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渦運動の非局所性と特異積分変換

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Singular integral transforms[1] are inherent in the nonlocal nature of vortex stretching in three-dimensional incompressible flows. Nonlocality appears as the pressure term in Euler equations and as the integral relationship from vorticity to strain in vorticity equations.[2–4] Pressure hessian[2,3,5,6] is its another form, contributing evolution of rate-of-strain. We intend to give a theoretical foundation for the vorticity-strain correlation with an explicit use of singular integral transforms.

There is a one-dimensional model for vorticity equation, the Constantin-Lax-Majda model[7];

$$\frac{\partial \omega}{\partial t} = H(\omega)\omega, \tag{1}$$

where

$$H(\omega) = \frac{1}{\pi} \oint \frac{\omega(y)}{x-y} \mathrm{d}y$$

is the Hilbert transform and $\oint$ denotes the principal-value integral. The "vorticity" ω and "rate-of-strain" $H[\omega]$ are Hilbert-conjugates and are real and imaginary parts of an analytic function in the upper-half plane (Note also that $H^2 = -1$). Actually, this model could mean more than it seems.

We consider motion of an inviscid fluid governed by three-dimensional Euler equations

$$\frac{Du_i}{Dt} = -\partial_i p,$$

together with the incompressible condition $\nabla \cdot \boldsymbol{u} = 0$. ($\partial_i = \partial/\partial x_i$.) Here $D/Dt = \partial/\partial t + (\boldsymbol{u} \cdot \nabla)$ denotes the Lagrangian time derivative, $\boldsymbol{u}$ the velocity and p the pressure. We treat the infinite space case with a fluid at rest at infinity. The velocity can be expressed as $\boldsymbol{u} = \nabla \times \boldsymbol{A}$ by the vector potential $\boldsymbol{A}$. If we take a curl under Coulomb gauge $\nabla \cdot \boldsymbol{A} = 0$, we have $\nabla^2 \boldsymbol{A} = -\boldsymbol{\omega}$, or

$$\boldsymbol{A}(\boldsymbol{x}) = \frac{-1}{4\pi} \int \frac{\boldsymbol{\omega}(\boldsymbol{y})}{|\boldsymbol{x}-\boldsymbol{y}|} \mathrm{d}\boldsymbol{y}. \tag{2}$$

Taking the curl of (2) yields the Biot-Savart formula. In order to differentiate (2) further, a formula for second derivative of the Newtonian potential is needed. That is, for a smooth function $g(\boldsymbol{x})$, we have[8]

$$\partial_i\partial_j g(\boldsymbol{x}) = \frac{-1}{4\pi}\int \frac{1}{|\boldsymbol{x}-\boldsymbol{y}|}\partial_i\partial_j \triangle g(\boldsymbol{y})\mathrm{d}\boldsymbol{y}$$
$$= \frac{\triangle g}{3}\delta_{ij} + K_{ij}[\triangle g](\boldsymbol{x}), \tag{3}$$

where

$$K_{ij}[f](\boldsymbol{x}) = \oint \frac{|\boldsymbol{x}-\boldsymbol{y}|^2\delta_{ij} - 3(x_i-y_i)(x_j-y_j)}{4\pi|\boldsymbol{x}-\boldsymbol{y}|^5} f(\boldsymbol{y})\mathrm{d}\boldsymbol{y}. \tag{4}$$

Here the principal value integral means $\oint = \lim_{\epsilon\to 0}\int_{|\boldsymbol{x}-\boldsymbol{y}|\geq\epsilon}$ (similar notaions will be used hereafter). The second derivative is made up of the local term due to Dirac delta function plus the nonlocal term in the form of singular integral. By using (3) and symmetrizing we find[2–4]

$$S_{ij}(\boldsymbol{x}) = \frac{3}{8\pi}\oint \frac{\epsilon_{ikl}r_k\omega_l(\boldsymbol{y})r_j + r_i\epsilon_{jkl}r_k\omega_l(\boldsymbol{y})}{r^5}\mathrm{d}\boldsymbol{y}, \tag{5}$$

where $\boldsymbol{r} = \boldsymbol{x} - \boldsymbol{y}$, and ϵ_{ijk} is the fully antisymmetric tensor.

