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A Single Facility Location Problem with respect to Minisum Criterion II

Masamichi Kon（金　正道）
Graduate School of Natural Science and Technology,
Kanazawa University

Shigeru Kushimoto（久志本　茂）
Faculty of Education, Kanazawa University

Abstract

The distance (the A-distance) which is determined by given orientations is proposed by P.Widmayer, Y.F.Wu and C.K.Wong[10]. In this article, we consider a single facility location problem under the A-distance with respect to minisum criterion, study properties of the optimal solution for the problem, and propose "The Iterative Algorithm" to find all optimal solutions. In $\boldsymbol{R}^2$, we consider the problem $\min_{\boldsymbol{x}\in\boldsymbol{R}^2} F(\boldsymbol{x}) = \sum_{i=1}^{n} w_i d_A(\boldsymbol{x}, \boldsymbol{y}_i)$, where d_A is the A-distance.

For each demand point and each given orientation, we draw a oriented line which passes the point. Then a plane is divided into regions. We call a point passing some lines an *intersection point.* It is shown that there exists an optimal solution in the set of *intersection points.* Let P be the smallest convex polygon including all demand points, which all boundary lines are given orientated lines. It is shown that any optimal solution is in P.

We propose "The Iterative Algorithm", where the solution in each step is an *intersection point.* This algorithm is as follows: We choose any demand point as an initial solution. The solution in the next step is determined as the adjacent *intersection point* in the steepest orientation among orientations according to lines which passes the present solution. We also propose the method to determine the adjacent *intersection point* easily by sorting lines.

Keywords : Location problem, Minisum criterion, A-metric, A-distance

1 Introduction

When we consider where a facility should be located on a plane, the problem will be a single facility location problem. In this paper, we assume that the facility can be located almost everywhere on a plane. Such a model is called a continuous model. We use minisum criterion. For example, minisum criterion is used when the facility is the public one. In this article, we consider a single facility location problem of a continuous model with respect to minisum criterion.

On the other hand, the location problem is different according to the distance used in it. So various distances are used [1,2,6,7,8,9]. The distance (the A-distance) which is determined by given orientations is proposed by P.Widmayer, Y.F.Wu and C.K.Wong[10]. We consider the above location problem under the A-distance.

In section 2, we give some definitions and results for the A-distance. In section 3, we formulate a single facility minisum location problem under the A-distance , and give some properties of the optimal solution, and propose an iterative algorithm for that problem.

2 The A-Metric

In $\boldsymbol{R}^2$, let

$$A = \{\alpha_1, \alpha_2, \cdots, \alpha_m\},\ m \geq 2$$

be a set of given orientations, where α_i's are angles for positive direction of x-axis, and we assume that $0 \leq \alpha_1 < \alpha_2 < \cdots < \alpha_m < \pi$.

If the orientation of a line (a half line, a line segment) belongs to A, we call the line (the half line, the line segment) A-oriented line (half line, line segment).

Let $B = \{L : A\text{-oriented line segment }\}$, and for $\boldsymbol{x}_1, \boldsymbol{x}_2 \in \boldsymbol{R}^2$, let $[\boldsymbol{x}_1, \boldsymbol{x}_2] = \{x = \lambda \boldsymbol{x}_1 + (1-\lambda)\boldsymbol{x}_2 : 0 \leq \lambda \leq 1\}$. The A-distance is defined as follows.

Definition 1 (The A-Distance) *For any $\boldsymbol{x}_1, \boldsymbol{x}_2 \in \boldsymbol{R}^2$, we define the A-distance between $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ $d_A(\boldsymbol{x}_1, \boldsymbol{x}_2)$ as*

$$d_A(\boldsymbol{x}_1, \boldsymbol{x}_2) = \begin{cases} d_2(\boldsymbol{x}_1, \boldsymbol{x}_2), & [\boldsymbol{x}_1, \boldsymbol{x}_2] \in B \\ \min\limits_{\boldsymbol{x}_3 \in \boldsymbol{R}^2} \left\{ d_2(\boldsymbol{x}_1, \boldsymbol{x}_3) + d_2(\boldsymbol{x}_3, \boldsymbol{x}_2) \,\middle|\, [\boldsymbol{x}_1, \boldsymbol{x}_3], [\boldsymbol{x}_3, \boldsymbol{x}_2] \in B \right\}, & \text{otherwise} \end{cases}$$

where d_2 is the Euclidean metric. We call d_A the A-metric. In fact, d_A is a metric in $\boldsymbol{R}^2$ [10].

