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POWERS AND COMMUTATIVITY OF SELFADJOINT OPERATORS

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1. Introduction

Let $B(\mathfrak{H})$ be the algebra of bounded operators on a Hilbert space $\mathfrak{H}$. If $T \in B(\mathfrak{H})$ satisfies $(Tx, x) \geq 0$ for every $x \in \mathfrak{H}$, then T is said to be nonnegative, and we denote it by $T \geq 0$. If $ReT = \frac{1}{2}(T + T^*)$ is nonnegative, then T is said to be accretive.

DePrima and Richard [2] showed the following:

Theorem A. *If T^n is accretive for $n = 1, 2, \ldots$, then $T \geq 0$.*

They have proved this theorem by using a mapping theorem for numerical ranges due to T.Kato. For completeness, we give another proof dependent on Sz.-Nagy's technique[8]. Since, for scalar $a > 0$, $(T + a)^n$ is accretive, we may assume that $Re\sigma(T) > 0$ and that $\|T\| < 1$. Since the inverse T^{-n} of T^n is accretive, $T^{-n}(I-T)^{-m}$ is accretive too for $n, m = 1, 2, \ldots$,because the coefficients of its power series expansion are nonnegative. Thus $ReT^n(I-T)^m \geq 0$. Using Bernstein's polynomials, for any polynomial $f \geq 0$ on the interval [0,1], we have $Ref(T) \geq 0$. Therefore the sequence $\{ReT^n\}_{n=0}^{\infty}$ satisfies the moment problem. Thus there is a nonnegative dilation $H \in B(\mathfrak{K})$ of T, that is $\mathfrak{H} \subset \mathfrak{K}$, $ReT^n = PH^n|_{\mathfrak{H}}$, where P is the orthogonal projection from $\mathfrak{K}$ to $\mathfrak{H}$. Hence we get Kadison's inequality $(ReT)^2 \leq Re(T^2)$, which implies $0 \leq (T - T^*)^2 = -4(ImT)^2$. Consequently we obtain $T \geq 0$.

2. Powers of Operators

In this section we shall extend Theorm A. For $X \in B(\mathfrak{H})$, a subspace $\mathfrak{L} \subset \mathfrak{H}$ is said to reduce X if $X\mathfrak{L} \subset \mathfrak{L}$ and $X^*\mathfrak{L} \subset \mathfrak{L}$. Then X can be represented as

$$X = X|_{\mathfrak{L}} \oplus X|_{\mathfrak{L}^\perp}.$$

An operator X is called completely non selfadjoint (c.n.s.) if there is no non-zero reducing subspace $\mathfrak{L}$ for X such that $X|_{\mathfrak{L}}$ is selfadjoint.

Let us remark that every operator X can be uniquely represented as a sum of a selfadjoint operator and a c.n.s. operator. In fact, $\mathfrak{L} := \{x \in \mathfrak{H} : X^n x = X^{*n} x, n = 1, 2, \ldots\}$ is a closed subspace of $\mathfrak{H}$; since $X^n(Xx) = X^{*(n+1)}x = X^{*n}Xx$ and $X^n X^* x = X^{*n} X^* x$ for any x in $\mathfrak{L}$, $\mathfrak{L}$ reduces X, so that $X|_{\mathfrak{L}}$ is selfadjoint and $X|_{\mathfrak{L}^\perp}$ is c.n.s.

Theorem 1. *Let X and Y be in $B(\mathfrak{H})$, and assume that $ReX \geq ReY$. If $X^n + Y^n$ is a selfadjoint operator for $n = 1, 2, \ldots$, then there is a subspace $\mathfrak{L}$ such that $\mathfrak{L}$ reduces both X and Y to selfadjoint operators and $(X|_{\mathfrak{L}^\perp})^* = Y|_{\mathfrak{L}^\perp}$, that is,*

