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Qualitative behavior of solutions to Ginzburg-Landau type systems

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1 Introduction

We are concerned with the time-dependent Ginzburg-Landau (G-L) equation on a circle:

$$U_t = U_{xx} + \lambda(1-|U|^2)U, \qquad x \in S^1 = \boldsymbol{R}/2\pi\boldsymbol{Z} \tag{1}$$

where

$$U = \begin{pmatrix} u \\ v \end{pmatrix}, \quad |U| = (u^2+v^2)^{1/2}, \quad U_t = \frac{\partial}{\partial t}U, \quad U_{xx} = \frac{\partial^2}{\partial x^2}U$$

and λ is a positive parameter. This equation has a Lyapunov function:

$$\mathcal{E}(U) := \int_{S^1} \{|U_x|^2 + \frac{\lambda}{2}(1-|U|^2)^2\} dx \tag{2}$$

and the semiflow in $H^1(S^1;\boldsymbol{R}^2)$ generated by solutions admits a global attractor $\mathcal{A}$, that is, the maximal compact invariant set. Considering this fact and that the semiflow is analytic, we see from [5] that every solution converges to an equilibrium solution, namely a solution to

$$U_{xx} + \lambda(1-|U|^2)U = 0, \qquad x \in S^1. \tag{3}$$

Hence if we could obtain all the solution to (3) and Morse index of them, we would be able to discuss the existence of connecting orbits between equilibrium solutions by applying the Conley index theory (see [3]). As seen below, however, it is not so easy to determine the Morse index of the solutions because Equation (3) has many secondary bifurcating solutions for large λ. In addition when we consider a little generalized system written as

$$\begin{aligned} &U_t = \tfrac{1}{a(x)}(a(x)U_x)_x + \lambda(1-|U|^2)U, \\ &a(\cdot) \in C^2(S^1;\boldsymbol{R}^2), \quad a(x) > 0, \end{aligned} \tag{4}$$

such an approach is not useful for studying of the dynamical structure.

Here we take account of some qualitaive behaivior of the solution to Equation (1) (or (4)) and give an insight into the structure of the attractor. Actually we propose a non-trivial Morse decompotion of the attractor, where the Morse decomposition means a decomposition of the attractor into a finite number of disjoint compact invariant sets. We believe that this decomposition will be useful when we discuss the existence of connecting orbits between the equilibrium solutions.

2 Equilibrium Solutions

We state a structure of equilibrium solutions to (3); all the solutions can be classified as follows:

i) constant solutions:

$$U = 0, \qquad U = \begin{pmatrix} \cos\xi \\ \sin\xi \end{pmatrix} \quad (\xi \text{ is any constant});$$

ii) collinear solutions:

$$U = \begin{pmatrix} \cos\xi \\ \sin\xi \end{pmatrix} w(x),$$

ξ is any given number and $w(x)$ is a non-constant solution to

$$w_{xx} + \lambda(1 - w^2)w = 0, \qquad x \in S^1. \tag{5}$$

In this case the projection of solution $U(x)$ into (u, v)-plane,

$$\mathcal{C}(U) := \{(u(x), v(x)) \in \boldsymbol{R}^2 : x \in S^1\}$$

make a line segment.

iii) winding solutions: $\mathcal{C}(U)$ for the solution $U(x)$ winds around $\mathcal{O} = (0, 0)$ clockwise or anti-clockwise, where the winding number of $\mathcal{C}(U)$ is defined as

$$W(\mathcal{C}(U)) := \frac{\theta(2\pi) - \theta(0)}{2\pi}$$

where $\theta(x)$ is the angle of the vector $(u(x), v(x))$.

Remark 2.1 *We easily see that for $\lambda > m^2$ Equation (3) has solutions:*

$$U(x) = a_m \begin{pmatrix} \cos(x+\xi) \\ \sin(x+\xi) \end{pmatrix}, \quad a_m = \sqrt{1 - m^2/\lambda}, \tag{6}$$

$$U(x) = \begin{pmatrix} \cos(\xi) \\ \sin(\xi) \end{pmatrix} \phi_\lambda(x), \tag{7}$$

where $\phi_\lambda(x)$ is a solution of (5) which bifurcated from origin at $\lambda = m^2$. In addition, as λ passes through $3m^2 - \lambda^2/2$ $(\lambda = 1, 2, \cdots, 2m-1)$, secondary bifurcations take place from the solution (6) and different type of winding solutions appear. Unfortunately it is difficult to solve the linearized eigenvalue problem of the secondary bifurcating solutions.

