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Perfect Isometries for Blocks with Abelian Defect Groups and the Inertial Quotients Isomorphic to D_6 , $Z_4 \times Z_2$ or $Z_3 \times Z_3$

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§1 Introduction

1.1 Let p be a prime number, k an algebraically closed field of characteristic p , O a complete discrete valuation ring with residue field k and quotient field K of characteristic zero, G a finite group , b a p-block of G (i.e. a primitive idempotent of $Z(kG)$), P a defect group of b , e a root of b in $C_G(P)$ and E the inertial quotient $N_G(P,e)/P \cdot C_G(P)$ of b. Let $\hat{b}$ be the unique primitive idempotent of $Z(OG)$ corresponding to b . We assume that K is large enough for groups which we consider. Let $\tilde{b}$ be the block of $N_G(P)$ which is a Brauer correspondent of b. When P is abelian , Alperin's weight conjecture in [1] announces that the number $l(b)$ of isomorphism classes of simple kGb-modules is equal to the number $l(\tilde{b})$ of isomorphism classes of simple $kN_G(P)\tilde{b}$-modules. This is known to be true if $|E| \leq 3$ by the results of Brauer [3] (Proposition (6G)) and Usami [13] , and if $|E| = 4$ by Puig and Usami [11],[12]. Our main result here proves it in the case the inertial quotient is isomorphic to a dihedral group of order 6 , $Z_4 \times Z_2$ ($p \geq 7$) , or $Z_3 \times Z_3$ ($p \neq 2$) ([14],[15],[16]).

1.2 Actually , when P is abelian , we expect deeper categorical equivalence between b and $\tilde{b}$ than merely an equality $1(b)=1(\tilde{b})$. Let $\tilde{\tilde{b}}$ be the unique primitive idempotent of $Z(O\mathrm{N}_G(P))$ corresponding to $\tilde{b}$. When P is abelian , Broué conjectures an equivalence of the derived module categories between block algebras $OG\hat{b}$ and $O\mathrm{N}_G(P)\tilde{\tilde{b}}$ as triangulated categories(cf. Problem 6.2 in [4]) , and in particular , the existence of a perfect isometry between the generalized ordinary characters in b and $\tilde{b}$ (See Definition 1.3). By Theorem 3.1 in [4] , in general , the derived category equivalence like this between two blocks guarantees the existence of a perfect isometry and then Broué says that a perfect isometry is a shadow of a derived category equivalence at the level of group character theory. (Since only a sketch of a proof of it is given in [4] , I have explained the details of a proof in [17].) In order to define a perfect isometry let $\mathrm{CF}_O(G,b)$ be the O-module of the O-valued class functions α on G such that $\alpha(g\hat{b}) = \alpha(g)$ for any $g \in G$. Also let $\mathrm{BCF}_K(G,b)$ be the K-vector space of K-valued class functions α on G vanishing on the p-singular conjugacy classes of G and satisfying $\alpha(g\hat{b}) = \alpha(g)$ for any $g \in G$.

Definition 1.3 Let b and f be blocks of a finite group G and a finite group H respectively. A perfect isometry between b and f is a bijective isometry between the generalized ordinary characters of G in b and those of H in f which induces a bijection between $\mathrm{CF}_O(G,b)$ and $\mathrm{CF}_O(H,f)$ and also induce a bijection between $\mathrm{BCF}_K(G,b)$ and $\mathrm{BCF}_K(H,f)$.

1.4 If there is a perfect isometry between b and f , then it preserves not only the number of irreducible ordinary characters and the number of irreducible Brauer characters (i.e. $l(b)=l(f)$), but also various properties between them by Theorem 1.5 in [4] as follows. First there is an algebra isomorphism between $Z(O\hat{Gb})$ and $Z(O\hat{Hf})$, where $\hat{f}$ denotes the unique primitive idempotent of $Z(OH)$ corresponding to f. Next it preserves defect of blocks and the height of irreducible ordinary characters. Moreover , Cartan matrices for b and f determine the same quadratic form over Z and then the elementary dividers of Cartan matrix are preserved.

