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REACHABLE SET OF SEMILINEAR RETARDED CONTROL SYSTEM

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1. Introduction

In this paper we deal with control problem for semilinear parabolic type equation in Hilbert space H as follows.

(1.1)
$$\begin{aligned}\frac{d}{dt}x(t) =&A_0x(t)+A_1x(t-h)+\int_{-h}^{0}a(s)A_2x(t+s)ds\\ &+f(t,x(t))+B_0u(t), \quad t\in(0,T].\end{aligned}$$

Let A_0 be the operator associated with a sesquilinear form defined on $V\times V$ satisfying Gårding's inequality:

$$(A_0u,v)=-a(u,v), \quad u,\ v\in V$$

where V is a Hilbert space such that $V\subset H\subset V^*$. Then A_0 generates an analytic semigroup in both H and V^* and so the equation (1.1) may be considered as an equation in both H and V^*. Let the operators A_1 and A_2 be a bounded linear operators from V to V^* and $a(\cdot)$ be Hölder continous. The nonlinear operator f from $\mathcal{R}\times V$ to H is Lipschitz continuous.

The first part of this paper is to give wellposedness and regularity in section 2. Our approach is closed to that in [1,2] mentioned above. For the semilinear system (1.2), we will give the result by using the intermediate property and contraction mapping principle. Next, under more generalized the range condition of the controller than of in [6,8,9], we establish that the approximate controllability for semilinear system is equivalent to that of its corresponding linear system. in section 3. This is to seek the equivalence between the reachable trajectory set of the semilinear system and that of the associated with linear system.

2. Wellposedness and regularity

We consider the problem of control for the following retarded functional differential equation of parabolic type with nonlinear term

(2.1)
$$\begin{aligned}\frac{d}{dt}x(t) =& A_0x(t) + A_2x(t-h) + \int_{-h}^{0} a(s)A_2x(t+s)ds \\ &+ f(t,x(t)) + B_0u(t),\end{aligned}$$

(2.2) $$x(0) = g^0, \quad x(s) = g^1(s), \quad s \in [-h, 0).$$

in Hilbert space in H. Let V be another Hilbert space such that $V \subset H \subset V^*$. The notations $|\cdot|$, $||\cdot||$ denote the norms of H, V respectively as usual. Let $a(u,v)$ be a bounded sesquilinear form defined in $V \times V$ satisfying Gårding's inequality

(2.3) $$\text{Re } a(u,u) \geq c_0||u||^2 - c_1|u|^2, \quad c_0 > 0, \quad c_1 \geq 0.$$

Let A_0 be the operator associated with a sesquilinear form

(2.4) $$(A_0u, v) = -a(u,v), \quad u, \ v \in V.$$

Then the operator A_0 is a bounded linear from V to V^*. The operators A_1 and A_2 are bounded linear operators from V to V^* such that they map $D(A_0)$ into H. We may assume that $(D(A_0), H)_{1/2,2} = V$ satisfying

(2.5) $$||u|| \leq C_1||u||_{D(A_0)}^{1/2}|u|^{1/2}$$

for some a constant $C_1 > 0$ where $(D(A_0), H)_{\theta,p}$ denotes the real interpolation space between $D(A_0)$ and H. The function $a(\cdot)$ is assumed to be a real valued Hölder continous in $[-h, 0]$ and the controller operator B_0 is a bounded linear operator from some Banach space U to H. Let f be a nonlinear mapping from $\mathcal{R} \times V$ into H. We assume that for any x_1, $x_2 \in V$ there exists a constant $L > 0$ such that

(2.6) $$|f(t,x_1) - f(f,x_2)| \leq L||x_1 - x_2||$$

(2.7) $$f(t,0) = 0.$$

Assume that (2.3) holds for $c_1 = 0$. Noting that $A_0 + c_1$ is an isomorphism from V to V^* if $c_1 \neq 0$.

