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Balanced Families in Compact Spaces

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1 Introduction

We shall denote by N the set $\{1,...,n\}$ and by $\mathcal{N}$ the family of the nonempty subsets of N. A subfamily $\{S_i\}_{i=1}^{p}$ of $\mathcal{N}$ is said to be *balanced* if there is a corresponding family $\{\lambda_i\}_{i=1}^{p}$ of nonnegative numbers such that $\sum_i \lambda_i \chi_{S_i} = \chi_N$, where χ_A denotes the characteristic vector of the set A, i.e., χ_A is an n-vector whose i-the coordinate is 1 if $i \in A$ and 0 if $i \notin A$.

The balancedness plays a crucial role in covering theorems of simplexes which are basic tools to prove the nonemptiness of the core of nontransferable utility games. (cf. [2], [3]) We shall examine the balancedness of a subfamily of $\mathcal{N}$ profoundly and extend the study to the case that a compact Hausdorff space is the substitute of the finite set N. The research would be expected to be a basis of the study of infinite dimensional game theory, that is, the game theory with infinitely many players.

We prepare mathematical background necessary for the arguments hereafter. Let Q be a compact Hausdorff space and let $C(Q)$ be the Banach space of all continuous real valued functions on Q with the supremum norm $\|\xi\| = \max_{q\in Q}|\xi(q)|$. Let $M(Q)$ be the Banach space of all regular signed Borel measures on Q with the norm $\|x\| = |x|(Q)$, where $|x|$ denotes the total variation of the regular signed Borel measure x on Q. Then we can regard $M(Q)$ as the dual Banach space $C(Q)'$ of $C(Q)$ by the bijection $x \mapsto \tilde{x}$ from $M(Q)$ onto $C(Q)'$ defined by

$$\tilde{x}(\xi) = \int \xi dx, \qquad \xi \in C(Q).$$

The space $M(Q)$ is equipped with the weak star topology throughout this note. We shall write $x(\xi)$ in place of $\int \xi dx$ when no confusion is likely to arise. We denote by $\mathbf{\Sigma}$ the σ-field of the Borel sets in Q. The *support* supp(x) of an element x of $M(Q)$ is defined by

$$\text{supp}(x) = Q \setminus \bigcup\{G : x(G) = 0,\ G \text{ is open}\}.$$

We introduce two binary relations $\geq$ and $\gg$ in $M(Q)$ by

$$x \geq y \text{ if } x(A) \geq y(A) \text{ for all } A \in \mathbf{\Sigma},$$

$$x \gg y \text{ if } x \geq y \text{ and } \text{supp}(x - y) = Q,$$

respectively. We shall use the symbol Δ to denote the convex subset

$$\{x \in M(Q) : \|x\| = x(1) = 1\}$$

of $M_+(Q) = \{x \in M(Q) : x \geq 0\}$, and the symbol Δ_{++} to denote the set $\{x \in \Delta : x \gg 0\}$. It may happen that the set Δ_{++} is empty. Consider a discrete uncountably infinite space Q and its one-point compactification Q^*. Let $x \in M(Q^*)$ and $x \geq 0$. Put $Q_n = \{q \in Q : x(\{q\}) \geq 1/n\}$. Since $|Q_n| \leq n\|x\|$, $\cup_{n=1}^{\infty} Q_n$ is countable and there is a point $q_0 \in Q \setminus \cup Q_n$. Thus, $x(\{q_0\}) = 0$ and $\{q_0\}$ is open. Therefore, Δ_{++} is empty.

Recall that Δ is compact and $M_+(Q)$ is closed. Moreover, if we correspond a point q in Q to the mass measure $\widehat{q}$ at q on Q, then the correspondence is into-homeomorphism. For any nonempty subsets A of Q, let Δ^A be the closed convex hull of $\{\widehat{q} : q \in A\}$. We shall use the same symbols as the finite dimensional case, but no confusion may occur.

2 Balanced families in compact spaces

We start with an examination of balanced subfamilies of $\mathcal{N}$. It is well known that a subfamily $\{S_i\}_{i=1}^{p}$ of $\mathcal{N}$ is balanced if and only if the vector χ_N/n is a convex combination of the vectors $\chi_{S_i}/|S_i|$. Geometrically this means the barycenter of the simplex Δ^N is contained in the polytope spanned by the barycenters of the faces Δ^{S_i}.

The concept of balancedness has been characterized in terms of the specific vectors such as χ_N or χ_N/n, but balancedness is free from the specification as shown in Proposition 1 below.