Theorem 1 ([10]) *For any A and any $\boldsymbol{x}_1, \boldsymbol{x}_2 \in \boldsymbol{R}^2$, $d_A(\boldsymbol{x}_1, \boldsymbol{x}_2)$ is always realized by the polygonal line which consists of at most two A-oriented line segments.*

In the following, we assume that A is given.

Definition 2 (The A-Circle) *For a point $\boldsymbol{y} \in \boldsymbol{R}^2$ and a constant $c > 0$,*

$$\{\boldsymbol{x} \in \boldsymbol{R}^2 | d_A(\boldsymbol{y}, \boldsymbol{x}) = c\}$$

is called the A-circle with radius c at center $\boldsymbol{y}$.

For the simplicity, let $\alpha_{m+k} = \pi + \alpha_k$ $(k = 1, 2, \cdots, m)$ and $\alpha_0 = \alpha_{2m} - 2\pi$, $\alpha_{2m+1} = \alpha_1 + 2\pi$. In this case, it follows that $0 \leq \alpha_1 < \alpha_2 < \cdots < \alpha_m < \pi \leq \alpha_{m+1} < \cdots < \alpha_{2m} < 2\pi$. Moreover, let $\boldsymbol{a}_j = (\cos\alpha_j, \sin\alpha_j)$ $(j = 0, 1, \cdots, 2m+1)$.

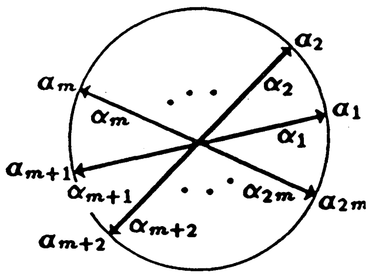

Figure 1.

By Definition 1, we can show the following lemma easily.

Lemma 1 ([6]) *For $\boldsymbol{x} = (x^1, x^2), \boldsymbol{y} = (y^1, y^2) \in \boldsymbol{R}^2$, if $\boldsymbol{x} \in \boldsymbol{y} + \mathcal{C}\{\boldsymbol{a}_j, \boldsymbol{a}_{j+1}\}$, where $\mathcal{C}\{\boldsymbol{a}_j, \boldsymbol{a}_{j+1}\} = \{\lambda \boldsymbol{a}_j + \mu \boldsymbol{a}_{j+1} | \lambda, \mu \geq 0\}$, then $d_A(\boldsymbol{x}, \boldsymbol{y})$ can be represented as follows.*

$$d_A(\boldsymbol{x}, \boldsymbol{y}) = \frac{(x^1 - y^1)(\sin\alpha_{j+1} - \sin\alpha_j) + (x^2 - y^2)(\cos\alpha_j - \cos\alpha_{j+1})}{\sin(\alpha_{j+1} - \alpha_j)} \tag{2.1}$$

Lemma 2 ([6]) *For each $\boldsymbol{y} \in \boldsymbol{R}^2$, $f(\boldsymbol{x}) = d_A(\boldsymbol{x}, \boldsymbol{y})$ is a convex function.*

3 The Minisum Location Problem

In this section, we consider a single facility minisum location problem under the A-distance.

In $\boldsymbol{R}^2$, let $\boldsymbol{y}_i = (y_i^1, y_i^2)$ for $i = 1, 2, \cdots, n$, be the location of n demand points, w_i be positive weight associated each demand point i, $\boldsymbol{x}$ be the location of the facility to be located. The problem is formulated as follows.

$$\min_{\boldsymbol{x} \in \boldsymbol{R}^2} F(\boldsymbol{x}) \tag{3.1}$$

where $F(\boldsymbol{x}) = \sum_{i=1}^{n} w_i d_A(\boldsymbol{x}, \boldsymbol{y}_i)$.