1.5 Since we can not compare b with $\tilde{b}$ directly , we take advantage of the following known result. By the result on blocks with normal defect groups in [8] and Proposition 14.6 in [9] , without any hypothesis on E and any hypothesis on P , a block $\tilde{b}$ of $N_G(P)$ and a block e of $N_G(P,e)$ are Morita equivalent to a suitable twisted group algebra over k of the evident extension L of E and P , simultaneously. Here we need a suitable k*-central extension $\hat{E}$ of E for this twisted group algebra. Let $\hat{L}$ denote the evident extension of $\hat{E}$ and P. Then using this $\hat{L}$ we can make $k_*\hat{L}$ denote the above twisted group algebra. When P is abelian , necessary $\hat{E}$ is defined rather easily in 2.4 in [11]. (See also Lemma 2.5 in [11].)

We would like to construct a perfect isometry between kGb (respectively $kN_G(P)\tilde{b}$) and $k_*\hat{L}$ in our case. In each known case ($|E| \leq 4$), a perfect isometry was constructed between kGb and $k_*\hat{L}$ explicitely or implicitely. When E is dihedral of order 6 ,

any k^*-central extension of E splits. When E is isomorphic to $Z_4 \times Z_2$ (or $Z_3 \times Z_3$) , there are exactly two possible k^*-central extensions of E ; splitting one and the unique non-splitting one. If $\hat{E}$ splits , then $k_*\hat{L} \cong kL$ and kL has the unique block. Whether $\hat{E}$ splits or not , fortunately there are a finite subgroup L' (a central extension of L by a p'-group) of $\hat{L}$ and a block b' of kL' such that $O_*\hat{L} \cong OL'\hat{b}'$ as algebras by Lemma 5.5 and Proposition 5.15 in [9], where $\hat{b}'$ is the primitive idempotent of Z(OL') corresponding to b' (cf. Remark 5 in section 1 in [6] and 2.13 in [11]).

1.6 Aside from categorical equivalence as general rings , we need a concept, an <u>isotypy</u> which goes beyond a perfect isometry by imposing local group theoretic condition. In fact in the known case ($|E| \leqq 4$), it was shown implicitely that b and $\tilde{b}$ are isotypic, and we also show that in our case. In the next section we will introduce definitions of an isotypy and some related concepts. Most of them are introduced in Broué's [4] and summalized in [7].

§2 Isotypy

2.1 We need some preparations. In this section we state some definitions in general situation.

Let G be a finite group. Let $\hat{b}$ and b be corresponding blocks of OG and kG. In Alperin and Broué's [2] , a b-<u>subpair</u> is a pair (Q,u) where Q is a p-subgroup of G, u is a block of $kC_G(Q)$ and

$(1,b) \subseteqq (Q,u)$ in G (i.e. $u^G = b$). A b-element is a pair (x,v) where x is a p-element of G , v is a block of $kC_G(x)$ and $(\langle x\rangle, v)$ is a b-subpair. As in [4] and [7] we also define $\hat{b}$-subpair to be a pair $(Q,\hat{u})$ where $\hat{u}$ is a block of $OC_G(Q)$ corresponding to u and (Q,u) is a b-subpair. We note that if (P,e) is a maximal b-subpair of G , then P is a defect group of b and e is a root of b. We also note that if a maximal b-subpair (P,e) is fixed, then each $Q \subseteqq P$ determines a unique block b_Q such that $(Q,b_Q) \subseteqq (P,e)$ in G. (i.e. $b_Q = e^{C_G(Q)}$)

Definition 2.2 The Brauer category $\mathbf{Br}_{\hat{b}}(G)$ has for objects the $\hat{b}$-subpairs $(Q,\hat{u})$ of G, and morphisms $(Q,\hat{u}) \longrightarrow (R,\hat{v})$ the maps in Hom(Q,R) induced by elements g of G such that $(Q,\hat{u})^g \subseteqq (R,\hat{v})$ in G. Let (P,e) be a maximal b-subpair of G and let $\mathbf{Br}_{\hat{b},P}(G)$ be the full subcategory of $\mathbf{Br}_{\hat{b}}(G)$ whose objects are the $\hat{b}$-subpairs $(Q,\hat{b}_Q)$ of G contained in $(P,\hat{e})$.