THEOREM 2.1. *Under the above assumptions for the nonlinear mapping f, then there exists a unique solution x of (2.1) and (2.2) such that*

$$x \in L^2(0,T;V) \cap W^{1,2}(0,T;V^*) \subset C([0,T];H).$$

for any $g = (g^0, g^1) \in Z = H \times L^2(-h,0;V)$. Moreover, there exists a constant C such that

$$||x||_{L^2(0,T;V)\cap W^{1,2}(0,T;V^*)} \leq C(|g^0| + ||g^1||_{L^2(-h,0;V)} + ||u||_{L^2(0,T;U)}),$$

where

$$||\cdot||_{L^2(0,T;V)\cap W^{1,2}(0,T;V^*)} = \max\{||\cdot||_{L^2(0,T;V)}, ||\cdot||_{W^{1,2}(0,T;V^*)}\}.$$

The proof will be shown a little later on. From now on, we consider the estimate of a solution of the problem (2.1) and (2.2) in accordance with the result of theorem 3.3 of [1] if it exists. For $T > 0$ it is easily seen that by interpolation theory

$$H = \{x \in V^* : \int_0^T ||A_0 e^{tA_0} x||_*^2 dt < \infty\},$$

where $||\cdot||_*$ is the norm of the element of V^*.

Identifying the antidual of H with H we may consider $V \subset H \subset V^*$. The realization of A_0 in H which is the restriction of A_0 to

$$D(A_0) = \{u \in V : A_0 u \in H\}$$

is also denoted by A_0. It is known that A_0 generates an analytic semigroup in both H and V^*. Replacing intermediate space F in the paper [1] with the space H, we can derive the results of G. Blasio, K. Kunisch and E. Sinestrari [1] regarding term by term to deduce the following result.

PROPOSITION 2.1. *Let $g = (g^0, g^1) \in Z = H \times L^2(-h,0;V)$ and $f \in L^2(0,T;V^*)$. Then for each $T > 0$, a solution x of the equation (2.1) and (2.2) belongs to*

$$L^2(0,T;V) \cap W^{1,2}(0,T;V^*) \subset C([0,T];H).$$

Moreover, for some constant C_T we have

$$\begin{aligned}\|x\|_{L^2(0,T;V)\cap W^{1,2}(0,T;V^*)} \le & C_T(|g^0| + \|g^1\|_{L^2(-h,0;V)} \\ & + \|f\|_{L^2(0,T;V^*)} + \|u\|_{L^2(0,T;U)}).\end{aligned}$$

The Proof of Theorem 2.1. Let us fix $T \in (0,h)$ such that

$$C_1 C_T L(T/\sqrt{2})^{\frac{1}{2}} < 1. \tag{2.8}$$

For $i = 1\ 2$, we consider the following equation.

$$\begin{aligned}\frac{d}{dt}y_i(t) =& A_0 y_i(t) + A_1 y_i(t-h) + \int_{-h}^{0} a(s)A_2 y_i(t+s)ds \\ & + f(t, x_i(t)) + B_0 u(t), \qquad t \in (0,T] \\ y_i(0) =& g^0, \quad y_i(s) = g^1(s), \qquad s \in [-h, 0).\end{aligned}$$

Then

$$\begin{aligned}\frac{d}{dt}(y_1(t) - y_2(t)) =& A_0(y_1(t) - y_2(t)) + A_1(y_1(t-h) - y_2(t-h)) \\ & + \int_{-h}^{0} a(s)A_2(y_1(t+s) - y_2(t+s))ds \\ & + f(t, x_1(t)) - f(t, x_2(t)), \qquad t \in (0,T] \\ y_1(0) - y_2(0) =& 0, \quad y_1(s) - y_2(s) = 0, \qquad s \in [-h, 0).\end{aligned}$$

From Theorem 3.3 of [1] and (2.6) it follows that

$$\|y_1 - y_2\|_{L^2(0,T;D(A_0))\cap W^{1,2}(0,T;H)} \le C_T \|f(\cdot, x_1) - f(\cdot, x_2)\|_{L^2(0,T;H)},$$
$$\|f(\cdot, x_1) - f(\cdot, x_2)\|_{L^2(0,T;H)} \le L\|x_1 - x_2\|_{L^2(0,T;V)}.$$