$$X = s.a. \oplus T,\ Y = s.a. \oplus T^*.$$

Proof. Since $ImX^n = -ImY^n$, $\mathfrak{L} := \{x \in \mathfrak{H} : X^n x = X^{*n} x, n = 1, 2, \ldots\}$ reduces X and Y to selfadjoint operators. We have only to show $(X|_{\mathfrak{L}^\perp})^* = Y|_{\mathfrak{L}^\perp}$. X and Y have the representations: $X = A + iB, Y = C - iB$, where A, B and C are selfadjoint operators. Then the assumption means $A \geq C$. Let us note that $\mathfrak{L}$ reduces A, B and C, and that $B|_{\mathfrak{L}} = 0$. We determine the sequences of operators $\{A_n\}, \{B_n\}, \{C_n\}$ and $\{D_n\}$ by $A_1 = A, B_1 = B, C_1 = C, D_1 = -B$, $A_{n+1} = AA_n - BB_n, B_{n+1} = AB_n + BA_n, C_{n+1} = CC_n + BD_n, D_{n+1} = CD_n - BC_n$. It is easy to see that they are selfadjoint; for instance, if they are selfadjoint for n, then

$A_{n+1}^* = A_n A - B_n B = (AA_{n-1} - BB_{n-1})A - (AB_{n-1} + BA_{n-1})B$

$= A(A_{n-1}A - B_{n-1}B) - B(B_{n-1}A + A_{n-1}B) = AA_n - BB_n = A_{n+1}$.

Thus we have $X^n = A_n + iB_n, Y^n = C_n + iD_n$. $B_2 + D_2 = 0$ means $(A - C)B + B(A - C) = 0$, from which it follows that $(A - C)B = B(A - C) = 0$, because $A \geq C$. Since $B_{n+1} + D_{n+1} = 0$, we have $(A - C)B_n + B(A_n - C_n) = 0$, from which it follows that $(A - C)B_n = B(A_n - C_n) = 0$, because the range of B is orthogonal to the one of $A - C$. Thus $(A - C)\mathfrak{H} \subset \mathfrak{L}$, and hence $(A - C)|_{\mathfrak{L}^\perp} = 0$. Consequently we obtain $(X|_{\mathfrak{L}^\perp})^* = Y|_{\mathfrak{L}^\perp}$. □

From this theorem and Theorem A we get the following :

Theorem 2. *Let X and Y be bounded operators satisfying $ReX \geq ReY$. If $X^n + Y^n \geq 0$ for $n = 1, 2, \ldots$, then X and Y are selfadjoint operators.*

Corollary 1. *If $X^n + Y^n \geq 0$ for $n = 1, 2, \ldots$, and $ReX \geq ReY \geq 0$, then $X \geq Y \geq 0$.*

In the above theorems the assumption $ReX \geq ReY$ is indispensable. For instance, take 2×2 matrices:

$$X = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, Y = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}.$$

Then $X^n + Y^n \geq 0$. However neither matrix is selfadjoint.

3. Commutativity

We consider applications of the above theorems to commutativity of selfadjoint operators.

Theorem 3. *Let A, B and C be selfadjoint operators satisfying $B \geq C$. If $AB^n + C^n A \geq 0$ for $n = 1, 2, ...,$ and if $A \geq 0$ or $C \geq 0$, then A commutes to B and C.*

Proof. Suppose that $A \geq 0$. Substituting $B + \delta$ and $C + \delta (\delta > 0)$ for B and C, respectively, we may assume that $B \geq C \geq 0$ and hence that A is invertible. $AB + CA \geq 0$ implies that $A(B - C)$ is selfadjoint, so that A commutes to $(B - C)$. We have

$$(A^{\frac{1}{2}} B A^{-\frac{1}{2}})^n + (A^{-\frac{1}{2}} C A^{\frac{1}{2}})^n \geq 0.$$

Since

$$Re(A^{\frac{1}{2}} B A^{-\frac{1}{2}}) = Re(A^{-\frac{1}{2}} C A^{\frac{1}{2}}) + (B - C),$$

by Theorem 2 we get $AB = BA, AC = CA$. Suppose next that $C \geq 0$. Then we may assume that $A \geq 0$. Therefore from the above it follows that A commutes to B and C. □

Corollary 2. *Let A and B be selfadjoint operators, and suppose that $A \geq 0$ or $B \geq 0$. If $AB^n + B^n A \geq 0$ for $n = 1, 2, ...,$ then A and B are commutative.*

