数理解析研究所講究録
903巻 1995年 97-105

Characterizations of operators satisfying chaotic order and its applications

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In what follows, a capital letter means a bounded linear operator on a complex Hilbert space H. An operator T is said to be positive (in symbol : $T \geq 0$) if $(Tx, x) \geq 0$ for all $x \in H$. Also an operator T is strictly positive (in symbol : $T > 0$) if T is positive and invertible. We write $A \gg B$ if $logA \geq logB$ which is called the chaotic order [FK][FFK2].

Results in this paper will appear in [F9] and [F10].

§1. Generalizations of Kosaki trace inequalities and related trace inequalities on chaotic order

By using an extension of the Furuta inequality and following after Kosaki's nice technique, we shall show generalizations of trace inequalities by Kosaki and related trace inequalities on chaotic order (i.e., $logA \geq logB$).

It is well known that $A \geq B \geq 0$ ensures $Tr(f(A)) \geq Tr(f(B))$, where Tr denotes the usual trace and f is a continuous increasing function on $\mathbf{R}_+$ with $f(0) = 0$. Kosaki [K] shows the following very interesting trace inequality as a generalization of the above mentioned trace inequality.

Theorem A [K]. *Assume* $A \geq B \geq 0$ *and* $p > 1$, $\alpha \geq Max\{-1, \frac{-p}{2}\}$.

(i) *Then there exists a partial isometry operator* U *satisfying*

$$A^{\frac{\alpha}{2}} B^p A^{\frac{\alpha}{2}} \leq U^* A^{p+\alpha} U.$$

(ii) *For a continuous increasing function* f *on* $\mathbf{R}_+$ *with* $f(0) = 0$*, we have*

$$Tr(f(A^{\frac{\alpha}{2}} B^p A^{\frac{\alpha}{2}})) \leq Tr(f(A^{p+\alpha}))$$

In the above statements the invertibility of A *is assumed when* $\alpha < 0$.

Recently Ando-Hiai [AH] established various log-majorization results to ensure excellent and useful inequalities for unitarily invariant norms.

On the other hand, as an extension of [H] and [L], we established the following Furuta inequality ([F1][F2][F3][FUJ] and [KA]).

Theorem B. *If $A \geq B \geq 0$, then for each $r \geq 0$,*

(i) $$(B^r A^p B^r)^{1/q} \geq (B^r B^p B^r)^{1/q}$$

and

(ii) $$(A^r A^p A^r)^{1/q} \geq (A^r B^p A^r)^{1/q}$$

hold for p and q such that $p \geq 0$ and $q \geq 1$ with $(1+2r)q \geq p+2r$.

In Furuta [F8] we established the following extension of Theorem B, which interpolates this log majorization results of Ando-Hiai and the Furuta inequality and moreover extends the results in [FFK1],[FFK2], [FK],[F4] and [F5].

Theorem C [F8]. *If $A \geq B \geq 0$ with $A > 0$, then for each $t \in [0,1]$ and $p \geq 1$,*

$$F_{p,t}(A,B,r,s) = A^{\frac{-r}{2}}\{A^{\frac{r}{2}}(A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^sA^{\frac{r}{2}}\}^{\frac{1-t+r}{(p-t)s+r}}A^{\frac{-r}{2}}$$

is a decreasing function of both r and s for any $s \geq 1$ and $r \geq t$ and the following inequality holds $F_{p,t}(A,A,r,s) \geq F_{p,t}(A,B,r,s)$, that is, for each $t \in [0,1]$ and $p \geq 1$,

$$A^{1-t+r} \geq \{A^{\frac{r}{2}}(A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^sA^{\frac{r}{2}}\}^{\frac{1-t+r}{(p-t)s+r}}$$

holds for any $s \geq 1$ and r such that $r \geq t$.

The following Theorem 1.1 is an extension of Theorem A [K].

Theorem 1.1 [F9]. *Let A and B be positive operators such that $A \geq B \geq 0$ with $A > 0$. Assume that $p \geq 1$, $s \geq 1$, $t \in [0,1]$ and $\beta \geq Max[t-1, \frac{1}{2}\{t(s+1)-ps\}]$. Then the following inequalities hold.*

(I) *There exists the a partial isometry operator U satisfying*

$$A^{\frac{\beta}{2}}(A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^sA^{\frac{\beta}{2}} \leq U^*A^{(p-t)s+\beta}U.$$

(II) *For a continuous increasing function f on $\mathbf{R}_+$ with $f(0)=0$,*

$$Tr\{f(A^{\frac{\beta}{2}}(A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^sA^{\frac{\beta}{2}})\} \leq Tr\{f(A^{(p-t)s+\beta})\}.$$

The following Theorem 1.2 is a parallel result to Theorem 1.1.