Using the Hölder inequality we also obtain that

(2.9)

$$\begin{aligned}\|y_1 - y_2\|_{L^2(0,T;H)} =& \{\int_0^T |y_1(t) - y_2(t)|^2 dt\}^{\frac{1}{2}} \\ \le & \{\int_0^T t \int_0^t |\dot{y}_1(\tau) - \dot{y}_2(\tau)|^2 d\tau dt\}^{\frac{1}{2}} \\ \le & \frac{\sqrt{T}}{2}\|y_1 - y_2\|_{W^{1,2}(0,T;H)}.\end{aligned}$$

Therefore, in terms of (2.5) and (2.9) we have

$$\begin{aligned}||y_1 - y_2||_{L^2(0,T;V)} &\le C_1||y_1 - y_2||^{\frac{1}{2}}_{L^2(0,T;D(A_0))}||y_1 - y_2||^{\frac{1}{2}}_{L^2(0,T;H)} \\ &\le C_1C_T(\frac{T}{\sqrt{2}})^{\frac{1}{2}}||f(\cdot,x_1) - f(\cdot,x_2)||_{L^2(0,T:H)} \\ &\le C_1C_TL(\frac{T}{\sqrt{2}})^{\frac{1}{2}}||x_1 - x_2||_{L^2(0,T;V)}.\end{aligned}$$

So by virtue of the condition (2.8) the contraction principle gives that the equation of (2.1) and (2.2) has a unique solution in $[-h,T]$.

Let $x(\cdot)$ be a solution of (2.1) and (2.2) and $y(\cdot)$ be a solution of following equation.

$$\begin{aligned}\frac{d}{dt}y(t) =& A_0y(t) + A_1y(t-h)\int_{-h}^{0} a(s)A_2y(t+s)ds \\ &+ B_0u(t), && t \in (0,T] \\ y(0) =& g^0, \quad y(s) = g^1(s), && s \in [-h,0).\end{aligned}$$

Consider the following problem:

$$\begin{aligned}\frac{d}{dt}(x(t) - y(t)) =& A_0(x(t) - y(t)) + A_1(x(t-h) - y(t-h)) \\ &+ \int_{-h}^{0} a(s)A_2(x(t+s) - y(t+s))ds + f(t,x(t)), \\ x(0) - y(0) =& 0, \quad x(s) - y(s) = 0 && s \in [-h,0).\end{aligned}$$

In virtue of Theorem 3.3 of [1] we have

$$\begin{aligned}||x - y||_{L^2(0,T;D(A_0))\cap W^{1,2}(0,T;H)} &\le C_T||f(\cdot,x)||_{L^2(0,T;H)} \\ &\le C_TL||x||_{L^2(0,T:V)} \\ &\le C_TL(||x - y||_{L^2(0,T;V)} + ||y||_{L^2(0,T;V)}).\end{aligned}$$

Combining (2.5), (2.9) and above inequality we have

$$\begin{aligned}||x - y||_{L^2(0,T;V)} &\le C_1||x - y||^{\frac{1}{2}}_{L^2(0,T;D(A_0))}||x - y||^{\frac{1}{2}}_{L^2(0,T;H)} \\ &\le C_1(\frac{T}{\sqrt{2}})^{\frac{1}{2}}C_TL(||x - y||_{L^2(0,T;V)} + ||y||_{L^2(0,T;V)}).\end{aligned}$$

Therefore, we have

$$
(2.10)\quad
\begin{aligned}
\|x-y\|_{L^2(0,T;V)} &\le \frac{C_1C_TL(\frac{T}{\sqrt{2}})^{\frac{1}{2}}}{1-C_1C_TL(\frac{T}{\sqrt{2}})^{\frac{1}{2}}}\|y\|_{L^2(0,T;V)},\\
\|x\|_{L^2(0,T;V)} &\le \frac{1}{1-C_1C_TL(\frac{T}{\sqrt{2}})^{\frac{1}{2}}}\|y\|_{L^2(0,T;V)}.
\end{aligned}
$$