Let r be a point of Δ^N such that $r \gg 0$. Define a vector r^S for $S \in \mathcal{N}$ by

$$r^S = \begin{cases} r_i / \sum_{j \in S} r_j & \text{for } i \in S \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 1 *For any vector r of Δ^N such that $r \gg 0$, a subfamily $\{S_i\}_{i=1}^p$ of $\mathcal{N}$ is balanced if and only if r is a convex combination of the points $\{r^{S_i}\}_{i=1}^p$.*

Proof. Suppose that the family $\{S_i\}_{i=1}^p$ is balanced. Then there is a corresponding family $\{\lambda_i\}_{i=1}^p$ of nonnegative numbers such that $\chi_N = \sum_{i=1}^p \lambda_i \chi_{S_i}$. Multiply the diagonal matrix $(a_{ij})_{i,j=1}^n$, where $a_{ij} = r_i$ if $i = j$ and $a_{ij} = 0$ otherwise, to both sides of the equality above. Then we have

$$r = \sum_{i=1}^{p} \lambda_i \Big(\sum_{j \in S_i} r_j \Big) r^{S_i}$$

and $\sum_{i=1}^p \lambda_i (\sum_{j \in S_i} r_j) = \sum_{k=1}^n r_k = 1$.

Conversely if r is represented as a convex combination of $\{r^{S_i}\}_{i=1}^p$ such as $r = \sum_{i=1}^p \mu_i r^{S_i}$, then we have the equation

$$\chi_N = \sum_{i=1}^{p} \Big(\mu_i / \sum_{j \in S_i} r_j\Big) \chi_{S_i}$$

by multiplying the diagonal matrix $(b_{ij})_{i,j=1}^n$, where $b_{ij} = r_i^{-1}$ if $i = j$ and $b_{ij} = 0$ otherwise, to both sides of the equality above. Therefore the family $\{S_i\}_{i=1}^p$ is balanced. □

Similar to the definition of r^S, we can define an element $\bar{x}^S$ of Δ for any $\bar{x} \in \Delta_{++}$ and any Borel subset S of Q with $\bar{x}(S) > 0$ by

$$\bar{x}^S(A) = \bar{x}(A \cap S)/\bar{x}(S), \quad A \in \Sigma.$$

Note that $\bar{x}^S$ belongs to Δ^S and $\bar{x}^S(\xi) = \int_S \xi d\bar{x}/\bar{x}(S)$ for any $\xi \in C(Q)$.

According to Proposition 1, we can define the balancedness of subfamilies of $\mathcal{N}$ by means of any vector r with $r \gg 0$. However, we cannot expect such uniformity in the infinite dimensional spaces. See the following example.

Example 1 Let m be the Lebesgue measure on $[0, 1]$, and consider the two elements $\bar{x} = m$ and $\bar{y} = m/2 + \hat{1}/2$ of $\Delta \subset M([0, 1])$. Let $S = [0, 1)$, and consider the family $\{S\}$. Then we have $\bar{x} = m = \bar{x}^S$ and $\bar{y} \neq m = \bar{y}^S$ in spite of the fact $\bar{x} \gg 0$ and $\bar{y} \gg 0$.

Inspired by Proposition 1 and Example 1, we define balancedness in compact Hausdorff spaces as follows:

Definition 1 Let Q be a compact Hausdorff space such that Δ_{++} is not empty, and let Σ be a Borel σ-field of Q. For an element $\bar{x}$ of Δ_{++} in $M(Q)$, let $\Sigma_{\bar{x}} = \{S \in \Sigma : \bar{x}(S) > 0\}$. A subfamily $\mathcal{B}$ of Σ is said to be *$\bar{x}$-balanced* if $\bar{x}$ belongs to the closed convex hull of the set $\{\bar{x}^S : S \in \mathcal{B} \cap \Sigma_{\bar{x}}\}$.

We probe the balancedness just defined hereafter. The following is the infinite dimensional version of the proposition obtained in Ichiishi[2].

Proposition 2 *Let $\bar{x}$ be an element of Δ_{++} and $\mathcal{B} = \{S_1, \dots, S_p\}$ be a finite subfamily of Σ such that $0 < \bar{x}(S_i) < 1$ for all $i = 1, \dots, p$. Then $\mathcal{B}$ is $\bar{x}$-balanced if and only if the family $\mathcal{B}' = \{Q \setminus S_1, \dots, Q \setminus S_p\}$ is $\bar{x}$-balanced.*

Proof. We need to prove only the "only if" part because of the symmetry of the statement. There are nonnegative numbers $\lambda_1, \dots, \lambda_p$ such that

$$\bar{x} = \sum_{i=1}^{p} \lambda_i \bar{x}^{S_i} \quad \text{and} \quad \sum_{i=1}^{p} \lambda_i = 1$$

by the hypothesis. Then we have $\sum_{i=1}^{p} \lambda_i (\bar{x} - \bar{x}^{S_i}) = 0$. On the other hand, we have $\bar{x} = \bar{x}(S_i)\bar{x}^{S_i} + \bar{x}(Q \setminus S_i)\bar{x}^{Q \setminus S_i}$; hence we have

$$\bar{x} - \bar{x}^{S_i} = -\frac{\bar{x}(Q \setminus S_i)}{\bar{x}(S_i)} (\bar{x} - \bar{x}^{Q \setminus S_i}).$$