This was shown in [7]. And then M.Fujii, R.Nakamoto, M.Nakamura[3] and S.Izumino[4] have given the other proofs of it. Let us remark that in this corollary we may exclude the condition :$A \geq 0$ or $B \geq 0$. Indeed, from $A(B^2)^n + (B^2)^n A \geq 0$ for $n = 1, 2, ...,$ it follows that $AB^2 = B^2 A \geq 0$. Since the closures of the ranges of B, B^2 and $|B|$ are equal, for the orthogonal projection P onto this space, we have $PA = AP \geq 0$. Therefore we get $PAPB^n + B^n PAP \geq 0$ for $n = 1, 2, ...,$ which implies B commutes to PAP. Thus $AB = APB = PAPB = BPAP = BPA = BA$.

Lemma. *Let A and B be nonnegative selfadjoint operators. If $AB + BA \geq 0$, then $AB^t + B^t A \geq 0$ for $0 \leq t \leq 1$.*

Proof. We may assume that B is invertible. For $0 < t < 1$ we have

$$B^t = \frac{sin(\pi t)}{\pi} \int_0^\infty \lambda^{t-1} (B + \lambda)^{-1} B d\lambda,$$

from which $AB^t + B^t A \geq 0$ follows. □

Let $\{E_\lambda\}$ and $\{F_\lambda\}$ be the spectral families corresponding to selfadjoint operators A and B, respectively. Then we denote $A \prec B$ if $\{E_\lambda\} \geq \{F_\lambda\}$ for every λ. For any

$B \geq 0$ and $C \geq 0$, there is the supremum $B \vee C$ of B and C in this order and it is equal to $\lim_{n\to\infty}(B^n + C^n)^{\frac{1}{n}}$ (see [6],[5],[1]).

Proposition. *Let A be a selfadjoint operator, and B and C nonnegative selfadjoint operators. If*

$$AB^n + C^n A \geq 0 \; for \; n = 1, 2, ...,$$

then A and $B \vee C$ are commutative.

Proof. We may assume that A is nonnegative. We can easily obtain

$$A(B^n + C^n) + (B^n + C^n)A \geq 0 \; for \; n = 1, 2, ...$$

For each m, we have

$$A(B^n + C^n)^{\frac{m}{n}} + (B^n + C^n)^{\frac{m}{n}} A \geq 0 \; for \; n = m, m+1, ...$$

Thus $A(B \vee C)^m + (B \vee C)^m A \geq 0$. By Corollary 2, A commutes to $B \vee C$. □

Corollary 3. *Let P and Q be orthogonal projections, and suppose that A is a selfadjoint operator. If $AP + QA \geq 0$, then A and $P \vee Q$ are commutative.*

Theorem 4. *Let A, B and C be selfadjoint operators satisfying $BC = CB$. If $AB^n + C^n A \geq 0$ for $n = 1, 2, ...$, and if $A \geq 0$ or $B, C \geq 0$, then A commutes to B and C.*

Proof. We may assume that A, B and C are nonnegative. Proposition implies that A commutes to $B \vee C$. Since $A(B - C)$ is selfadjoint, A commutes to $B - C$. From $B \vee C = \lim_{n\to\infty}(B^n + C^n)^{\frac{1}{n}}$, it follows that $B \vee C$ commutes to B and C, and hence we gain
$B \vee C = \min\{X : X \geq B, X \geq C, XB = BX, XC = CX\} = \frac{1}{2}(B + C + |B - C|)$.
Consequently A commutes to B and C. □

Corollary 4. *Let P and Q be commutative orthogonal projections, and suppose that A is a selfadjoint operator. If $AP + QA \geq 0$, then A commutes to P and Q.*

At the end of this paper we give a counter example so that in Corollary 4 we can not exclude the condition: P and Q are commutative. Take 2×2 matrices:

$$P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, Q = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, A = \begin{pmatrix} 4 & -1 \\ -1 & 2 \end{pmatrix}.$$

Then $AP + QA \geq 0$, but $AP \neq PA$.

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