Combining Proposition 2.1 and (2.10) we obtain

$$
\begin{aligned}
\|x\|_{L^2(0,T;V)\cap W^{1,2}(0,T;V^*)} \le& C_T(|g^0| + \|g^1\|_{L^2(0,T;V)} + L\|x\|_{L^2(0,T;V)}\\
&+ \|u\|_{L^2(0,T;U)})\\
\le& C_T(|g_0| + \|g^1\|_{L^2(0,T;V)} + \|u\|_{L^2(0,T:U)}\\
&+ \frac{L}{1-C_1C_TL(\frac{T}{\sqrt{2}})^{\frac{1}{2}}}\|y\|_{L^2(0,T;V)})\\
\le& C_T(|g_0| + \|g^1\|_{L^2(0,T;V)} + \|u\|_{L^2(0,T:U)})\\
&+ \frac{LC_T}{1-C_1C_TL(\frac{T}{\sqrt{2}})^{\frac{1}{2}}}(|g^0| + \|g^1\|_{L^2(0,T;V)}\\
&+ \|u\|_{L^2(0,T:U)})\\
\le& C(|g_0| + \|g^1\|_{L^2(0,T;V)} + \|u\|_{L^2(0,T:U)}).
\end{aligned}
$$

Since the condition (2.8) is independent of initial value, the solution of (2.1) and (2.2) can be extended to the interval $[-h, nT]$ for n is a natual number, and so the proof is complete.

3. Approximate controllability for linear system

In this section we consider the approximate controllability of retarded system with nonmlinear term. The fundamental solution $W(t)$ of the equation (2.1) and (2.2) is defined as follows:

$$
\begin{aligned}
&\frac{d}{dt}W(t) = A_0W(t) + A_1W(t-h) + \int_{-h}^{0} a(s)A_2W(t+s)ds, \ t > 0,\\
&W(0) = I, \quad W(s) = 0, \ s \in [-h, 0).
\end{aligned}
$$

Since we are assuming that $a(\cdot)$ is Hölder continuous, as is seen in [13] the fundamental solution exists. It also is known that $W(t)$ is strongly continuous and $AW(t)$ and $dW(t)/dt$ are strongly continuous except at $t = nr$, $n = 0, 1, 2, \ldots$. Therefore we may assume that

$$|W(t)| \leq M, \quad t \geq 0$$

where M is a constant. The solution of (2.1) and (2.2) is expressed by

$$x(t) = W(t)g^0 + \int_{-h}^{0} U_t(s)g^1(s)ds + \int_0^t W(t-\tau)f(\tau, x(\tau))d\tau,$$

$$U_t(s) = W(t-s-h)A_1 + \int_{-h}^{s} W(t-s+\sigma)a(\sigma)A_2 d\sigma$$

(cf. S. Nakagiri [10]).

LEMMA 3.1. *Let $f \in L^2(0,T;H)$ and $x(t) = \int_0^t W(t-s)f(s)ds$. Then there exists a constant C such that*

$$||x||_{L^2(0,T;V)} \leq C\sqrt{T}||f||_{L^2(0,T;H)}.$$

Proof. By the similary way of Theorem 2.3 of [1] it holds that

$$||x||_{L^2(0,T;D(A_0))} \leq C_T||f||_{L^2(0,T;H)}. \tag{3.1}$$

By using Hölder inequality,

$$\begin{aligned} ||x||^2_{L^2(0,T;H)} &= \int_0^T |\int_0^t W(t-s)f(s)ds|^2 dt \\ &\leq M^2 \int_0^T (\int_0^t |f(s)|ds)^2 dt \\ &\leq M^2 \int_0^T t \int_0^t |f(s)|^2 ds dt \\ &\leq M^2 \frac{T^2}{2} \int_0^t |f(s)|^2 ds. \end{aligned}$$