Therefore we have

$$\sum_{i=1}^{p} \frac{\lambda_i \bar{x}(Q \setminus S_i)}{\bar{x}(S_i)} (\bar{x} - \bar{x}^{Q \setminus S_i}) = 0.$$

If we put $\mu = \sum_{i=1}^{p} \frac{\lambda_i \bar{x}(Q \setminus S_i)}{\bar{x}(S_i)}$ and $\mu_i = \sum_{i=1}^{p} \frac{\lambda_i \bar{x}(Q \setminus S_i)}{\mu \bar{x}(S_i)}$, then we have the desired result $\bar{x} = \sum_{i=1}^{p} \mu_i \bar{x}^{Q \setminus S_i}$. □

We cannot expect the corresponding result for infinite families as shown in the following examples.

Example 2 Let N^* be the one-point compactification of the positive integers and $\bar{x}$ the Borel measure on N^* defined by $\bar{x}(n) = 1/2^{(n+1)}$ for $n = 1, 2, \dots$, and $\bar{x}(\infty) = 1/2$. Let $S_n = N^* \setminus \{n\}$ and consider the family $\mathcal{B} = \{S_n : n = 2, 3, \dots\}$. Then $\mathcal{B}$ is $\bar{x}$-balanced because $\bar{x}^{S_n}$ converges to $\bar{x}$. On the other hand, it is trivial that the family $\mathcal{B}' = \{\{2\}, \{3\}, \dots\}$ is not $\bar{x}$-balanced.

We need the following lemma to present the next example and we shall also use it later.

Lemma 1 *Let $\{x_\alpha\}$ be a net in Δ and x an element of Δ. Then $x_\alpha(A) \to x(A)$ for every $A \in \Sigma$ implies $x_\alpha \to x$.*

Proof. Let ξ be an element of $C(Q)$. Since ξ is bounded, for any $\varepsilon > 0$, there is a measurable simple function σ on Q such that $\|\xi - \sigma\| < \varepsilon/3$. Since $x_\alpha(\sigma) \to x(\sigma)$ by the hypothesis, there is α_0 such that $|x_\alpha(\sigma) - x(\sigma)| < \varepsilon/3$ for $\alpha \geq \alpha_0$. Therefore, for any $\alpha \geq \alpha_0$, we have

$$\begin{aligned} |x_\alpha(\xi) - x(\xi)| &= |x_\alpha(\xi) - x_\alpha(\sigma)| + |x_\alpha(\sigma) - x(\sigma)| + |x(\sigma) - x(\xi)| \\ &< \|\xi - \sigma\| + \varepsilon/3 + \|\sigma - \xi\| \\ &< \varepsilon. \end{aligned}$$

□

Example 3 Consider the compact Hausdorff space $Q = \{0,1\}^N$ with the product topology, where $N = \{1, 2, \ldots.\}$ and $\{0,1\}$ has the usual topological group structure, and let $\bar{x}$ be the Haar measure on Q. For any two disjoint finite subsets A and B of N, define the subset $H^{A,B}$ of Q by

$$H^{A,B} = \{q \in Q : q(n) = 0 \text{ for } n \in A, q(n) = 1 \text{ for } n \in B\}.$$

Then it is easily seen that $\bar{x}(H^{A,B}) = 1/2^{|A|+|B|}$. Define a sequence S_n by

$$S_1 = H^{\{1\},\emptyset}, \text{ and } S_{n+1} = H^{\{n+1\},\{1,\ldots,n\}} \cup S_n.$$

Then we have $\bar{x}(S_n) = 1 - 1/2^n$ and $S_n \nearrow Q \setminus \{(1,1,\ldots,1,\ldots)\}$. Therefore, we have

$$\bar{x}^{S_n}(A) = \frac{\bar{x}(A \cap S_n)}{\bar{x}(S_n)} \to \bar{x}(A) \quad \text{for all } A \in \Sigma;$$

and hence, $\bar{x}^{S_n}$ converges to $\bar{x}$ by Lemma 1. Therefore the family $\{S_n\}$ is $\bar{x}$-balanced. On the other hand, since $Q \setminus S_n = H^{\emptyset,\{1,\ldots,n\}} \subset Q \setminus S_1 \subset H^{\emptyset,\{1\}}$, $\bar{x}^{Q \setminus S_n}$ belongs to $\Delta^{H^{\emptyset,\{1\}}}$, i.e. $\text{supp}(\bar{x}^{Q\setminus S_n}) \subset H^{\emptyset,\{1\}}$ for all $n = 1, 2, \ldots.$ Therefore, every point of $\overline{\text{co}}\{\bar{x}^{Q\setminus S_n} : n = 1, 2, \ldots\}$ has the support in $H^{\emptyset,\{1\}}$. However, since $\text{supp}(\bar{x}) = Q$, we have $\bar{x} \notin \overline{\text{co}}\{\bar{x}^{Q\setminus S_n} : n = 1, 2, \ldots\}$ and $\mathcal{B}' = \{Q \setminus S_n : n = 1, 2, \ldots\}$ is not $\bar{x}$-balanced.

