Spectrum and connectivity of graphs

A.E. Brouwer

(talk in Kyoto, 931119)

**Problem** Let $\Gamma$ be a nice graph. Show that $\Gamma$ is very connected.

In this talk I would like to give three examples of results about the connectivity of a graph that follow by considering its spectrum.

Three measures of connectivity play a rôle here:

(i) is $\Gamma$ connected or not?
(ii) $\kappa(\Gamma)$, the vertex connectivity of $\Gamma$, that is, the minimum number of vertices that one has to remove in order to disconnect $\Gamma$.
(iii) $t(\Gamma)$, the toughness of $\Gamma$, is defined as

$$\min_{S} \frac{|S|}{c(\Gamma \setminus S)}$$

where $S$ runs over all sets of vertices such that $\Gamma \setminus S$ is disconnected, and $c(\Gamma \setminus S)$ is its number of connected components.

The graph $K_0$ without vertices is not connected (we have $c(K_0) = 0$, while $c(\Gamma) = 1$ for connected graphs $\Gamma$) but I shall leave undefined whether it is disconnected, and hence do not define $\kappa(\Gamma)$ and $t(\Gamma)$ when $\Gamma$ is complete.

For example, for the Petersen graph we find $\kappa(\Gamma) = 3$ and $t(\Gamma) = \frac{4}{3}$. More generally, we clearly have $\kappa(\Gamma) \leq k(\Gamma)$ if $k(\Gamma)$ is the (minimal) valency of $\Gamma$. One may also ask about the size of ‘nonlocal’ cut sets. For example,

(1) ('unimodality') Is it true that if $S$ is a cut set of $\Gamma$, with separation $\Gamma \setminus S = A + B$, then $\min(|\Gamma(S) \cap A|, |\Gamma(S) \cap B|) \leq |S|$? (Here $\Gamma(S)$ denotes the set of all vertices adjacent to some vertex of $S$.)
[Jack Koolen remarks that some condition is necessary: for each $i$, $0 \leq i \leq 17$, the Biggs-Smith graph has a cut set $S$ of size 17 such that $|\Gamma(S) \cap A| = 17 + i$, $|\Gamma(S) \cap B| = 34 - i$.]

(2) Show that $|S|$ is substantially larger than $k$ when $S$ is nonlocal (say, given a lower bound on the size or the minimum valency of each component of $\Gamma \setminus S$).

1 The connectivity of strongly regular graphs

**Theorem 1.1** (Brouwer & Mesner [4]) Let $\Gamma$ be strongly regular of valency $k$. Then $\kappa(\Gamma) = k$, and the only cut sets of size $k$ are the point neighbourhoods.

Open problems are for example:

(3) Prove the above result for distance-regular graphs.

(4) Let $\Gamma$ be strongly regular with parameters $(v, k, \lambda, \mu)$, and let $S$ be a disconnecting set not containing any point neighbourhood $\Gamma(x)$. Show that $|S| \geq 2k - 2 - \lambda$.

(5) Let $S$ be a disconnecting set such that $|S \cap \Gamma(x)| \leq \alpha k$ for some fixed $\alpha$, $0 < \alpha < 1$, and all vertices $x$ of $\Gamma$. Prove a superlinear (in $k$) lower bound for $|S|$.

**Note** (added July '94): Brouwer & Mulder [5] showed $\kappa(\Gamma) = k$ for graphs with the property that any two distinct vertices have either 0 or 2 common neighbours. This settles (3) in the case $\lambda \in \{0, 2\}$, $\mu = 2$.

2 The connectedness of generic pieces of generalized polygons

**Theorem 2.1** (Brouwer [2]) Let $\Gamma$ be the point graph or the flag graph of a finite generalized polygon. Then the subgraph $\Delta$ of $\Gamma$ induced on the set of all vertices far away from ('in general position w.r.t.') a point or flag is connected, except in the cases $G_2(2)$, $^2F_4(2)$ and (for the flag graph) $B_2(2)$, $G_2(3)$. A similar result holds more generally for the complement of a geometric hyperplane.
Open problems:
(6) Generalize this to near polygons.
(7) Generalize this to distance-regular graphs.