By Lemma 2, F is a convex function. Therefore, (3.1) is the convex programming problem. Furthermore, there exists an optimal solution for (3.1). Let S^* be a set of optimal solutions for (3.1).

First, for each demand point, we draw all A-oriented lines which passes the point. Let $L_{ij} = \{\boldsymbol{y}_i + \gamma \boldsymbol{a}_j | \gamma \in \boldsymbol{R}\}$ $(i = 1, 2, \cdots, n; j = 1, 2, \cdots, m)$. We call an element of

$$\bigcup_{i,i'} \bigcup_{j \neq j'} (L_{ij} \bigcap L_{i'j'})$$

an *intersection point* (Figure 2). Let I be a set of all *intersection points.* We call a convex polygon $S \subset \boldsymbol{R}^2$ *a region* if all boundary lines of S are some of L_{ij}'s and $intS \neq \emptyset$ and

$$(intS) \bigcap L_{ij} = \emptyset, \quad i = 1, 2, \cdots, n; j = 1, 2, \cdots, m,$$

where $intS$ is the interior of S (Figure 2).

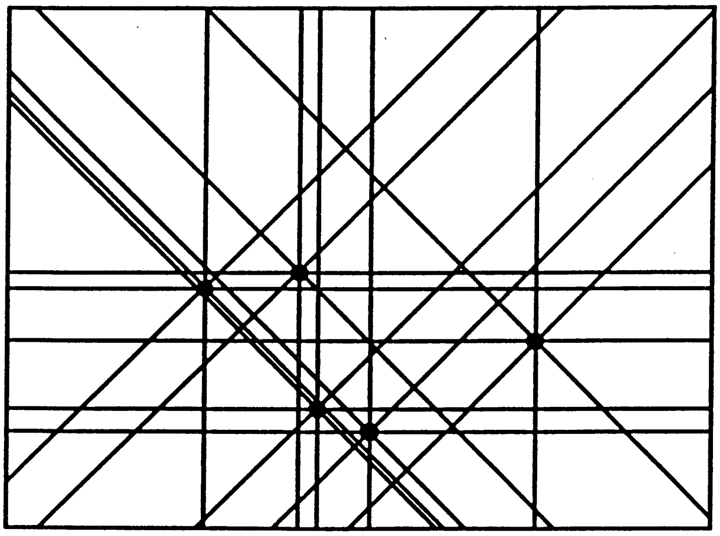

Figure 2. $A = \{0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}\}$

Since, for some j_i $(1 \leq j_i \leq 2m)$, $i = 1, 2, \cdots, n$, any *region* S is represented as

$$S = \bigcap_{i=1}^{n} (\boldsymbol{y}_i + C\{\boldsymbol{a}_{j_i}, \boldsymbol{a}_{j_i+1}\}),$$

F is linear on each *region* S by (2.1). On the other hand, for any $x \notin \bigcup_{i,j} L_{ij}$, *the region* S whose interior contains x is determined uniquely as follows.

$$S = \bigcap_{i=1}^{n} (y_i + C\{a_{j_i}, a_{j_i+1}\})$$

where

$$x \in y_i + C\{a_{j_i}, a_{j_i+1}\}, \quad i = 1, 2, \cdots, n.$$

In this case, we call *the region* S *the region characterized by* x, and write $S(x)$.

Theorem 2

$$S^* \bigcap I \neq \emptyset$$

Theorem 3 *Let* S_1, S_2 *be adjacent bounded regions.*

$$S_1 \subset S^* \Longrightarrow S^* \bigcap (int S_2) = \emptyset$$

Next let $\{P_\lambda, \lambda \in \Lambda\}$ be a set of all convex polygons including $\{y_1, y_2, \cdots, y_n\}$, which all boundary lines are A-oriented lines. Let

$$P = \bigcap_{\lambda \in \Lambda} P_\lambda.$$

P is the smallest convex polygon including all demand points, which all boundary lines are A-oriented lines(Figure 3). Note that boundary lines of P are A-oriented supporting lines to $\{y_1, y_2, \cdots, y_n\}$.