The bilateral relationship between vorticity and strain is best seen in terms of the vorticity tensor $\Omega_{ij} \equiv (\partial_j u_i - \partial_i u_j)/2 = -(1/2)\epsilon_{ijk}\omega_k$, which decomposes the velocity gradient as $\partial_j u_i = S_{ij} + \Omega_{ij}$. Note that $\boldsymbol{\Omega}$ and $\boldsymbol{S}$ do not commute in general. With $\boldsymbol{\Omega}$ we can write eq.(5) as

$$S_{ij}(\boldsymbol{x}) = \frac{3}{4\pi}\oint \frac{r_k\Omega_{ki}(\boldsymbol{y})r_j - r_i\Omega_{jk}(\boldsymbol{y})r_k}{r^5}\mathrm{d}\boldsymbol{y}, \tag{6}$$

$\equiv T_{ij}[\boldsymbol{\Omega}]$, say.

Now, we seek for the inverse transform which expresses vorticity in terms of strain. By the definition of $\boldsymbol{S}$ and $\boldsymbol{A}$ we have

$$S_{ij} = \frac{1}{2}(\partial_i\epsilon_{jkl}\partial_k A_l + \partial_j\epsilon_{ikl}\partial_k A_l).$$

Take a divergence and a curl for i and j respectively we obtain

$$\triangle^2 A_p = -2\epsilon_{pqj}\partial_q\partial_i S_{ij}.$$

Again using (3) under $\nabla \cdot \boldsymbol{A} = 0$, we have

$$\omega_i = -\triangle A_i = -\frac{3}{2\pi} \oint \mathrm{d}\boldsymbol{y} \frac{\epsilon_{ijk}(x_j - y_j) S_{kl}(\boldsymbol{y})(x_l - y_l)}{|\boldsymbol{x} - \boldsymbol{y}|^5}.$$

In terms of $\boldsymbol{\Omega}$ this becomes

$$\Omega_{ij}(\boldsymbol{x}) = -\frac{3}{4\pi} \oint \frac{r_k S_{ki}(\boldsymbol{y}) r_j - r_i S_{jk}(\boldsymbol{y}) r_k}{r^5} \mathrm{d}\boldsymbol{y}. \tag{7}$$

A crucial observation is that $\boldsymbol{\Omega}$ and $\boldsymbol{S}$ are connected with each other through the *identical singular integral tramsform* (up to minus sign), that is,

$$\boldsymbol{T}[\boldsymbol{T}[\boldsymbol{\Omega}]] = -\boldsymbol{\Omega}. \tag{8}$$

Vorticity and rate-of-strain tensors are conjugates under the transform $\boldsymbol{T}$. Because of $\mathrm{tr}(\boldsymbol{S} \cdot \boldsymbol{S}) + \mathrm{tr}(\boldsymbol{\Omega} \cdot \boldsymbol{\Omega}) = -\triangle p$ we have

$$\langle S_{ij} S_{ij} \rangle = \langle \Omega_{ij} \Omega_{ij} \rangle, \tag{9}$$

where the brackets denote the spatial average. The identity (9) can be regarded as the Parseval formula for the transform $\boldsymbol{T}$. The apparently trivial shift from $\boldsymbol{\omega}$ to $\boldsymbol{\Omega}$ makes manifest the conjugate relationship between vorticity and strain.

The transform $\boldsymbol{T}$ can also be defined for general 3×3 matrices. Some of its properties are as follows. $\boldsymbol{T}$ is traceless; $\mathrm{tr}(\boldsymbol{T}) = 0$, where tr denotes trace. Also, $\boldsymbol{T}[c(\boldsymbol{x})\boldsymbol{I}] \equiv 0$ for arbitrary scalar $c(\boldsymbol{x})$ and $\boldsymbol{I}$ is the identity matrix.

It should be noted that $\boldsymbol{T}^2 = -\boldsymbol{I}$ does not hold for general matrices. In fact, we can show by a direct computation using Fourier transform (eq.(11) below) that $\boldsymbol{T}[\boldsymbol{T}^2[\boldsymbol{X}] + \boldsymbol{X}] = 0$ for any $\boldsymbol{X}$. It can be shown generally from this that

$$T_{ij}^2[X] = -X_{ij} + R_i R_j [R_k R_l [X_{kl}]] - \epsilon_{ipq} \epsilon_{jkl} R_p R_k [X_{ql}],$$

where R_i denotes the Riesz transform. (Basic facts about R_i summerized at the end of this paper is needed in what follows.) We also have the adjoint formulae; for 3×3 matrices $\boldsymbol{f}$ and $\boldsymbol{g}$,

$$\langle \mathrm{tr}(\boldsymbol{T}[\boldsymbol{f}] \cdot \boldsymbol{g}) \rangle = \pm \langle \mathrm{tr}(\boldsymbol{f} \cdot \boldsymbol{T}[\boldsymbol{g}]) \rangle, \tag{10}$$

where + should be taken when one of $\boldsymbol{f}$ and $\boldsymbol{g}$ is symmetric and the other is antisymmetric and − when both are symmetric or antisymmetric. These can be verified by writing both sides explicitly (proofs omitted).