Theorem 1.2 [F9]. *Let A and B be positive invertible operators such that $A \gg B$* (i.e., $logA \geq logB$). *Assume that $p \geq u > 0$, $s \geq 1$, $\alpha \in [0,1]$ and $\beta \geq -u\alpha$. Then the following inequalities hold.*

(I) *There exists the a partial isometry operator U satisfying*

$$A^{\frac{\beta}{2}}(A^{\frac{u\alpha}{2}}B^pA^{\frac{u\alpha}{2}})^sA^{\frac{\beta}{2}} \leq U^*A^{(u\alpha+p)s+\beta}U.$$

(II) *For a continuous increasing function f on $\mathbf{R}_+$ with $f(0) = 0$,*

$$Tr\{f(A^{\frac{\beta}{2}}(A^{\frac{u\alpha}{2}}B^pA^{\frac{u\alpha}{2}})^sA^{\frac{\beta}{2}})\} \leq Tr\{f(A^{(u\alpha+p)s+\beta})\}.$$

Corollary 1.3 [F9]. *Let A and B be positive invertible operators such that $A \gg B$* (i.e., $logA \geq logB$). *Assume that $p > 0$, and $\beta \geq 0$. Then the following inequalities hold.*

(I) *There exists the a partial isometry operator U satisfying*

$$A^{\frac{\beta}{2}}B^pA^{\frac{\beta}{2}} \leq U^*A^{p+\beta}U.$$

(II) *For a continuous increasing function f on $\mathbf{R}_+$ with $f(0) = 0$,*

$$Tr\{f(A^{\frac{\beta}{2}}B^pA^{\frac{\beta}{2}})\} \leq Tr\{f(A^{p+\beta})\}.$$

§2. Extensions of Ando's characterization of operators satisfying $logA \geq logB$ and its applications

It is shown in Ando [A] that $A \gg B$ holds if and only if $A^p \geq (A^{\frac{p}{2}}B^pA^{\frac{p}{2}})^{\frac{1}{2}}$ holds for all $p \geq 0$.

In this section , we state an extension of this result and its application.

Theorem 2.1 [F10]. *Let A and B be positive invertible operators. Then the following* (I) *and* (II) *holds.*

(I) *If $A \gg B$* (i.e., $logA \geq logB$) , *then for each $\alpha \in [0,1]$,and all $p \geq 0$ and $u \geq 0$,*

$$G_{p,u,\alpha}(A,B,r,s) = A^{\frac{-ur}{2}}\{A^{\frac{ur}{2}}(A^{\frac{u\alpha}{2}}B^pA^{\frac{u\alpha}{2}})^sA^{\frac{ur}{2}}\}^{\frac{u(\alpha+r)}{(u\alpha+p)s+ur}}A^{\frac{-ur}{2}}$$

is a decreasing function of both s and r such that $s \geq 1$ and $r + \alpha \geq 1$.

(II) $A \gg B$ *holds if and only if* $g(s,r) = A^{\frac{-r}{2}}(A^{\frac{r}{2}}B^sA^{\frac{r}{2}})^{\frac{r}{s+r}}A^{\frac{-r}{2}}$ *is a decreasing function of both* $r \geq 0$ *and* $s \geq 0$.

Theorem 2.2 [F10]. *Let A and B be positive invertible operators. Then the following assertions are mutually equivalent.*

(I) $A \gg B$ (i.e., $log A \geq log B$).

(II) *For each* $\alpha \in [0,1]$, *and all* $p \geq 0$ *and* $u \geq 0$,

$$A^{u(\alpha+r)} \geq \{A^{\frac{ur}{2}}(A^{\frac{u\alpha}{2}}B^pA^{\frac{u\alpha}{2}})^sA^{\frac{ur}{2}}\}^{\frac{u(\alpha+r)}{(u\alpha+p)s+ur}}$$

holds for any $s \geq 1$ *and* r *such that* $r + \alpha \geq 1$.

(III) *For each* $\alpha \in [0,1]$, *and all* $p \geq 0$,

$$A^{p(\alpha+r)} \geq \{A^{\frac{pr}{2}}(A^{\frac{p\alpha}{2}}B^pA^{\frac{p\alpha}{2}})^sA^{\frac{pr}{2}}\}^{\frac{\alpha+r}{(\alpha+1)s+r}}$$

holds for any $s \geq 1$ *and* r *such that* $r + \alpha \geq 1$.

(IV) *For each* $\alpha \in [0,1]$, *and all* $p \geq 0$,

$$A^{\alpha+r} \geq \{A^{\frac{r}{2}}(A^{\frac{\alpha}{2}}B^pA^{\frac{\alpha}{2}})^sA^{\frac{r}{2}}\}^{\frac{\alpha+r}{(\alpha+p)s+r}}$$

holds for any $s \geq 1$ *and* r *such that* $r + \alpha \geq 1$.