We expect that suitable partitions of Q satisfy the balancedness we have defined. The following proposition assures us our definition of balancedness is appropriate.

Proposition 3 *Let $\bar{x}$ be an element of Δ_{++}. Let $\{A_i\}$ be a countable covering of a compact Hausdorff space Q such that $A_i \in \Sigma$ for all i and $\bar{x}(A_i \cap A_j) = 0$ for $i \neq j$. Then $\{A_i\}$ is $\bar{x}$-balanced. In particular, any countable partition of Q consisting of Borel sets is $\bar{x}$-balanced for any $\bar{x} \in \Delta_{++}$.*

Proof. Define a disjoint countable covering $\{B_j\}$ of Q by $B_j = A_j \setminus \bigcup_{i>j} A_i$. Then it is easily seen that $\bar{x}(B_j) = \bar{x}(A_j)$ and $\bar{x}^{B_j} = \bar{x}^{A_j}$. Therefore, for any $A \in \Sigma$,

$$\begin{aligned}\bar{x}(A) &= \sum \bar{x}(A \cap B_j) \\ &= \sum \bar{x}(B_j)\bar{x}^{B_j}(A) \\ &= \sum \bar{x}(B_j)\bar{x}^{A_j}(A).\end{aligned}$$

Since $\{B_j\}$ is a disjoint covering of Q, we have $\sum \bar{x}(B_j) = 1$. If the sum is essentially finite, then the proof is completed. Suppose the sum has infinite terms essentially. We can assume $\bar{x}(B_1) \neq 0$ without loss of generality. For any $n = 1, 2, \ldots$, define an element x_n of $\text{co}\{\bar{x}^{A_j} : j = 1, 2, \ldots\}$ by $x_n = \sum_{j=1}^{n} (\bar{x}(B_j)/\lambda_n)\bar{x}^{A_j}$, where $\lambda_n = \sum_{j=1}^{n} \bar{x}(B_j)$. Then we have the equations

$$\begin{aligned}\bar{x}(A) &= (\lambda_n x_n)(A) + \sum_{j>n} \bar{x}(B_j)\bar{x}^{A_j}(A) \\ &= x_n(A) + (\lambda_n - 1)x_n(A) + \sum_{j>n} \bar{x}(B_j)\bar{x}^{A_j}(A).\end{aligned}$$

Therefore we have

$$\begin{aligned}|\bar{x}(A) - x_n(A)| &\leq (1 - \lambda_n)x_n(A) + \sum_{j>n} \bar{x}(B_j) \\ &\leq 2(1 - \lambda_n).\end{aligned}$$

We can conclude $x_n \to \bar{x}$ from Lemma 1 since $\lambda_n \to 1$. Therefore we have $\bar{x} \in \overline{\text{co}}\{\bar{x}^{A_j} : j = 1, 2, \ldots\}$. □

We give another example of a balanced family such that any two sets of the family have a nonempty intersection.

Example 4 Let N^* be the one point compactification of the positive integers, and $\bar{x}$ the element defined in Example 2 above. Consider the family

$\{A, B, C\}$ of the subsets of N^* defined by $A = \{1, 2\}$, $B = \{2, 3, \ldots, \infty\}$, and $C = \{3, 4, \ldots, \infty, 1\}$. Then the family $\{A, B, C\}$ is $\bar{x}$-balanced.

In fact, we have

$$\bar{x}^A(n) = \begin{cases} 2/3 & \text{for } n = 1 \\ 1/3 & \text{for } n = 2 \\ 0 & \text{otherwise} \end{cases}, \qquad \bar{x}^B(n) = \begin{cases} 0 & \text{for } n = 1 \\ 2/3 & \text{for } n = \infty \\ 1/(3 \times 2^{(n-1)}) & \text{otherwise} \end{cases},$$

$$\bar{x}^C(n) = \begin{cases} 2/7 & \text{for } n = 1 \\ 0 & \text{for } n = 2 \\ 4/7 & \text{for } n = \infty \\ 1/(7 \times 2^{(n-2)}) & \text{otherwise} \end{cases}$$

and

$$\bar{x} = \frac{3}{16}\bar{x}^A + \frac{3}{8}\bar{x}^B + \frac{7}{16}\bar{x}^C.$$

References

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