Further properties of $\boldsymbol{T}$ can be seen in the Fourier transform[9] (designated by $\tilde{}$);

$$\tilde{T}_{ij}[\tilde{\Omega}] = \tilde{S}_{ij} = \frac{1}{|\boldsymbol{k}|^2}(k_i k_l \tilde{\Omega}_{jl} - k_j k_l \tilde{\Omega}_{li}), \tag{11}$$

where $\boldsymbol{k}$ is the wavenumber. Using Riesz transform R_i, we can write

$$T_{ij}[\boldsymbol{\Omega}] = -R_i R_l \Omega_{jl} + R_j R_l \Omega_{li}.$$

Because of boundedness(from L^p to itself) of R_i and eq.(8) $\boldsymbol{\Omega}$ and $\boldsymbol{S}$ are comparable[10] in L^p-norm[11];

$$A_p^{-1}||\boldsymbol{\Omega}||_p \le ||\boldsymbol{T}[\boldsymbol{\Omega}]||_p \le A_p||\boldsymbol{\Omega}||_p,$$

with some constants A_p for $1 < p < \infty$ ($A_2 = 1$). As in the case of Riesz transform, by analytic extension the vorticity and rate-of-strain tensors can be regarded as the boundary value of the pairs of conjugate harmonic functions in (3+1)-dimensional space; $\mathbf{R}_+^{3+1} = \{(\boldsymbol{x}, y)|\boldsymbol{x} \in \mathbf{R}^3, y > 0\}$. Let

$$\begin{cases} u_{ij}(\boldsymbol{x}, y) = (P_y * \Omega_{ij})(\boldsymbol{x}, y) \\ v_{ij}(\boldsymbol{x}, y) = (P_y * S_{ij})(\boldsymbol{x}, y) \\ = -(\Omega_{jl} * R_l R_i[P_y]) + (\Omega_{li} * R_l R_j[P_y]) \end{cases} \tag{12}$$

where P_y is defined by (17) below and the last line follows by (19). Here $*$ denotes convolution and u_{ij} and v_{ij} are respectively antisymmetric and symmetric; $u_{ij} = -u_{ji}$ $v_{ij} = v_{ji}$. Then we have by (16)

$$\triangle u_{ij} = \left(\frac{\partial^2}{\partial y^2} + \sum_{k=1}^{3} \frac{\partial^2}{\partial x_k^2} u_{ij}\right) = 0, \ \triangle v_{ij} = 0.$$

Similar as (18), it can be shown by the Fourier transform that

$$S_{ij} = T_{ij}[\Omega] \Longleftrightarrow \begin{cases} \dfrac{\partial^2 u_{jl}}{\partial x_i \partial x_l} - \dfrac{\partial^2 u_{li}}{\partial x_j \partial x_l} = -\dfrac{\partial^2 v_{ij}}{\partial y^2} \\ \dfrac{\partial^2 v_{jl}}{\partial x_i \partial x_l} - \dfrac{\partial^2 v_{li}}{\partial x_j \partial x_l} = \dfrac{\partial^2 u_{ij}}{\partial y^2} \end{cases}, \tag{13}$$

with $\partial_i(u_{ij} + v_{ij}) = 0$. Equation(13) corresponds to the generalized Cauchy-Riemann condition underlying vorticity-strain conjugation. Moreover, $\lim_{y\to 0} u_{ij}(\boldsymbol{x}, y) = \Omega_{ij}(\boldsymbol{x})$, $\lim_{y\to 0} v_{ij}(\boldsymbol{x}, y) = S_{ij}(\boldsymbol{x})$. Therefore, the vorticity and rate-of-strain tensors can be regarded as the boundary values of conjugate harmonic functions in $\mathbf{R}_+^{3+1}$.

As an application of this transform we note the relationship between three of the Siggia's invariants $I_1 = \langle (S_{ij}S_{ij})^2 \rangle$, $I_2 = \langle S_{ij}S_{ij}|\boldsymbol{\omega}|^2 \rangle$, $I_3 = \langle \omega_i S_{ij} S_{jk} \omega_k \rangle$, $I_4 = \langle |\boldsymbol{\omega}|^4 \rangle$, which describe vorticity-strain correlation.[12−15] By subtracting singularity it can be shown for a smooth function $\alpha(\boldsymbol{x})$ that

$$T_{ij}[\alpha \boldsymbol{X}] = \alpha T_{ij}[\boldsymbol{X}] - U_{ij}[\boldsymbol{X}; \alpha], \tag{14}$$

where we have set

$$U_{ij}[\boldsymbol{X}; \alpha](\boldsymbol{x}) = \frac{3}{4\pi} \oint \frac{r_k X_{ki}(\boldsymbol{y}) r_j - r_i X_{jk}(\boldsymbol{y}) r_k}{r^5} (\alpha(\boldsymbol{x}) - \alpha(\boldsymbol{y})) \mathrm{d}\boldsymbol{y}.$$