2.3 Let $CF_K(G)$ be the K-vector space of K-valued class functions on G, and let $CF_K(G,b)$ be the subspace of $CF_K(G)$ of class functions α such that $\alpha(g\hat{b}) = \alpha(g)$ for any $g \in G$. For each b-element (x,v) we define the decomposition map

$$d_G^{(x,v)} : CF_K(G) \longrightarrow BCF_K(C_G(x),v)$$

by $d_G^{(x,v)}(\alpha)(x') = \alpha(xx'\hat{v})$ for α in $CF_K(G)$ and p'-element x' in $C_G(x)$. If (x,v) is a b-element, then $d_G^{(x,v)}(\alpha) \neq 0$ only if $\alpha \in CF_K(G,b)$ by Brauer's Second Main Theorem.

Definition 2.4 (Definition 4.3 and Definition 4.6 in [4])
Let G and H be finite groups, let b and f be a block of G and a block of H, and P be a defect group of both b and f. (Here we fix a maximal b-subpair (P,e) of G setting $b_P = e$. We also fix a maximal b-subpair (P,f_P) of H.) The blocks b and f are isotypic if the following conditions hold;

(i) The inclusion of P into G and H induces an equivalence of the Brauer categories $\mathbf{Br}_{b,P}^{\wedge}(G)$ and $\mathbf{Br}_{f,P}^{\wedge}(H)$.

(ii) There exists a perfect isometry I^Q from the generalized ordinary characters of $C_H(Q)$ in f_Q to the generalized ordinary characters of $C_G(Q)$ in b_Q for each cyclic subgroup Q of P such that

$$d_G^{(x,b_Q)} \circ I^{<1>} = I_{p'}^{Q} \circ d_H^{(x,f_Q)} \qquad 2.4.1$$

for all generators x of Q, where $I_{p'}^{Q}$ is an induced K-linear map from I^Q between $BCF_K(C_H(Q),f_Q)$ and $BCF_K(C_G(Q),b_Q)$.

Note that $I^{<1>}$ is a perfect isometry between f and b . With this family of perfect isometries we call $I^{<1>}$ an isotypy between f and b.

Definition 2.5 (Remark 2 after Definition 4.6 in [4])
Broué also proposed a "good definition" of isotypic , which requires condition (i) above and following condition (ii'):

(ii') There exists a perfect isometry I^Q from the generalized ordinary characters of $C_H(Q)$ in f_Q to the generalized ordinary characters of $C_G(Q)$ in b_Q for each subgroup Q of P such that

$$d_{C_G(Q)}^{(z,b_{Q<z>})} \cdot I^Q = I_{p'}^{Q<z>} \cdot d_{C_H(Q)}^{(z,f_{Q<z>})} \qquad 2.5.1$$

for any element z in $C_P(Q)$.

2.6 There are typical examples of an isotypy. A p-nilpotent block b of G with a defect group P and the unique block of P are isotypic by Theorem 5.2 in [4]. (Actually in this case , the O-algebra $O\mathrm{G}\hat{b}$ is isomorphic to a full matrix algebra over OP and then $O\mathrm{G}\hat{b}$ is Morita equivalent to OP by Puig's 1.4.1 in [10]. If b is a block of G with a cyclic defect group P , then b , $\tilde{b}$ (a Brauer correspondent of b) and e , considered as blocks of G, $N_G(P)$ and $N_G(P,e)$ are isotypic, where e is a root of b by Linckelmann's Theorem 5.3 in [4].

Note that if a defect group of b is abelian, then above condition (i) for b and $\tilde{b}$ (respectively a block e of $N_G(P,e)$) always holds, since $N_G(P,e)$ controles the fusion of b-subpairs by Proposition 4.21 in [2]. Then Broué has posed the following conjecture:

Broué's Conjecture 2.7 (Conjecture 6.1 in [4]) Let b be a block of a finite group G with abelian defect group P and let (P,e) be a maximal b-subpair of G. Then b and e , considered as blocks of G and $N_G(P,e)$ are isotypic.