3 Main Theorem

Before stating main results obtained in [4], we introduce notations. We write

$$U^{\gamma}(t,x) = \begin{pmatrix} u^{\gamma}(t,x) \\ v^{\gamma}(t,x) \end{pmatrix} := \begin{pmatrix} \cos\gamma & \sin\gamma \\ -sin\gamma & \cos\gamma \end{pmatrix} U(t,x),$$

and define

$$Z_N(U(t,\cdot);\gamma) := \{\text{the number of zeros of } x \longmapsto u^{\gamma}(t,x)\}$$

provided that $u^{\gamma}(t,x)$ is not identically zero. Denote an alpha and omefa-limit sets by

$$\alpha(U_0) = \cap_{\tau\le 0}\mathcal{C}\ell\left(\cup_{\tau\ge t}U(t,\cdot;U_0)\right), \quad \omega(U_0) = \cap_{\tau\ge 0}\mathcal{C}\ell\left(\cup_{\tau\le t}U(t,\cdot;U_0)\right),$$

where $U(t,\cdot;U_0)$ is a solution of (1) with $U(0,\cdot) = U_0$. Set

$$\begin{aligned} M_{2k} = \;& \{U_0 \in \mathcal{A} : Z_N(U(t,\cdot;U_0);\gamma) = 2k \;\; (-\infty < t < \infty) \\ & \text{for any } \gamma \;\text{ and }\; 0 \notin \alpha(U_0)\}, \\ M^c_{2k} = \;& \{U \in \mathcal{A} : U = \begin{pmatrix} \cos\gamma \\ \sin\gamma \end{pmatrix} \phi_k(x) \;\; (0 \le \gamma < 2\pi), \\ & \phi_k(x) \text{ is a bifurcating solution from zero at} \lambda = k^2\}. \end{aligned}$$

Note that M_{2k} contains the winding solutions with winding number k or $-k$.

Lemma 3.1 *If $k^2 < \lambda$, then $M(2k) := M_{2k} \cup M^c_{2k}$ is non-empty and compact invariant set in $\mathcal{A}$. Moreover $M(2k) \cap M(2j) = \emptyset$ for $k \neq j, 0 < k^2, j^2 < \lambda$.*

This lemma leads us to a Morse decompotion as follows:

Theorem 3.2 *Given any λ in $(k^2,(k+1)^2)$, set $\Lambda = \{0,2,4,\cdots,2k,2k+1\}$ and*

$$M(0) = \{U = \begin{pmatrix} \cos\gamma \\ \sin\gamma \end{pmatrix} : 0 \le \gamma < 2\pi\}, \qquad M(2k+1) = \{0\}.$$

Then $\mathcal{A}$ is decomposed as $\mathcal{A} = \mathcal{M} \cup \mathcal{C}$, where $\mathcal{M} = \cup_{K\in\Lambda} M(K)$ and C is the set of connecting orbits; hence for $U_0 \in \mathcal{C}$ there exists $K, L \in \Lambda, K > L$ such that $\alpha(U_0) \subset M(K), \omega(U_0) \subset M(L)$.

Remark 3.3 *With a little modification of the definition of the Morse sets $M(K)$, we also obtain quite a similar Morse decompotion of the global attractor for (4).*

Let $\Omega(\epsilon)$ be a family of parametrized domain as defined by

$$\Omega(\epsilon) = \{x = (r\cos\theta, r\sin\theta) \in \boldsymbol{R}^2 : 1 < |x| < 1 + \epsilon a(\theta), \;\; 0 \le \theta < 2\pi\},$$

and consider the G-L equation in $\Omega(\epsilon)$ with Neumann boundary condition:

$$U_t = \Delta U + \lambda(1 - |U|^2)U \quad \text{in } \Omega(\epsilon), \qquad \frac{\partial u}{\partial n} = 0 \quad \text{on } \partial\Omega(\epsilon). \tag{8}$$

Then Equation (4) can be regarded as a limit equation of (8) when $\epsilon \to 0$. Indeed the dynamical system (of finite dimension) on the global attractor $\mathcal{A}_\epsilon$ for (8) is a nice perturbed system of the one for (4) in some sense (see [1]). Hence the Morse decompotion obtained above persists under the perturbation caused by this deformation of the domain.

References

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