That is, a smooth function $\alpha(\boldsymbol{x})$ can be passed in and out of $\boldsymbol{T}$ by introducing a smoothing operator.[16] Letting $\alpha(\boldsymbol{x}) = \mathrm{tr}(\boldsymbol{\Omega} \cdot \boldsymbol{\Omega}) = -|\boldsymbol{\omega}|^2/2$, $\beta(\boldsymbol{x}) = \mathrm{tr}(\boldsymbol{S} \cdot \boldsymbol{S})$ we find

$$\langle \mathrm{tr}(\boldsymbol{T}[\boldsymbol{\Omega}] \cdot \boldsymbol{T}[\boldsymbol{\Omega}]) \mathrm{tr}(\boldsymbol{\Omega} \cdot \boldsymbol{\Omega}) \rangle = \langle \mathrm{tr}(\boldsymbol{T}[\boldsymbol{\Omega}] \cdot \alpha \boldsymbol{T}[\boldsymbol{\Omega}]) \rangle$$

$$= \langle \mathrm{tr}(\boldsymbol{\Omega} \cdot \boldsymbol{T}[\alpha \boldsymbol{T}[\boldsymbol{\Omega}]]) \rangle \text{ by(10)}$$

$$= -\langle \mathrm{tr}(\boldsymbol{\Omega} \cdot \boldsymbol{\Omega}) \mathrm{tr}(\boldsymbol{\Omega} \cdot \boldsymbol{\Omega}) \rangle - \langle \mathrm{tr}(\boldsymbol{\Omega} \cdot \boldsymbol{U}[\boldsymbol{S}; \alpha]) \rangle \text{ by(14)},$$

or

$$\tfrac{1}{2} I_2 = \tfrac{1}{4} I_4 + \langle \mathrm{tr}(\boldsymbol{\Omega} \cdot \boldsymbol{U}[\boldsymbol{S}; \alpha]) \rangle .$$

Similarly, we have

$$\langle \mathrm{tr}(\boldsymbol{S} \cdot \boldsymbol{S}) \mathrm{tr}(\boldsymbol{\Omega} \cdot \boldsymbol{\Omega}) \rangle$$

$$= -\langle \mathrm{tr}(\boldsymbol{S} \cdot \boldsymbol{S}) \mathrm{tr}(\boldsymbol{S} \cdot \boldsymbol{S}) \rangle + \langle \mathrm{tr}(\boldsymbol{S} \cdot \boldsymbol{U}[\boldsymbol{\Omega}; \beta]) \rangle ,$$

that is,

$$\tfrac{1}{2} I_2 = I_1 - \langle \mathrm{tr}(\boldsymbol{S} \cdot \boldsymbol{U}[\boldsymbol{\Omega}; \beta]) \rangle .$$

(Note

that $\langle \mathrm{tr}(\boldsymbol{S}\cdot\boldsymbol{U}[\boldsymbol{\Omega};\beta])\rangle = -\langle \mathrm{tr}(\boldsymbol{\Omega}\cdot\boldsymbol{U}[\boldsymbol{S};\beta])\rangle$, $\langle \mathrm{tr}(\boldsymbol{\Omega}\cdot\boldsymbol{U}[\boldsymbol{S};\alpha])\rangle = -\langle \mathrm{tr}(\boldsymbol{S}\cdot\boldsymbol{U}[\boldsymbol{\Omega};\alpha])\rangle$.) It seems worthwhile to examine further kinematic constraints imposed by the conjugate character upon the vorticity-strain correlation.

