as $t \longrightarrow +\infty$) will be of pure liquid (resp. solid). On the other hand, (ii) and (iii) mean that if the temperature is kept near the phase transition temperature, then the limit state possibly includes a mushy region. In particular, in the case of (iii), all step functions $w^\infty$ with range in $\{\xi_-, \xi_0, \xi_+\}$ are solutions of ($\mathrm{P}^\infty$) and hence ($\mathrm{P}^\infty$) has in general infinitely many solutions.

Our main result is stated in the following theorem.

**Theorem 3.1.** *Suppose that conditions (A1)-(A6) and (3.1) hold, and let $\{u, w\}$ be the solution to (CP) on $R_+$. Further, suppose that for each $p \in R$ the (algebraic) inclusion*

$$\beta(r) + g(r) \ni p$$

*has a finite number of solutions $r$ in $\overline{D(\beta)}$. Then,*

$$u(t) \longrightarrow u^\infty \text{ weakly in } H^1(\Omega) \text{ as } t \longrightarrow +\infty, \tag{3.6}$$

*where $u^\infty$ is the unique solution of (3.4), and there exists a function $w^\infty \in L^\infty(\Omega)$ such that*

$$\beta(w^\infty(x)) + g(w^\infty(x)) \ni u^\infty(x) \quad \text{for a.e. } x \in \Omega$$

*and*

$$w(t,x) \longrightarrow w^\infty(x) \text{ for a.e. } x \in \Omega \text{ as } t \longrightarrow +\infty.$$

We prove the theorem by the following four lemmas.

**Lemma 3.1.** *Under the same assumptions of Theorem 3.1, for the solution $\{u, w\}$ to (CP) on $R_+$, we have*

$$u - u_\infty \in L^2(R_+; H^1(\Omega)),\ w' \in L^2(R_+; L^2(\Omega)) \text{ and } \hat{\beta}(w) \in L^\infty(R_+; L^1(\Omega)), \tag{3.7}$$

$$u \in L^\infty([1, +\infty); H^1(\Omega)). \tag{3.8}$$

**Proof.** Multiplying the difference of (1.1) and (3.2) by $u(t)-u^\infty$ and (1.2) by $w'(t)$, we get

$$\frac{d}{dt}\{\int_\Omega \rho^*(\rho(u(t)))dx - (\rho(u(t))+w(t),u^\infty)\} + (w'(t),u(t)) +$$
$$+|\nabla(u(t)-u^\infty)|^2_{L^2(\Omega)} + \int_\Gamma \alpha_N |u(t)-u^\infty|^2 d\Gamma$$
$$= (f(t)-f^\infty, u(t)-u^\infty) + \int_\Gamma (h(t)-h^\infty)(u(t)-u^\infty)d\Gamma$$

and

$$\nu|w'(t)|^2_{L^2(\Omega)} + \frac{d}{dt}\{\int_\Omega \hat\beta(w(t))dx + \int_\Omega \hat g(w(t))dx\} = (u(t),w'(t))$$

for a.e. $t\ge 0$. Adding these two equalities we have

$$\frac{d}{dt}\{\int_\Omega \rho^*(\rho(u(t)))dx - (\rho(u(t))+w(t),u^\infty) + \int_\Omega \hat\beta(w(t))dx + \int_\Omega \hat g(w(t))dx\} +$$
$$+\nu|w'(t)|^2_{L^2(\Omega)} + |\nabla(u(t)-u^\infty)|^2_{L^2(\Omega)} + \int_\Gamma \alpha_N|u(t)-u^\infty|^2 d\Gamma$$
$$= (f(t)-f^\infty, u(t)-u^\infty) + \int_\Gamma (h(t)-h^\infty)(u(t)-u^\infty)d\Gamma$$

for a.e. $t\ge 0$, so that there are positive constants $C_1$ and $C_2$ such that

$$\frac{d}{dt}\{\int_\Omega \rho^*(\rho(u(t)))dx - (\rho(u(t))+w(t),u^\infty) + \int_\Omega \hat\beta(w(t))dx + \int_\Omega \hat g(w(t))dx\} +$$
$$+\nu|w'(t)|^2_{L^2(\Omega)} + C_1|u(t)-u^\infty|^2_{H^1(\Omega)}$$
$$\le C_2\{|f(t)-f^\infty|^2_{L^2(\Omega)} + |h(t)-h^\infty|^2_{L^2(\Gamma)}\}$$
$$\text{for a.e. } t\ge 0.$$