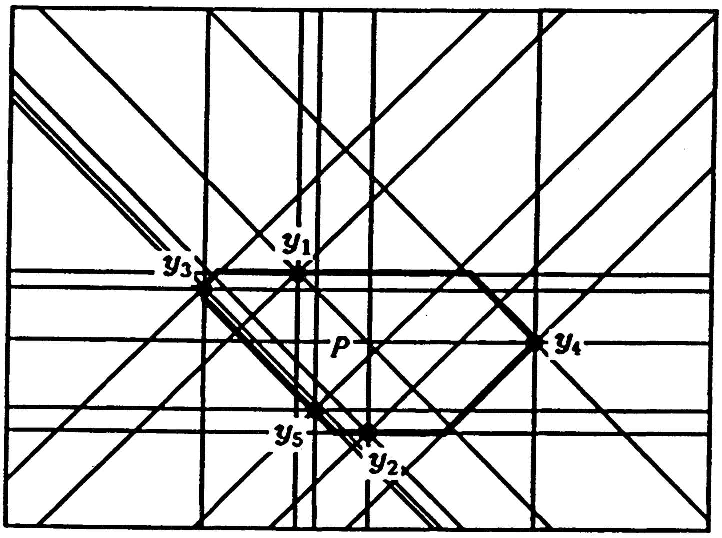

Figure 3. P for $\{y_1, y_2, \cdots, y_5\}$, $A = \{0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}\}$

Theorem 4

$$S^* \subset P$$

By Theorem 2 and 4, there exists an optimal solution which is an *intersection point* on P. So we consider to determine such an optimal solution by iterative method that traces only *intersection points* where a initial point is any demand point. Now we assume that a solution after the rth iteration is $\boldsymbol{x}^{(r)}$. Of cource, $\boldsymbol{x}^{(r)} \in I$.

We say $\boldsymbol{a}_j$ $(1 \leq j \leq 2m)$ holds condition (Q) for $\boldsymbol{x}^{(r)}$ if

$$\exists \boldsymbol{y}_i \ (1 \leq i \leq n) \text{ s.t. } \boldsymbol{y}_i = \boldsymbol{x}^{(r)} + \gamma \boldsymbol{a}_j \text{ for some } \gamma \in \boldsymbol{R}$$

and

$$\exists \varepsilon > 0 \text{ s.t. } \boldsymbol{x}^{(r)} + \varepsilon \boldsymbol{a}_j \in P.$$

Let

$$J = \{j | \boldsymbol{a}_j \text{ holds condition (Q) for } \boldsymbol{x}^{(r)}\}.$$

For the objective function F, we represent the right differential coefficient of F at $\boldsymbol{x}_0 \in \boldsymbol{R}^2$ with respect to $\boldsymbol{o} \neq \boldsymbol{a} \in \boldsymbol{R}^2$ as $\partial_+ F(\boldsymbol{x}_0; \boldsymbol{a})$, and let

$$u^{(r)} = \min_{j \in J} \left\{ \partial_+ F(\boldsymbol{x}^{(r)}; \boldsymbol{a}_j) \right\}. \tag{3.2}$$

If $u^{(r)} \geq 0$ then $\boldsymbol{x}^{(r)}$ is optimal by the convexity of the objective function F.

By Theorem 3 and 4, the set of optimal solutions S^* is an *intersection point* or A-oriented line segment, whose end points are adjacent *intersection points*, or *a region*.

Before we state the algorithm, we consider the determination of P. First, we sort L_{ij}'s according to x-intercept or y-intercept. For each j, if $\alpha_j \neq \frac{\pi}{2}$, L_{ij} is $-x \tan \alpha_j + y = y_i^2 - y_i^1 \tan \alpha_j$, let $b_{ij} = y_i^2 - y_i^1 \tan \alpha_j$. Otherwise, i.e. $\alpha_j = \frac{\pi}{2}$, L_{ij} is $x = y_i^1$, let $b_{ij} = y_i^1$.

For each j, we sort all different lines among L_{ij}, $i = 1, 2, \cdots, n$ according to b_{ij}, $i = 1, \cdots, n$ in ascending order. Let those lines be $\ell_1^j, \ell_2^j, \cdots, \ell_{n_j}^j$, where ℓ_i^j is the ith line among α_j-oriented sorted lines. Note that $n_j \leq n$.