On the other hand, in terms of $\boldsymbol{\Omega}$ and $\boldsymbol{S}$ the equations of motion become

$$\frac{D\boldsymbol{\Omega}}{Dt} = -\boldsymbol{\Omega}\cdot\boldsymbol{S} - \boldsymbol{S}\cdot\boldsymbol{\Omega},$$

$$\frac{D\boldsymbol{S}}{Dt} = -\boldsymbol{\Omega}\cdot\boldsymbol{\Omega} - \boldsymbol{S}\cdot\boldsymbol{S} - \boldsymbol{P}.$$

Here $P_{ij} = \partial_i\partial_j p$ is the pressure hessian which is expressed as

$$P_{ij}(\boldsymbol{x}) = \frac{\triangle p}{3}\delta_{ij} + K_{ij}[\triangle p](\boldsymbol{x}),$$

by (4) or

$$P_{ij} = R_i R_j [\mathrm{tr}(\boldsymbol{\Omega}\cdot\boldsymbol{\Omega} + \boldsymbol{T}[\boldsymbol{\Omega}]\cdot\boldsymbol{T}[\boldsymbol{\Omega}])],$$

by using the Riesz transform (eq.(15) below). Pursuit of the similarity with the model (1) regarding dynamics may be useful for understanding a putative singularity formation in Euler flows[17,18] and small-scale motion in Navier-Stokes turbulence. Finally, we note that a similar conjugation is also seen in two dimensions.

The Riesz transform.[1] It is defined by

$$R_i[f](\boldsymbol{x}) = c_n \oint \frac{y_i}{|\boldsymbol{y}|^{n+1}} f(\boldsymbol{x}-\boldsymbol{y})\mathrm{d}\boldsymbol{y}$$

for $i = 1, 2, ...n$, $\boldsymbol{x} \in \mathbf{R}^n$ with $c_n = \Gamma\left(\frac{n+1}{2}\right)/\pi^{(n+1)/2}$, where Γ is the gamma function. This is a generalization of the Hilbert transform into $n-$dimensions. Its Fourier transform is given by $\widetilde{R}_j = \mathrm{i}k_j/|\boldsymbol{k}|$ and thus

$$\partial_i\partial_j f = -R_i R_j \triangle f. \tag{15}$$

We recall how the Riesz transform is related with harmonic functions. A function of the form

$$u(\boldsymbol{x}, y) = \int \widetilde{f}(\boldsymbol{t})\exp(-2\pi\mathrm{i}\boldsymbol{t}\cdot\boldsymbol{x})\exp(-2\pi|\boldsymbol{t}|y)\mathrm{d}\boldsymbol{t},$$

(with $\boldsymbol{x} \in \mathbf{R}^n$ and $y > 0$) is hamonic in $\mathbf{R}_+^{n+1}$;

$$\triangle u \equiv \frac{\partial^2 u}{\partial y^2} + \sum_{k=1}^{n} \frac{\partial^2 u}{\partial x_k^2} = 0, \tag{16}$$

We write Poisson integral of f as

$$u(\boldsymbol{x}, y) = \int P_y(\boldsymbol{x}) f(\boldsymbol{x} - \boldsymbol{t}) \mathrm{d}\boldsymbol{t} = (P_y * f)(\boldsymbol{x}),$$

where

$$\begin{aligned} P_y(\boldsymbol{x}) &= \int \exp(-2\pi \mathrm{i} \boldsymbol{t} \cdot \boldsymbol{x}) \exp(-2\pi |\boldsymbol{t}| y) \mathrm{d}\boldsymbol{t} \\ &= \frac{c_n y}{(|\boldsymbol{x}|^2 + y^2)^{(n+1)/2}} \end{aligned} \tag{17}$$

is called the Poisson kernel. Note that as $y \to 0$, $P_y(\boldsymbol{x}) \to \delta^{(n)}(\boldsymbol{x})$ ($n-$dimensional Dirac's delta). Let $u_0 = P_y * f, u_1 = P_y * f_1, \ldots, u_n = P_y * f_n$, then the connection between the Riesz transform and harmonic function lies in the following fact[1];

$$f_j = R_j[f] \text{ for } j = 1, \ldots, n \Longleftrightarrow \begin{cases} \sum_{j=0}^{n} \frac{\partial u_j}{\partial x_j} = 0 \\ \frac{\partial u_j}{\partial x_k} = \frac{\partial u_k}{\partial x_j} \end{cases} \tag{18}$$

Equations (15) is the called generalized Cauchy-Riemann condition (or the M. Riesz system). On the other hand,

$$Q_y^{(j)}(\boldsymbol{x}) \equiv R_j[P_y](\boldsymbol{x}) = \frac{c_n x_j}{(|\boldsymbol{x}|^2 + y^2)^{(n+1)/2}}$$

is called the conjugate Poisson kernel which satisfies

$$R_j[P_y] * f = P_y * R_j[f]. \tag{19}$$

The function f and its Riesz transforms are regarded as boundary values of these conjugates harmonics; $\lim_{y\to 0} u_0(\boldsymbol{x}, y) = f(\boldsymbol{x})$, $\lim_{y\to 0} u_j(\boldsymbol{x}, y) = f_j(\boldsymbol{x})$.

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