Therefore

$$\|x\|_{L^2(0,T;H)} \leq MT\|f\|_{L^2(0,T;H)}. \tag{3.2}$$

Combining (3.1) and (3.2) we have that

$$\|x\|^2_{L^2(0,T;V)} \leq C_T MT\|f\|^2_{L^2(0,T;H)}.$$

Let $Z = H \times L^2(-h, 0; V)$ be the state space and be a product Hilbert space with the norm

$$\|g\|_Z = (|g^0|^2 + \int_{-h}^{0} \|g^1(s)\|^2 ds)^{\frac{1}{2}}, \quad g = (g^0, g^1) \in Z.$$

Let $g \in Z$ and $x(t; g, f, B_0 u)$ be a solution of the equation (2.1) and (2.2) associated with nonlinear term f and control $B_0 u$ at time t. In view of the result of Theorem 2.1, we can define the solution semigroup for the problem (2.1) and (2.2) as follows:

$$S(t)g = (x(t; g, 0, 0), x_t(\cdot; g, 0, 0))$$

where $g = (g^0, g^1) \in Z$, $x(t; g, 0, 0)$ is the solution of (2.1) and (2.2) with $f(t, x) = 0$ and $B_0 = 0$ and $x_t(s; g, 0, 0) = x(t + s; g, 0, 0)$ defined in $[-h, 0]$. Then we have the following proposition which can show just as Theorem 4.2 of [1].

PROPOSITION 3.1. *(i) The operator $S(t)$ is a C_0-semigroup on Z. (ii) The intinitesimal generator A of $S(t)$ is characterized by*

$$\begin{aligned} D(A) =& \{g = (g^0, g^1) : g^0 \in H, \ g^1 \in L^2(-h, 0; V), \\ & g^1(0) = g^0, \ A_0 g^0 + A_1 g^1(-h) + \int_{-h}^{0} a(s) A_2 g^1(s) ds \in H\}, \\ Ag =& (A_0 g^0 + A_1 g^1(-h) + \int_{-h}^{0} a(s) A_2 g^1(s) ds, \dot{g}^1). \end{aligned}$$

Note that $a(\cdot)$ need not be Hölder continuous for the above results to hold. It has only to belong to $L^2(-h, 0)$.

For the sake of simplicity, we assume that $S(t)$ is uniformly bounded, that is, there exists a constant $M \geq 1$ such that

$$||S(t)||_Z \leq M.$$

As is seen in [7], the equation (2.1) and (2.2) can be transformed into an abstract equation

$$z(t) = Az(t) + F(z(t)) + Bu(t), \tag{3.3}$$

$$z(0) = g, \tag{3.4}$$

where $z(t) = (x(t), x_t(\cdot))$ belongs to the Hilbert space Z and $g = (g^0, g^1) \in Z$. The operator A is the infinitesimal generator of C_0-semigroup $S(t)$, $F(z(t)) = (f(t, x(t)), 0)$ and $Bu = (B_0 u, 0)$. The mild solution of initial problem (3.3) and (3.4) is the following form:

$$z(t; g, f, Bu) = S(t)g + \int_0^t S(t-s)F(z(s))ds + \int_0^t S(t-s)Bu(s)ds.$$

LEMMA 3.2. *Let $z_u(t) = z(t; g, f.u)$. Then for $0 < t < T$ there exists a constant C such that*

(1) $||F(z_u)||_{L^2(0,T;Z)} \leq C(||g||_Z + ||u||_{L^2(0,T;U)})$,

(2)

$$\begin{aligned} ||F(z_{u_1}) - F(z_{u_2})||_{L^2(0,T;Z)} &(= ||f(\cdot, x_{u_1}) - f(\cdot, x_{u_2})||_{L^2(0,T;H)}) \\ &\leq LC\sqrt{T}/(1 - LC\sqrt{T})||B(u_1 - u_2)||_{L^2(0,T;U)}. \end{aligned}$$