Now we assume that $0 \leq \alpha_1, \cdots, \alpha_{q-1} < \frac{\pi}{2}$, $\alpha_q = \frac{\pi}{2}$, $\frac{\pi}{2} < \alpha_{q+1}, \cdots, \alpha_m$. Next we arrange ℓ_i^j, $i = 1, 2, \cdots, n_j; j = 1, 2, \cdots, m$ as follows.

$$\ell_1^1, \ell_1^2, \cdots, \ell_1^{q-1}, \ell_{n_q}^q, \ell_{n_{q+1}}^{q+1}, \cdots, \ell_{n_m}^m, \ell_{n_1}^1, \ell_{n_2}^2, \cdots, \ell_{n_{q-1}}^{q-1}, \ell_1^q, \ell_1^{q+1}, \cdots, \ell_1^m \tag{3.3}$$

Now coefficients of L_{ij}'s are stored. When we consider P, "$=$" in ℓ_1^j $(1 \leq j \leq m)$ is replaced by "$\geq$", and "$=$" in $\ell_{n_j}^j (1 \leq j \leq m)$ is replaced by "$\leq$". P is the region determined by its system of inequalities. Note that this system of inequalities may contain reduntant inequalities.

For the simplicity of the representation, let lines in (3.5) be $\ell(1), \ell(2), \cdots, \ell(2m)$. Especially, let $\ell(2m+1) = \ell(1), \ell(2m+2) = \ell(2)$.

The procedure for the determination of P

Step 1. Determine an *intersection point* of $\ell(1)$ and $\ell(2)$, and let its *intersection point* be $\boldsymbol{z}_1$. Set $j = 2$.

Step 2. Determine an *intersection point* of $\ell(j)$ and $\ell(j+1)$, and let its *intersection point* be $\boldsymbol{z}_j$.

Step 3. If $\boldsymbol{z}_j = \boldsymbol{z}_{j-1}$ then remove $\ell(j)$.

Step 4. If $j = 2m + 1$ then stop otherwise set $j = j + 1$ and go to Step 2.

Let $\ell(j_1)$, $\ell(j_2)$, $\cdots$, $\ell(j_p)$ be lines left after the above procedure. P is represented by the system of inequalities corrsponding to those lines. Now its system of inequalities does not contain redundant inequalities.

The Iterative Algorithm

Step 1. Choose any demand point as an initial solution $\boldsymbol{x}^{(0)}$. (We choose the demand point with the largest weight.) Set $r = 0$.

Step 2. Calculate $u^{(r)}$.

Step 3. If $u^{(r)} > 0$ then stop. $\boldsymbol{x}^{(r)}$ is an optimal solution.

Step 4. If $u^{(r)} = 0$ then stop. If the number of $\boldsymbol{a}_k$'s which hold $u^{(r)} = 0$, i.e. $\partial_+ F(\boldsymbol{x}^{(r)}; \boldsymbol{a}_k) = 0$, is

1. one, then any point on $[\boldsymbol{x}^{(r)}, \boldsymbol{x}_k^{(r)}]$, where $\boldsymbol{x}_k^{(r)}$ is an $\boldsymbol{a}_k$-oriented adjacent it intersection point to $\boldsymbol{x}^{(r)}$, is optimal.
2. two, then for sufficiently small $\varepsilon > 0$ and $\boldsymbol{a}_{k_1}, \boldsymbol{a}_{k_2}$ which hold $u^{(r)} = 0$, any point $\boldsymbol{x} \in S(\boldsymbol{x}^{(r)} + \varepsilon(\boldsymbol{a}_{k_1} + \boldsymbol{a}_{k_2}))$ is optimal.

Step 5. Otherwise, i.e. $u^{(r)} < 0$, choose any $\boldsymbol{a}_k$ which holds $u^{(r)} = \partial_+ F(\boldsymbol{x}^{(r)}; \boldsymbol{a}_k)$, and let $\boldsymbol{x}^{(r+1)}$ be an $\boldsymbol{a}_k$-oriented adjacent *intersection point* to $\boldsymbol{x}^{(r)}$. Set $r = r + 1$, and go to Step 2.