It is very easy to see that in a strongly regular graph the subgraph on the vertices far away from a point is connected (except when the graph is complete multipartite).

3 The toughness of a regular graph

**Theorem 3.1** (Alon-Brouwer, cf. [1, 3]) Let $\Gamma$ be a graph on $v$ vertices, regular of valency $k$, and with eigenvalues $k = \theta_1 \geq \theta_2 \geq \ldots \geq \theta_v$. Put $$\lambda := \max_{2 \leq j \leq v} |\theta_j|.$$ Then $$t(\Gamma) > \frac{k}{\lambda} - 2.$$ 

Open problems:
(8) Prove $t(\Gamma) > \frac{k}{\lambda} - 1$. (I conjecture that this is the right bound.)
(9) Prove $t(\Gamma) = \frac{k}{\lambda}$ in many cases.

**Examples** We have bipartite graphs of small toughness, so the ‘$-1$’ would be best possible. The Delsarte-Hoffman bound for cocliques $C$ in strongly regular graphs states $$|C| \leq \frac{v}{1 + k/(-\theta_v)}.$$ If equality holds, and $\lambda = -\theta_v$ (as is often the case), then we find with $S = \Gamma \setminus C$: $t(\Gamma) \leq (v - |C|)/|C| = \frac{k}{\lambda}$.

4 Tools

How are these results proved? Essentially, only interlacing (cf. Haemers [6]) is used. Interlacing comes in two main forms:

(i) If $\Delta$ is an induced subgraph of a graph $\Gamma$, then the eigenvalues $\eta_j$ ($1 \leq j \leq u$) of $\Delta$ interlace the eigenvalues $\theta_i$ ($1 \leq i \leq v$) of $\Gamma$: we have $\theta_i \geq \eta_i$ ($1 \leq i \leq u$) and $\eta_{u-j} \geq \theta_{v-j}$ ($0 \leq j \leq u - 1$).
(ii) Given a partition $\Pi$ of the index set of a symmetric matrix $A$, let $B = (B_{R,S})_{R,S \in \Pi}$ be the matrix of average row sums of the corresponding submatrices of $A$. Then the eigenvalues of $B$ interlace those of $A$.

**Examples**

**Lemma 4.1** The average valency of a graph is not more than its largest eigenvalue.

**Proof:** Use a partition with 1 part. $\square$

**Lemma 4.2** Let $\Gamma$ be regular of valency $k$ on $v$ vertices, and let the graph induced on the $r$-set $R$ have average valency $k_R$. Then

$$\theta_2 \geq (vk_R - rk)/(v - r) \geq \theta_v,$$

(and hence

$$r \leq v(k_R - \theta_v)/(k - \theta_v).$$

For $k_R = 0$ we find the Delsarte-Hoffman bound).

**Proof:** Use a partition with 2 parts. $\square$

**Lemma 4.3** Let $\Gamma$ and $R$ be as before. Put $\lambda = \max(\vert\theta_2\vert, \vert\theta_v\vert)$. Then

$$\sum_x (\vert\Gamma(x) \cap R\vert - \frac{rk}{v})^2 \leq \lambda^2 r(v - r)/v.$$ 

**Proof:** Use a partition with 2 parts, and apply to $A^2$, the square of the adjacency matrix of $\Gamma$. $\square$

## 5 Proofs of the results in sections 1,2,3

Let $\Gamma$ have eigenvalues $k = \theta_1 \gg \theta_2 \geq \ldots$. If $\Delta$ is a disconnected subgraph, then its spectrum is the union of the spectra of its components. Each component has a largest eigenvalue at least as large as its average degree, and by interlacing it follows that all components except perhaps one have average degree at most $\theta_2$, but this is much too small (except when $\Gamma$ is very small).

This proves the results of Sections 1 and 2. For those of Section 3, use the above three Lemmata and compute.
References