Proof. (1) From Theorem 2.1 it follows that

$$\begin{aligned} ||F(z_u)||_{L^2(0,T;Z)} &= ||f(t, x(t))||_{L^2(0,T;H)} \\ &\leq L||x||_{L^2(0,T;V)} \\ &\leq LC(||g||_Z + ||u||_{L^2(0,T;U)}). \end{aligned}$$

(2) From Lemma 3.1 it follows that

$$
\begin{aligned}
||F(z_{u_1}) - F(z_{u_2})||_{L^2(0,T;Z)} &= ||f(\cdot, x_{u_1}) - f(\cdot, x_{u_2})||_{L^2(0,T;H)} \\
&\leq L||x_{u_1} - x_{u_2}||_{L^2(0,T;V)} \\
&\leq L|| \int_0^t W(t-s)\{f(s, x_{u_1}(s)) - f(s, x_{u_2}(s))\}ds||_{L^2_t(0,T;V)} \\
&\quad + L|| \int_0^t W(t-s)B\{u_1(s) - u_2(s)\}ds||_{L^2_t(0,T;V)} \\
&\leq LC\sqrt{T}||f(\cdot, x_{u_1}) - f(\cdot, x_{u_2})||_{L^2(0,T;H)} \\
&\quad + LC\sqrt{T}||B(u_1 - u_2)||_{L^2(0,T;U)}
\end{aligned}
$$

where we set $||f(t)||_{L^2_t(0,T;V)} = ||f||_{L^2(0,T;V)}$. Since $||f(\cdot, x_u)||_{L^2(0,T;H)}$ $= ||F(z_u)||_{L^2(0,T;Z)}$ the proof is complete.

We define reachable sets for the system (3.3) and (3.4) as follows:

$$
\begin{aligned}
L_T(g) &= \{z(T; g, 0, Bu) : \ u \in L^2(0,T;U)\}, \\
R_T(g) &= \{z(T; g, f, Bu) : \ u \in L^2(0,T;U)\}.
\end{aligned}
$$

It is known that $L_T(0)$ is independent of T (see Lemma 7.4.1 in [12]).

We denote the bounded linear operator $L^2(0,T;Z)$ to Z by

$$
\hat{S}p = \int_0^T S(T-s)p(s)ds
$$

for $p \in L^2(0,T;Z)$. The system (3.3) and (3.4) is approximately controllable on $[0,T]$ if $\overline{R_T(g)} = Z$, that is, for any $\varepsilon > 0$ and $z \in Z$ there exists a control $u \in L^2(0,T;U)$ such that

$$
||z - S(T)g - \hat{S}F(z_u) - \hat{S}Bu|| < \varepsilon
$$

where $||\cdot||$ is a norm on Z.

We need the following hypothesis:

(B)

For any $\varepsilon > 0$ and $p^0 \in L^2(0,T;H)$ there exists a $u \in L^2(0,T;U)$ such that

$$||\hat{S}(p^0,0) - \hat{S}Bu|| < \varepsilon,$$
$$||Bu||_{L^2(0,T;Z)}(= ||B_0u||_{L^2(0,T;H)}) \le q_1||p^0||_{L^2(0,T;H)}.$$

where q_1 is a constant independent of p.

It is easily seen that if the range of the operator B is dense in Z then the condition is satisfied. Our concern is based on more general assumption than that in [6,8,9]. In [8; Example 2] it is introduced a simple example of the control operator B that satisfies assumption (B).

THEOREM 3.1. *Let us assume hypothesis (B). Then we have that* $\overline{R_T(g)} = \overline{L_T(g)}$.