The Iterative Algorithm is convergent because $\boldsymbol{x}^{(r)}$ in The Iterative Algorithm is different from $\boldsymbol{x}^{(0)}, \boldsymbol{x}^{(1)}, \cdots, \boldsymbol{x}^{(r-1)}$ and the number of *intersection points* is finite. Since the number of *intersection poins* is $\mathcal{O}(n^2)$ and the complexity of caluculation of $F(\boldsymbol{x}^{(r)})$ is $\mathcal{O}(n)$, the complexity of The Iterative Algorithm is $\mathcal{O}(n^3)$.

If $\boldsymbol{a}_k$ which holds $u^{(r)} = \partial_+ F(\boldsymbol{x}^{(r)}; \boldsymbol{a}_k)$ is determined in Step 5 of The Iterative Algorithm, we need to determine $\boldsymbol{x}^{(r+1)}$ which is an $\boldsymbol{a}_k$-oriented adjacent *intersection point* to $\boldsymbol{x}^{(r)}$. Next we consider the procedure to determine $\boldsymbol{x}^{(r+1)}$. For each j, let $f_j(x, y)$ be the left side of L_{ij}, i.e.

$$f_j(x,y) = \begin{cases} -x \tan \alpha_j + y & \text{if } \alpha_j \neq \frac{\pi}{2}, \\ x & \text{if } \alpha_j = \frac{\pi}{2}. \end{cases}$$

If $\alpha_j \neq \frac{\pi}{2}$ then $\nabla f_j(x, y) = (-\tan \alpha_j, 1)$ otherwise $\nabla f_j(x, y) = (1, 0)$.

We assume that an initial solution $\boldsymbol{x}^{(0)} = \boldsymbol{y}_{i_0}$ is given. Set $r = 0$ where r is a counter. Next, we determine

$$\ell^j_{s_r(j)}, \ j = 1, 2, \cdots, m; 1 \leq s_r(j) \leq n_j$$

corresponding to $L_{i_0 j}$, $j = 1, 2, \cdots, m$ (e.g. binary search). Note that $\boldsymbol{x}^{(r)}$ is an *intersection point* of $\ell^j_{s_r(j)}$'s, i.e. $\boldsymbol{x}^{(r)}$ can be represented by $s_r(j)$'s. We concentrate on $s_r(j)$, $j = 1, 2, \cdots, m$. We assume that $\alpha_k (1 \leq k \leq 2m)$ is determined in Step 5 of The Iterative Algorithm. Let

$$j' = \begin{cases} k & \text{if } 1 \leq k \leq m, \\ k - m & \text{if } m < k \leq 2m. \end{cases}$$

For $j \neq k, j \neq k - m (1 \leq j \leq m)$, let

$$t_{kj} = < \nabla f_j(x, y), \boldsymbol{a}_k >$$

where $< \boldsymbol{x}, \boldsymbol{y} >$ is inner product of $\boldsymbol{x}, \boldsymbol{y} \in \boldsymbol{R}^2$.

Next, for each $j \neq j'$ $(1 \leq m)$, we determine an *intersection point* of $\ell^{j'}_{s_r(j')}$ and $\ell^{j}_{s_r(j)+sign(t_{kj})}$, where, for $x \neq 0$,

$$sign(x) = \begin{cases} +1 & \text{if } x > 0, \\ -1 & \text{if } x < 0. \end{cases}$$

For $j \neq j'$ $(1 \leq j \leq m)$, let $\boldsymbol{z}_{kj}$ be an *intersection point* of

$$\ell^{j'}_{s_r(j')} \quad \text{and} \quad \ell^{j}_{s_r(j)}. \tag{3.4}$$

$\boldsymbol{z}_{kj}$'s are candidates for $\boldsymbol{x}^{(r+1)}$. Let

$$J^{(r)} = \left\{ j : d_2(\boldsymbol{x}^{(r)}, \boldsymbol{z}_{kj}) = \min_{j \neq j', 1 \leq j \leq m} \left\{ d_2(\boldsymbol{x}^{(r)}, \boldsymbol{z}_{kj}) \right\} \right\}. \tag{3.5}$$

$\boldsymbol{x}^{(r+1)}$ is an *intersection point* of $\ell^{j}_{s_r(j)+sign(t_{kj})}$, $j \in J^{(r)} \bigcup \{j'\}$. Let

$$s_{r+1}(j) = \begin{cases} s_r(j) & \text{if } j = j', \\ s_r(j) + sign(t_{kj}) & \text{if } j \in J^{(r)}, \\ s_r(j) + 0.5 sign(t_{kj}) & \text{otherwise.} \end{cases} \tag{3.6}$$

Set $r = r + 1$, and go to the next step.