2.8 Here we introduce recent results on Broué's Conjecture 2.7. Fong and Harris have proved that if G is a finite group

with abelian Sylow 2-subgroup P , then the principal 2-blocks of G and $N_G(P)$ are isotypic.([7]) (They used "Classification of finite simple groups".) Broué, Malle and Michel have proved the following result in [5] ; Let $\mathbf{G}^F$ be a finite reductive group.(Let $\mathbf{G}$ be a connected reductive algebraic group over an algebraic closure of a finite field $\mathbf{F}_q$ and $F: \mathbf{G} \longrightarrow \mathbf{G}$ be a Frobenius endomorphism defining a rational structure on this finite field and $\mathbf{G}^F$ be the group of rational points.) If r is a " large" prime number which does not divide q , then for any unipotent r-block of $\mathbf{G}^F$ Broué's Conjecture 2.7 holds. (A prime number r is "large" means that there exists a unique positive integer d such that r is a diviser of $\Phi_d(q)$ where $\Phi_d(x)$ is a cyclotomic factor of the "polinomial order" of $\mathbf{G}^F$.) Now we add our theorem.

Theorem 2.9 Let G be a finite group, b be a p-block with an abelian defect group P and E ($= N_G(P,e)/P \cdot C_G(P)$) be the inertial quotient of b where e is a root of b in $C_G(P)$. If E is isomorphic to a dihedral group of order 6 , $Z_4 \times Z_2$ ($p \geq 7$ in this case) , or $Z_3 \times Z_3$ ($p \neq 2$ in this case) , then b , its Brauer correspondent $\tilde{b}$ and e, considered as blocks of G , $N_G(P)$ and $N_G(P,e)$ respectively , are all isotypic.

§3 (G,b)-local system

3.1 As we mentioned in 1.5 , we will construct an isotypy between b and the corresponding block b' of L' determined by $k_*\hat{L}$.

Whether $\hat{E}$ splits or not, we can treat $\hat{L}$ and a pair L' and b' equivalently. Note that P is the normal p-Sylow subgroup of L' and P is a defect group of b' and for any p-subgroup Q of P, $Br_Q(b')$ is still a block of $C_{L'}(Q)$, where Br_Q is a Brauer homomorphism (cf. section 2 in [2]). Consequently we identify the b'-subpairs with the corresponding p-subgroups of L' (i.e. all the p-subgroups of P) , and we omit to mention each block. (See 2.13 in [11].)

3.2 Now for an isotypy in " good definition " , we have to construct a family of perfect isometries satisfying condition (2.5.1) in (ii').A family of bijections $\{ I_p^Q, \mid$ all cyclic subgroups $Q \subseteqq P\}$ satisfying (2.4.1) in (ii) is called a local system by Broué in Definition 4.3 in [4]. But we employ a slightly different notation as below.

We have developed " general part " of our method in section 3 in [11] , assuming only that P is abelian. Here we summalize it briefly. Before we start , we need some notation. Let $BCF_K(G)$ be a subspace of $CF_K(G)$ of the class functions vanishing on the p-singular conjugacy classes of G. For a p-element x we define the twisted restriction $d_G^x : CF_K(G) \longrightarrow BCF_K(C_G(x))$ by $d_G^x(\alpha)(x')= \alpha(xx')$ for $\alpha \in CF_K(G)$ and any p'-element $x' \in C_G(x)$, and let $e_G^x : BCF_K(C_G(x)) \longrightarrow CF_K(G)$ be its adjoint map. Similarly we also use $d_{\hat{L}}^x$, interpreting it as one for a finite group by 3.1.