Proof. Under assumption (B) it is known that $\overline{L_T(0)} = Z$ (see K. Naito [8; Lemma 2]). Therefore, we have that $S(T)g \in \overline{L_T(0)}$ and hence, $\overline{L_T(0)} = \overline{L_T(g)}$ for any initial value $g \in Z$. Now we will show that $\overline{L_T(g)} \subset \overline{R_T(g)}$. Let $z_T \in \overline{L_T(g)}$. Then for any given $\epsilon > 0$ there exists $u \in L^2(0,T;U)$ such that

$$||z_T - S(T)g - \hat{S}Bu|| \le \frac{\epsilon}{2^3}. \tag{3.5}$$

Let $v_1 \in L^2(0,T;U)$ is arbitrarily fixed. By assumption (B) there exists $v_2 \in L^2(0,T;U)$ such that

$$||\hat{S}(B_0u - f(\cdot, x_{v_1})) - \hat{S}B_0v_2|| \le \frac{\epsilon}{2^3}.$$

Since $B_0u - f(\cdot, x_{v_1}) \in L^2(0,T;H)$ is the first component of the $Bu - F(z_{v_1}) \in L^2(0,T;Z)$, we have

$$||\hat{S}(Bu - F(z_{v_1})) - \hat{S}Bv_2|| \le \frac{\epsilon}{2^3}. \tag{3.6}$$

From (3.5) and (3.6) it follows that

$$||z_T - S(T)g - \hat{S}F(z_{v_1}) - \hat{S}Bv_2|| \le \frac{\epsilon}{2^2}. \tag{3.7}$$

We can choose $w_2 \in L^2(0,T;U)$ such that

$$\|\hat{S}(F(x_{v_2}) - F(z_{v_1})) - \hat{S}Bw_2\| \leq \frac{\epsilon}{2^3}. \tag{3.8}$$

Therefore, from Lemma 3.2 it obtains that

$$\begin{aligned}
\|Bw_2\|_{L^2(0,T;Z)} &\leq q_1 \|F(x_{v_2}) - F(z_{v_1})\|_{L^2(0,T;Z)} \\
&\leq q_1 \frac{LC\sqrt{T}}{1-LC\sqrt{T}} \|Bv_2 - Bv_1\|_{L^2(0,T;Z)}.
\end{aligned}$$

Let us define $v_3 = v_2 - w_2$ in $L^2(0,T;U)$. Then from (3.7) and (3.8)

$$\|z_T - S(T)g - \hat{S}F(z_{v_2}) - \hat{S}Bv_3\| \leq (\frac{1}{2^2} + \frac{1}{2^3})\epsilon.$$

Define $v_n = v_{n-1} - w_{n-1}$ by induction. Then we have

$$\|z_T - S(T)g - \hat{S}(F(z_{v_n}) - Bv_{n+1}\| \leq (\frac{1}{2^2} + \ \dots \ + \frac{1}{2^{n+1}})\epsilon \leq \frac{1}{2}\epsilon$$

and

$$\begin{aligned}
&\|Bv_{n+1} - Bv_n\|_{L^2(0,T;Z)} \\
&\leq q_1 \frac{LC\sqrt{T}}{1-LC\sqrt{T}} \|Bv_n - Bv_{n-1}\|_{L^2(0,T;Z)}.
\end{aligned}$$

For sufficiently small T such that $LC\sqrt{T} < \min\{1/2, 1/(q_1+1)\}$, the sequence $\{Bv_n\}$ is Cauchy sequence and hence converges in $L^2(0,T;Z)$. Thus there exists some integer N such that for all $n \geq N$ we have that

$$\|\hat{S}Bv_{n+1} - \hat{S}Bv_n\| \leq \frac{1}{2}.$$

Therefore it follows that

$$\begin{aligned}
&\|z_t - S(T)g - \hat{S}F(z_{v_n}) - \hat{S}Bv_n\| \\
&\leq \|z_t - S(T)g - \hat{S}F(z_{v_n}) - \hat{S}Bv_{n+1}\| \\
&+ \|\hat{S}Bv_{n+1} - \hat{S}Bv_n\| \\
&\leq \frac{1}{2}\epsilon + \frac{1}{2}\epsilon \leq \epsilon
\end{aligned}$$

for all $n \geq N$. Hence for sufficiently small T, we have proof that $\overline{L_T(g)} \subset \overline{R_T(g)}$. But since $\overline{L_T(g)}$ is independent of the time T and initial value g, we conclude that $\overline{L_T(g)} = \overline{R_T(g)}$.

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