In the next step, a solution $\boldsymbol{x}^{(r)}$ is an *intersection point* of $\ell^{j}_{s_r(j)}$'s such that $s_r(j) \in \boldsymbol{N}$ where $\boldsymbol{N}$ is a set of natural numbers. For j such that $s_r(j) \notin \boldsymbol{N}$, it means that $\boldsymbol{x}^{(r)}$ lies between $\ell^{j}_{[s_r(j)]}$ and $\ell^{j}_{[s_r(j)]+1}$ where $[\cdot]$ is Gauss' symbol.

In The Iterative Algorithm, we condider the representation of a solution after the rth iteration $\boldsymbol{x}^{(r)}$ as

$$\boldsymbol{x}^{(r)},\ \ell^{j}_{s_r(j)},\ j = 1, 2, \cdots, m.$$

The above is only the case of $r = 0$. If we consider the case of $r \geq 1$ in The Iterative Algorithm, $sign(t_{kj})$ in (3.4) and the second equation in (3.6) is replaced by

$$\begin{array}{ll} [s_r(j) - 0.5] & \text{if } sign(t_{kj}) = -1, \\ [s_r(j) + 1] & \text{if } sign(t_{kj}) = 1, \end{array}$$

and $sign(t_{kj})$ in the third equation in (3.6) is replaced by

$$sign(t_{kj})([s_r(j)] - [s_r(j) - 0.5]).$$

4 Numerical Example

In the problem (2), let $n = 5, A = \{0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}\}, \boldsymbol{y}_1 = (63, 97), \boldsymbol{y}_2 = (102, 7), \boldsymbol{y}_3 = (10, 90), \boldsymbol{y}_4 = (197, 57), \boldsymbol{y}_5 = (73, 20), w_1 = w_2 = w_3 = w_4 = w_5 = 1$. We set $\boldsymbol{x}^{(0)} = \boldsymbol{y}_1$ as an initial solution, by The Iterative Algorithm, we have $\boldsymbol{x}^{(0)} = (63, 97), \boldsymbol{x}^{(1)} = (63, 90), \boldsymbol{x}^{(2)} = (63, 57), \boldsymbol{x}^{(3)} =$

$(73, 57), x^{(4)} = (73, 36)$. The optimal solution is $x^{(4)} = (73, 36)$, and the optimal value is $F(x^{(4)}) = 340.22$ (Figure 4).

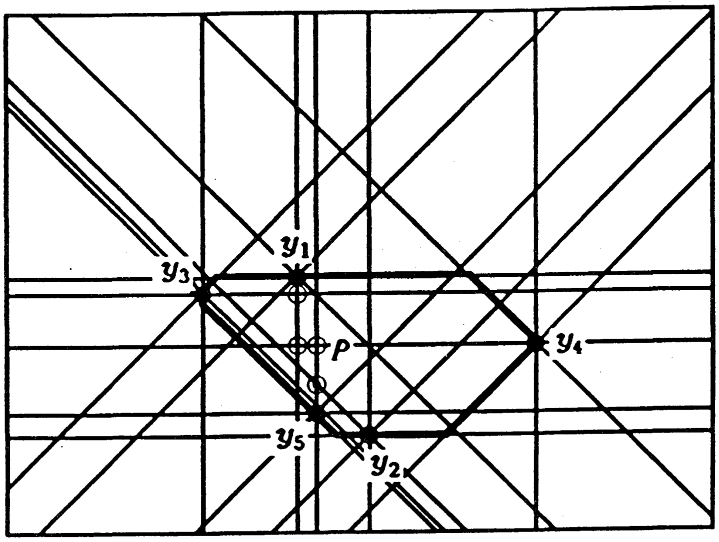

Figure 4. $\odot$: the optimal solution

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