Theorem 2.3 [F10]. *Let A and B be positive invertible operators. If* $A \geq (A^{\frac{1}{2}}BA^{\frac{1}{2}})^{\frac{1}{2}}$, *then for each* $\alpha \in [0,1]$, *and all* $p \geq 1$ *and* $u \geq 1$,

$$H_{p,u,\alpha}(A,B,r,s) = A^{\frac{-ur}{2}}\{A^{\frac{ur}{2}}(A^{\frac{u\alpha}{2}}B^pA^{\frac{u\alpha}{2}})^sA^{\frac{ur}{2}}\}^{\frac{u(\alpha+r)}{(u\alpha+p)s+ur}}A^{\frac{-ur}{2}}$$

is a decreasing function of both s *and* r *such that* $s \geq 1$ *and* $r + \alpha \geq 1$.

Theorem 2.4 [F10]. *Let A and B be positive invertible operators. Then the following assertions are mutually equivalent.*

(I) $A \geq (A^{\frac{1}{2}}BA^{\frac{1}{2}})^{\frac{1}{2}}$.

(II) *For each* $\alpha \in [0,1]$, *and all* $p \geq 1$ *and* $u \geq 1$,

$$A^{u(\alpha+r)} \geq \{A^{\frac{ur}{2}}(A^{\frac{u\alpha}{2}}B^pA^{\frac{u\alpha}{2}})^sA^{\frac{ur}{2}}\}^{\frac{u(\alpha+r)}{(u\alpha+p)s+ur}}$$

holds for any $s \geq 1$ *and* r *such that* $r + \alpha \geq 1$.

(III) *For each* $\alpha \in [0,1]$, *and all* $p \geq 1$,

$$A^{p(\alpha+r)} \geq \{A^{\frac{pr}{2}}(A^{\frac{p\alpha}{2}}B^{p}A^{\frac{p\alpha}{2}})^{s}A^{\frac{pr}{2}}\}^{\frac{\alpha+r}{(\alpha+1)s+r}}$$

holds for any $s \geq 1$ *and* r *such that* $r + \alpha \geq 1$.

(IV) *For each* $\alpha \in [0,1]$, *and all* $p \geq 1$,

$$A^{\alpha+r} \geq \{A^{\frac{r}{2}}(A^{\frac{\alpha}{2}}B^{p}A^{\frac{\alpha}{2}})^{s}A^{\frac{r}{2}}\}^{\frac{\alpha+r}{(\alpha+p)s+r}}$$

holds for any $s \geq 1$ *and* r *such that* $r + \alpha \geq 1$.

Remark 2.1. When we replace the hypothesis $A \gg B$ in Theorem 2.2 by $A \geq (A^{\frac{1}{2}}BA^{\frac{1}{2}})^{\frac{1}{2}}$ in Theorem 2.4, then all the results in Theorem 2.2 remain valid if we replace $p \geq 0$ and $u \geq 0$ in Theorem 2.2 by $p \geq 1$ and $u \geq 1$ in Theorem 2.4.

Also we remark that when we replace the hypothesis $A \gg B$ in (I) of Theorem 2.1 by $A \geq (A^{\frac{1}{2}}BA^{\frac{1}{2}})^{\frac{1}{2}}$ in Theorem 2.3, but the result in (I) of Theorem 2.1 remain valid if we replace $p \geq 0$ and $u \geq 0$ in Theorem 2.1 by $p \geq 1$ and $u \geq 1$ in Theorem 2.3.

Remark 2.2. Put $r + \alpha = 1$ and $s = 1$ in (IV) of Theorem 2.4, then $A^{u} \geq (A^{\frac{u}{2}}B^{p}A^{\frac{u}{2}})^{\frac{u}{p+u}}$ holds for any $p \geq 1$ and $u \geq 1$, which has been shown in [Theorem 2,FFW]. Also put $r + \alpha = 1$ and $s = 1$ in (IV) of Theorem 2.2, then we have $A^{u} \geq (A^{\frac{u}{2}}B^{p}A^{\frac{u}{2}})^{\frac{u}{p+u}}$ holds for any $p \geq 0$ and $u \geq 0$, and this result is already obtained in [Theorem 1,FFK2] and [Theorem 1,F5].

§3. Characterizations of operators satisfying $logA \geq logB$ associated with some operator equations and parallel results

Theorem 3.1 [F10]. *Let* A *and* B *be positive invertible operators. Then the following assertions are mutually equivalent.*

(I) $A \gg B$ (i.e., $logA \geq logB$) .