Therefore, for all $T>0$, we have

$$\int_\Omega \rho^*(\rho(u(T)))dx - (\rho(u(T))+w(T),u^\infty) + \int_\Omega \hat\beta(w(T))dx + \int_\Omega \hat g(w(T))dx +$$
$$+\nu\int_0^T |w'(t)|^2_{L^2(\Omega)}dt + C_1\int_0^T |u(t)-u^\infty|^2_{H^1(\Omega)}dt$$
$$\le C_2\{\int_0^T |f(t)-f^\infty|^2_{L^2(\Omega)}dt + \int_0^T |h(t)-h^\infty|^2_{L^2(\Gamma)}dt\} +$$
$$+\int_\Omega \rho^*(\rho(u_0))dx - (\rho(u_0)+w_0,u^\infty) + \int_\Omega \hat\beta(w_0)dx + \int_\Omega \hat g(w_0))dx.$$

Hence (3.7) is obtained. Also, (3.8) is a direct consequence of (3.7) and a standard regularity result for parabolic equations. □

**Lemma 3.2.** *Under the same assumptions of Theorem 3.1, put*

$$U^t(x) := \int_t^{t+1} |w'(\tau,x)|^2 d\tau \quad \textit{for } x\in\Omega.$$

*Then,* $U^t(x) \longrightarrow 0$ *as* $t \longrightarrow +\infty$ *for a.e.* $x \in \Omega$.

**Proof.** By Lemma 3.1, we have

$$\lim_{T \nearrow +\infty} \int_T^{+\infty} dt \int_\Omega |w'(t,x)|^2 dx = 0,$$

so that

$$\lim_{T \nearrow +\infty} \int_\Omega dx \int_T^{+\infty} |w'(t,x)|^2 dt = 0.$$

Hence,

$$\int_T^{+\infty} |w'(t,x)|^2 dt \longrightarrow 0 \text{ as } T \longrightarrow +\infty \text{ for a.e. } x \in \Omega.$$

This implies the lemma. □

**Lemma 3.3.** *Under the same assumptions of Theorem 3.1, (3.6) holds.*

**Proof.** Let $\{u, w\}$ be a solution to (CP) and $u^\infty$ be the solution to (3.4).
Let $\{t_n\}$ be any sequence with $t_n \longrightarrow +\infty$ as $n \longrightarrow +\infty$, and put

$$u_n(t) := u(t_n + t),\ w_n(t) := w(t_n + t),\ f_n(t) := f(t_n + t),\ h_n(t) := h(t_n + t)$$
$$\text{for } 0 \le t \le 1.$$

Since by Lemma 3.1, $u - u^\infty$ and $w'$ are in $L^2(R_+; H^1(\Omega))$ and $L^2(R_+; L^2(\Omega))$, respectively, we see that

$$u_n \longrightarrow u^\infty \text{ in } L^2(0,1; H^1(\Omega)), \tag{3.9}$$

and

$$w_n' \longrightarrow 0 \text{ in } L^2(0,1; L^2(\Omega)), \tag{3.10}$$

as $n \longrightarrow +\infty$. Moreover since by Lemma 3.1, $u$ is bounded in $H^1(\Omega)$ on $[1, +\infty)$, we may assume that for a function $\tilde{u}^\infty$ in $H^1(\Omega)$

$$u_n(0) = u(t_n) \longrightarrow \tilde{u}^\infty \text{ weakly in } H^1(\Omega) \tag{3.11}$$

as $n \longrightarrow +\infty$. Now, consider the Cauchy problem for each $n$

$$\begin{cases} \rho(u_n)'(t) + \partial\Phi_n^t(u_n(t)) = f_n(t) - w_n'(t) & \text{in } L^2(\Omega),\ 0 \le t \le 1, \\ u_n(0) = u(t_n) \end{cases}$$

where $\Phi_n^t$ is a proper l.s.c. and convex function on $L^2(\Omega)$ such that for each $n$ and $t \in [0,1]$

$$\Phi_n^t(z) := \begin{cases} \frac{1}{2}|z - h_n(t)|_V^2 & \text{if } z \in V, \\ +\infty & \text{otherwise,} \end{cases}$$

and $\partial\Phi_n^t$ is the subdifferential of $\Phi_n^t$ in $L^2(\Omega)$. From (3.9), (3.10) and (3.11), we see that

$$\Phi_n^t \longrightarrow \Phi_\infty \quad \text{on } L^2(\Omega) \text{ in the sense of Mosco for every } t \in [0,1],$$

$$f_n - w_n' \longrightarrow f^\infty \quad \text{in } L^2(0,1;L^2(\Omega))$$

and

$$u_n(0) \longrightarrow \tilde{u}^\infty \quad \text{in } L^2(\Omega)$$

as $n \longrightarrow +\infty$, where $\Phi_\infty$ is a proper l.s.c. and convex function on $L^2(\Omega)$ such that

$$\Phi_\infty(z) := \begin{cases} \dfrac{1}{2}|z - h^\infty|_V^2 & \text{if } z \in V, \\ +\infty & \text{otherwise.} \end{cases}$$

