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On a Minimization Theorem

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完備距離空間における Ekeland の ε 変分不等式（定理 1），Caristi の不動点定理（定理 2），Takahashi の最小値定理の一般化（定理 3）について，簡単な証明を紹介します。

以上の 3 定理がすべて本質的に同じアイデアで証明できるのが興味のあるところです。なお，本稿の内容は高橋渉氏との共同研究によるものです。

以下，X は距離 d の完備距離空間，$f(x)$ は X 上で定義された下半連続かつ下に有界な任意の実数値関数を表すものとします。

定理 1. *For any* $\varepsilon > 0$ *and* $u \in X$ *with* $f(u) \leqq \inf_{x \in X} f(x) + \varepsilon$ *there exists* $v \in X$ *such that* $f(v) \leqq f(u)$, $d(u, v) \leqq 1$ *and*

$$f(x) + \varepsilon d(v, x) > f(v)$$

for all $x \in X$ *distinct from* v.

証明. *We define* $s(w) = \inf \{ f(x) \mid f(x) + \varepsilon d(w, x) \leqq f(w) \}$.

By induction we can construct a sequence $\{v_n\}$ such that $v_1 = u$ and

$$\begin{cases} f(v_{n+1}) + \varepsilon d(v_n, v_{n+1}) \leqq f(v_n), \\ f(v_{n+1}) \leqq s(v_n) + \frac{1}{n}. \end{cases}$$

Then $\{f(v_n)\}$ is a decreasing sequence which is bounded below, and converges to some α. Since for any m and n with $n \leqq m$ the inequality $\varepsilon d(v_n, v_m) \leqq f(v_n) - f(v_m)$ holds, $\{v_n\}$ is a Chauchy sequence and converges to some v. Since $f(x)$ is lower semicontinuous, $f(v) \leqq \alpha \leqq f(u)$. Since $f(v_m) \leqq f(u) \leqq \inf_{x \in X} f(x) + \varepsilon$, $\varepsilon d(u, v_m) \leqq f(u) - f(v_m) \leqq \varepsilon$ for all m. Hence $d(u, v) \leqq 1$. Suppose that $f(x) + \varepsilon d(v, x) \leqq f(v)$ for some x distinct from v. Since $\varepsilon d(v_n, v_m) \leqq f(v_n) - f(v)$ for any fixed n and all m with $n \leqq m$, $f(v) + \varepsilon d(v_n, v) \leqq f(v_n)$ for all n. Hence $f(x) + \varepsilon d(v_n, x) \leqq f(v_n)$ for all n. By the definition of $s(v_n)$ and v_{n+1}, $f(v) \leqq f(v_{n+1}) \leqq f(x) + \frac{1}{n}$ for all n. Hence $f(v) \leqq f(x)$. This is a contradiction.

定理 2. Let T be a self-map of X. If

$$f(Tx) + d(x, Tx) \leqq f(x)$$

for all $x \in X$, then T has a fixed point.

証明. We define $s(w) = \inf\{f(x) \mid f(x) + d(w, x) \leqq f(w)\}$. By induction we can construct a sequence $\{v_n\}$ such that v_1 is arbitrary and

$$\begin{cases} f(v_{n+1}) + d(v_n, v_{n+1}) \leqq f(v_n), \\ f(v_{n+1}) \leqq S(v_n) + \frac{1}{n}. \end{cases}$$

Then $\{v_n\}$ converges to some v and $f(v) + d(v_n, v) \leqq f(v_n)$ for all n. Since $f(Tv) + d(v, Tv) \leqq f(v)$, $f(Tv) + d(v_n, Tv) \leqq f(v_n)$ for all n. By the definition of $S(v_n)$ and v_{n+1}, $f(v) \leqq f(v_{n+1}) \leqq f(Tv) + \frac{1}{n}$ for all n. Hence $f(v) \leqq f(Tv)$, and then $d(v, Tv) = 0$.

定理 3. Let $g(x)$ be a real function defined on X which is bounded below. If for each $v \in X$ with $\inf_{x \in X} g(x) < g(v)$ there exists $x \in X$ distinct from v such that $f(x) + d(v, x) \leqq f(v)$, then $g(x)$ attains its infimum.

証明. We define $S(w) = \inf\{f(x) \mid f(x) + d(w, x) \leqq f(w)\}$. By induction we can construct a sequence $\{v_n\}$ such that v_1 is arbitrary and

$$\begin{cases} f(v_{n+1}) + d(v_n, v_{n+1}) \leqq f(v_n), \\ f(v_{n+1}) \leqq S(v_n) + \frac{1}{n}. \end{cases}$$

Then $\{v_n\}$ converges to some v and $f(v) + d(v_n, v) \leqq f(v_n)$ for all n. If $\inf_{x \in X} g(x) < g(v)$, there exists some x distinct from v such that $f(x) + d(v, x) \leqq f(v)$, and then $f(x) + d(v_n, x) \leqq f(v_n)$ for all n. By the definition of $S(v_n)$ and v_{n+1}, $f(v) \leqq f(v_{n+1}) \leqq f(x) + \frac{1}{n}$ for all n. Hence $f(v) \leqq f(x)$. This is a contradiction.

参考文献

[Caristi] *Fixed point theorems for mappings satisfying inwardness conditions.* Trans. Amer. Math. Soc., 215 (1976), 241-251.

[Ekeland] *Nonconvex minimization problems.* Bull. Amer. Math. Soc., 1 (1979), 443-474.

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