(II) *For each* $\alpha \in [0,1]$ *and for any* $p \geq u > 0$ *and* $s \geq 1$, *there exists a unique invertible positive contraction* $T_{p,u,\alpha}(s)$ *such that*

$$T_{p,u,\alpha}(s)A^{ps+u\alpha(s-1)}T_{p,u,\alpha}(s) = A^{\frac{-u\alpha}{2}}(A^{\frac{u\alpha}{2}}B^pA^{\frac{u\alpha}{2}})^sA^{\frac{-u\alpha}{2}}.$$

(III) *For any* $p \geq 0$, *there exists a unique invertible positive contraction* T_p *such that* $T_pA^pT_p = B^p$.

Next we state the following parallel result to Theorem 3.1.

Theorem 3.2 [F10]. *Let* A *and* B *be positive invertible operators. Then the following assertions are mutually equivalent.*

(I) $A \geq (A^{\frac{1}{2}}BA^{\frac{1}{2}})^{\frac{1}{2}}$.

(II) *For each* $\alpha \in [0,1]$ *and for any* $p \geq u \geq 1$ *and* $s \geq 1$, *there exists a unique invertible positive contraction* $T_{p,u,\alpha}(s)$ *such that*

$$T_{p,u,\alpha}(s)A^{ps+u\alpha(s-1)}T_{p,u,\alpha}(s) = A^{\frac{-u\alpha}{2}}(A^{\frac{u\alpha}{2}}B^pA^{\frac{u\alpha}{2}})^sA^{\frac{-u\alpha}{2}}.$$

(III) *For any* $p \geq 1$, *there exists a unique invertible positive contraction* T_p *such that* $T_pA^pT_p = B^p$.

Related to Theorem 3.1 and Theorem 3.2, we have the following parallel result by the same way as proof of Theorem 3.1.

Theorem 3.3 [F10]. *Let* A *and* B *be positive invertible operators. Then the following assertions are mutually equivalent.*

(I) $A \geq B \geq 0$.

(II) *For each* $t \in [0,1]$ *and for any* $p \geq 2$ *and* $s \geq 1$, *there exists a unique invertible positive contraction* $T_{p,t}(s)$ *such that*

$$T_{p,t}(s)A^{(p-t)s+t-1}T_{p,t}(s) = A^{\frac{t-1}{2}}(A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^sA^{\frac{t-1}{2}}.$$

(III) *For any* $p \geq 1$, *there exists a unique invertible positive contraction* T_p *such that* $T_pA^pT_p = A^{\frac{-1}{2}}B^{p+1}A^{\frac{-1}{2}}$.

(IV) $A^{1+p} \geq (A^{\frac{p}{2}}B^{p+2}A^{\frac{p}{2}})^{\frac{1}{2}}$ *for all* $p \geq 0$.

Results in this scetion are extensions of [Theorem 2, F6] and [Theorem 2.1, F7].

§4. Addendum . An extension of Theorem B

Recently we have the following result as an extension form of Theorem B.

Theorem D [FFK3]. *If $A \geq B > 0$, then $(B^\gamma A^\alpha B^\gamma)^\beta \geq (B^\gamma B^\alpha B^\gamma)^\beta$ holds under any one of the following conditions ;*

(i) $$\frac{1}{\beta} \leq \alpha,\ 0 < \beta < 1,\ \textit{and}\ \gamma = \frac{\alpha\beta - 1}{2(1-\beta)}$$

(ii) $$\frac{1}{\beta} \leq \alpha \leq 1,\ 1 < \beta \leq 2,\ \textit{and}\ \gamma = \frac{\alpha\beta - 1}{2(1-\beta)}$$

(iii) $$\frac{1}{2} \leq \alpha \leq 1,\ 2 \leq \beta,\ \textit{and}\ \gamma = \frac{\alpha\beta - 1}{2(1-\beta)}.$$

Remark 4.1. (i) and (ii) are announced in [Y, p 61], but we remark that (i) is nothing but exchange of parameters p, q and r in Theorem B , that is, put $(1+2r)q = p+2r$ for $r \geq 0$, $p \geq 1$ and $q \geq 1$ in Theorem B, then we easily obtain $p \geq q \geq 1$ and we have only to replace p by α, r by γ and $\frac{1}{q}$ by β, then we have (i) which is nothing but another expression of (i) in Theorem B.

Moreover a simple proof of (ii) is obtained in [FFK3] along a mothod of [F2] by using polar decomposition. Also in [FFK3] we obtained (iii) along a method of [F2]. We have to assume invertibilty of A and B in the cases (ii) and (iii) since $\gamma \leq 0$.

Remark 4.2. We should mention that Tanahashi [TA] has obtained several interesting results closely related to Theorem D.

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