Therefore, by a general theory in [8],

$$u_n \longrightarrow \tilde{u} \quad \text{in } C([0,1];L^2(\Omega)) \text{ as } t \longrightarrow +\infty, \tag{3.12}$$

where $\tilde{u}$ is the solution of

$$\begin{cases} \rho(\tilde{u})'(t) + \partial\Phi_\infty(\tilde{u}(t)) = f^\infty & \text{in } L^2(\Omega), 0 \le t \le 1, \\ \tilde{u}(0) = \tilde{u}^\infty. \end{cases}$$

From (3.9) and (3.12) it follows that $\tilde{u} = u^\infty$ on $[0,1]$. Consequently (3.6) holds. □

**Lemma 3.4.** *Under the same assumptions of Theorem 3.1, put*

$$V(x) := \{r \in \overline{D(\beta)}; w(t_n, x) \to r \text{ for some } t_n \text{ with } t_n \to +\infty\} \quad \text{for } x \in \Omega.$$

*Then,*

*(1)* $V(x) \neq \emptyset$ *for a.e.* $x \in \Omega$*;*

*(2)* $\beta(r) + g(r) \ni u^\infty(x)$ *for all* $r \in V(x)$ *and a.e.* $x \in \Omega$*;*

*(3)* $V(x)$ *is a singleton for a.e.* $x \in \Omega$*.*

**Proof.** (1) is clear by the boundedness of $w(t,x)$ on $R$.
Let $x \in \Omega$ with $\lim_{t\to+\infty} \int_t^{t+1} |w'(\tau,x)|^2 d\tau = 0$ (cf. Lemma 3.2) and $r \in V(x)$. Then, there exists a sequence $\{t_n\}$ such that

$$t_n \longrightarrow +\infty \text{ and } w(t_n, x) \longrightarrow r \quad \text{as } n \longrightarrow +\infty.$$

Fixing $x$, put

$$w_n(t) := w(t_n + t, x),\ u_n(t) := u(t_n + t, x) \quad \text{for } 0 \le t \le 1.$$

By Lemma 3.3 and (A2), we may assume that

$$u_n - g(w_n) \longrightarrow u^\infty(x) - g(r) \quad \text{in } L^2(0,1) \text{ as } t \longrightarrow +\infty$$

Now consider a sequence of ODEs:

$$\begin{cases} w_n'(t) + \beta(w_n(t)) + g(w_n(t)) \ni u_n(t) & \text{for } 0 \le t \le 1, \\ w_n(0) = w(t_n, x). \end{cases}$$

By a general theory in [8] again, $w_n$ converges in $C([0,1])$ to the solution $\tilde{w}$ of

$$\begin{cases} \tilde{w}'(t) + \beta(\tilde{w}(t)) + g(\tilde{w}(t)) \ni u^\infty(x) & \text{for } 0 \le t \le 1, \\ \tilde{w}(0) = r. \end{cases} \tag{3.13}$$

But, since $\tilde{w}' \equiv 0$ i.e. $\tilde{w} \equiv r$ by assumption, we see from (3.13) that

$$\beta(r) + g(r) \ni u^\infty(x).$$

Thus, (2) is proved. At last, we show (3). Suppose that $V(x)$ has more than two elements for some $x \in \Omega$, say $r_1, r_2 \in V(x)$, $r_1 < r_2$. By definition, there exists two sequence $\{s_n\}$ and $\{t_n\}$ such that

$$s_n \longrightarrow +\infty, \ w(s_n, x) \longrightarrow r_1;$$
$$t_n \longrightarrow +\infty, \ w(t_n, x) \longrightarrow r_2$$

as $n \longrightarrow +\infty$, and

$$s_n < t_n < s_{n+1} < t_{n+1} \ \text{ for } n = 1, 2, 3, \cdots.$$

From the continuity of $w$ with respect to $t$, for any $r \in (r_1, r_2)$, there exists a sequence $\{\tau_n\}$ with $\tau_n \longrightarrow +\infty (n \longrightarrow +\infty)$ such that

$$s_n < \tau_n < t_n \ \text{ for } n = 1, 2, 3, \cdots \text{ and } w(\tau_n, x) = r \ \text{ for large } n.$$

This implies that $r \in V(x)$ and hence $[r_1, r_2] \subset V(x)$. This contradicts the assumption that $\beta(r) + g(r) \ni u^\infty(x)$ has a finite number of solutions $r$ in $\overline{D(\beta)}$. Thus, $V(x)$ must be a singleton for a.e. $x \in \Omega$. □

In particular, (2) and (3) of Lemma 3.4 imply that $w(t, x)$ converges to a solution $w^\infty(x)$ for a.e. $x \in \Omega$ as $t \longrightarrow +\infty$ and the limit $w^\infty$ is a solution of ($P^\infty$). Thus we complete the proof of Theorem 3.1.

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