3.3 Let X be an E-stable non-empty set of subgroups of P satisfying the following condition

If $Q \in X$ and $Q \subseteqq R \subseteqq P$, then $R \in X$,

and let $\Gamma = (\Gamma_Q)_{Q \in X}$ be a family of isometries

$$\Gamma_Q : BCF_K(C_{\hat{L}}(Q)) \longrightarrow BCF_K(C_G(Q), b_Q).$$

For any subgroup Q of P , let T_Q be a transversal for the orbits of $C_E(Q)$ on P. Then for $Q \in X$ the map

$$\Delta_Q : CF_K(C_{\hat{L}}(Q) \longrightarrow CF_K(C_G(Q), b_Q),$$

$$\eta \longmapsto \sum_{z \in T_Q} e^z_{C_G(Q)}(\Gamma_{Q<z>}(d^z_{C_{\hat{L}}(Q)}(\eta))), \qquad 3.3.1$$

is a bijiective isometry. (In (3.3.1) making use of the twisted restrictions and their adjoint maps ,we have glued up a subset of $\{\Gamma_{Q<z>}\}_{Q<z> \in X}$.) We say that Γ is E-equivalent if for any $Q \in X$, any $\eta \in BCF_K(C_{\hat{L}}(Q))$ and any $s \in E$ we have $\Gamma_Q(\eta)^s = \Gamma_{Q^s}(\eta^s)$. (This condition guarantees that Δ_Q does not depend on the choice of T_Q.) We call the family $\Gamma = \{\Gamma_Q\}_{Q \in X}$ a (G,b)-<u>local system</u> over X if it is E-equivalent and if Δ_Q , for $Q \in X$, maps generalized characters to generalized characters. From (3.3.1) also notice that for $Q \in X$ Γ_Q is an induced K-linear map from Δ_Q between $BCF_K(C_{\hat{L}}(Q))$ and $BCF_K(C_G(Q), b_Q)$. Now we can easily prove that (3.3.1) and the definition of a (G,b)-local system imply that if Γ is a (G,b)-local system over X , then for any $Q \in X$, Δ_Q is a perfect isometry and it can be I^Q satisfying (2.5.1) in (ii') (cf.3.3 in [11] and 3.1). For $X = \{P\}$, a (G,b)-local system over X exists. The idea therefore is to extend an arbitrary (G,b)-local system to one over the set of all subgroups of P (with some modification , if necessary).

3.4 Thus suppose that X is arbitrary, but does not contain all subgroups of P. We choose a subgroup Q of P maximal with respect to $Q \notin X$, and X' be the union of X with the E-orbits of Q. Then the map

$$\Delta_Q^o : CF_K(C_{\hat{L}}(Q)) \longrightarrow CF_K(C_G(Q), b_Q)$$
$$\eta \longmapsto \sum_{z \in T_Q \setminus Q} e^z_{C_G(Q)}(\Gamma_{Q<z>}(d^z_{C_{\hat{L}}(Q)}(\eta))),$$

induces a bijective isometry

$$\bar{\Delta}_Q^o : CF_K^o(\bar{C}_{\hat{L}}(Q)) \longrightarrow CF_K^o(\bar{C}_G(Q), \bar{b}_Q)$$

between the sets of K-valued class functions on $\bar{C}_{\hat{L}}(Q) = C_{\hat{L}}(Q)/Q$ and $\bar{C}_G(Q) = C_G(Q)/Q$, respectively, vanishing on the p'-conjugacy classes; here $\bar{b}_Q$ denotes the image of b_Q in $k\bar{C}_G(Q)$. Moreover $\bar{\Delta}_Q^o$ maps generalized characters to generalized characters by Proposition 3.7 in [11]. In order to show that Γ extends to a (G,b)-local system over X' it suffices to show that $\bar{\Delta}_Q^o$ extends to an $N_E(Q)$-stable isometry

$$\bar{\Delta}_Q : CF_K(\bar{C}_{\hat{L}}(Q)) \longrightarrow CF_K(\bar{C}_G(Q), \bar{b}_Q)$$

mapping generalized characters to generalized characters.(Then it will follow that $\bar{\Delta}_Q$ is bijective.) (cf. Proposition 3.11 [11])

Only the construction of $\bar{\Delta}_Q$ is the task of each case and we use induction on $|G|$ and a case-by-case analysis according to the structure of $C_E(Q)$ and its action on P/Q. (Note that $\bar{b}_Q$ has an abelian defect group P/Q and the inertial quotient $C_E(Q)$. Hence if $|C_E(Q)| \leqq 4$, then we can make use of the known results.)

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